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# THE ROLE OF CLEANER FISH IN A PREDATOR-PREY MODEL: DYNAMICS AND OPTIMAL HARVESTING

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**Abstract.** This study focuses on a predator-prey model that includes a Cleaner Fish. It highlights the crucial role of the Cleaner Fish in the system dynamics, as well as the effect of fishing on the three species. The analysis begins by studying the positivity and boundedness of the solutions to ensure that the populations remain present and limited. The stability of the system is examined around the interior equilibrium point, which represents a state where the populations of prey, predators, and Cleaner Fish maintain balance. The optimal harvesting policy is also investigated, aiming to find the fishing strategy that maximizes the dynamic profit of the species while preserving their sustainability. Finally, numerical simulations using Matlab software are conducted to illustrate the theoretical results obtained.

**Keywords:** predator-prey; stability analysis; Hopf bifurcation; optimal harvesting policy; fishing effort.

**2020 AMS Subject Classification:** 91B05, 91A06, 91B02, 91B50.

## INTRODUCTION

Prey-predator models, also known as Lotka-Volterra models, are widely used as mathematical tools to study the interactions between prey and predator populations in ecosystems.

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As highlighted by renowned evolutionary biologist Richard Levins, ”prey-predator models are abstract representations of the dynamic relationships between species, highlighting the mutual evolution and adaptation that occur in natural communities.” These models describe cyclic fluctuations in populations, where an increase in the prey population leads to a subsequent increase in the predator population, which, in turn, reduces the prey population, creating a feedback loop. As noted by complexity scientist John H. Holland, ”these dynamic models reveal the delicacy and a fragility of ecosystems, emphasizing the importance of stability and adaptation in ensuring species survival in a constantly changing world.” Prey-predator models provide valuable theoretical foundations for understanding ecological interactions and can be applied to a range of areas, such as natural resource management, biodiversity conservation, and even epidemic forecasting.

Cleaner fish, also known as cleaning fish, are intriguing and important players in aquatic ecosystems [1–5] . These fish, such as cleaner wrasses and cleaner gobies, have a specialized role in removing parasites and dead tissue from other fish species. Their unique cleaning behavior involves feeding on parasites found on the skin, gills, and even inside the mouths of host fish. As emphasized by renowned primatologist and conservationist Jane Goodall, ”these cleaner fish provide a valuable service in maintaining the health and hygiene of fish by eliminating harmful parasites.” However, in certain situations, cleaner fish occasionally exhibit a behavior known as ”cheating” where they may consume the offspring or eggs of their client fish. But, in general their presence and cleaning work are crucial for the survival and well-being of fish, thereby contributing to the stability of aquatic ecosystems.

Fishing of marine species has a profound impact on marine ecosystems. Both commercial and recreational fishing activities can lead to significant changes in the composition, the abundance, and the structure of fish populations and other marine species. Overfishing, in particular, can have detrimental consequences, including the depletion of fish stocks, disruption of food chains, and degradation of marine habitats, which leads researchers to seek effective solutions to conserve marine biodiversity through the construction of bioeconomic models [11–13]. The importance of these models lies in their ability to understand the complex interactions between biological resources and economic activities. These models integrate biological, ecological, and economic data to inform informed decision-making. They enable the assessment of long-term consequences of human activities on natural resources and the environment, then help identify sustainable strategies for ecosystem management.

In the seas and oceans, various ecological models demonstrate the crucial importance of cleaner fish in marine ecosystems [14]. Among these models, the example of the prey-predator relationship between the sardine (Sardina pilchardus) and the bluefin tuna (Thunnus thynnus), with the intervention of the cleaner fish Labroides dimidiatus, highlights the interdependence and impact of these species exploited by fishing. The sardine, as a prey species, is abundant in the oceans. It serves as an essential food source for many marine predators, including the bluefin tuna. The bluefin tuna, a large predator sought after for its prized flesh in the commercial fishing industry, has suffered from overfishing, leading to an imbalance in this ecosystem. This is where the role of the cleaner fish, the Labroides dimidiatus, becomes crucial. The cleaner fish feeds on external parasites and dead tissue present on the skin of fish, including the bluefin tuna. By cleaning the parasites, the cleaner fish promotes the health of predators and contributes to maintaining the ecological balance of the marine ecosystem. Sustainable fisheries management becomes essential to preserve this ecosystem and maintain the complex interactions between these species. Strict regulations, such as catch quotas and closed seasons, have been implemented to ensure the conservation of sardine and bluefin tuna populations, as well as the preservation of cleaner fish like the Labroides dimidiatus.

To better understand the behavior of the prey-predator system in the presence of cleaner fish, it is common to construct a biomathematical model. This model allows for the mathematical representation of interactions among the prey, predator, and cleaner fish, facilitating the analysis of system stability, particularly in the presence of fishing. The objective of this modeling and stability analysis is to comprehend how different species interact with each other and how human intervention, such as fishing, can impact the system's equilibrium. By analyzing the equations and conducting numerical simulations, one can study the effects of fishing on the populations of prey, predators, and cleaner fish, as well as the interactions among these populations.

The structure of the document is as follows: After the introduction, we present the proposed bioeconomic model in section 1. Section 2 is dedicated to studying the positivity, and boundedness of the system solutions. In section 3, we analyze the stability of the interior equilibrium point and discuss the occurrence of Hopf bifurcation. Section 4 is devoted to calculating the effort required to maximize fishermen's profits. Finally, we present numerical simulations of the theoretical results obtained.

## 1. PRESENTATION OF THE MODEL

Our bioeconomic model consists of a prey, a predator, and a predator cleaner fish, where the logistic growth function is used to describe the growth of prey, predators, and predator cleaner fish populations, taking into account environmental limitations. It considers resource availability, space, and competitive interactions among individuals to determine the growth of each species. The logistic growth function is often mathematically represented by the Verhulst equation:

$$\frac{dN}{dt} = RN\left(1 - \frac{N}{K}\right)$$

In this model, predators play a crucial role by providing a food source for cleaner fish. Cleaner fish feed on parasites and debris present on predators, thus benefiting from their existence. This mutualistic relationship between predators and cleaner fish is advantageous for both species. Cleaner fish find an abundant food source, while predators benefit from regular cleaning, which can improve their health and physical condition. Therefore, the mortality rate of predators depends on the cleaner fish and will be expressed as $-\frac{m}{1+\delta z}y$, such that in the absence of z , it will be in the form $-my$.

However, there is a negative aspect to this interaction. Cleaner fish may also feed on the eggs of predators, which can have an impact on their growth and reproduction. By consuming the eggs, cleaner fish reduces the opportunity for predators to successfully reproduce, which can affect the size of the predator population, this complex interaction between predators and cleaner fish demonstrates that their relationship is not solely beneficial. While cleaner fish benefit from the presence of predators by feeding on parasites, their consumption of predator eggs can influence the dynamics of the predator population. So, predator growth will be expressed as follows

$$\frac{dy}{dt} = R_2 y \left( \frac{1}{1+\lambda z} - \frac{y}{K} \right) - \frac{m}{1+\delta z} y$$

Regarding predation, the Lotka-Volterra equations are employed to model the interactions between prey and predators. These equations represent the changes in prey population based on predation by predators, as well as the changes in predator population based on their reproduction rate and predation success. These equations are based on the notion that prey population growth is constrained by predation from predators, while predator population growth depends on prey availability. We also incorporate the effect of fishing on all three species: the prey, the predator, and the cleaner fish. Fishing can have significant consequences on population dynamics and ecosystem balance. By incorporating this effect, It helps us assess sustainable fishing practices and make informed decisions to maintain population balance and overall health of the marine ecosystem. The captured quantity of each species is expressed as $-qEN$, where ”q” represents the catchability rate, ”E” represents the fishing effort and N the biomass of species. Thus, by exploiting all the preceding data, we obtain the following system

$$\begin{cases} \dot{x}(t) = R_1 x \left( 1 - \frac{x}{K} \right) - g_1 xy - q_1 E_1 x \\ \\ \dot{y}(t) = R_2 y \left( \frac{1}{1+\lambda z} - \frac{y}{K} \right) + g_2 xy - \frac{my}{1+\delta z} - q_2 E_2 y \\ \\ \dot{z}(t) = R_3 z \left( 1 - \frac{z}{K} \right) + \beta yz - q_3 E_3 z \end{cases} \tag{1}$$

Where x, y and z respectively represent the biomass of prey, predator and cleaner fish. The table below presents a summary of the parameters and their corresponding explanations.

| Parameter | Meaning |
|---|---|
| $R_i$ | Intrinsic growth rates |
| $K$ | Carrying capacities for the species |
| $g_1$ | Mortality rates due to predation effect |
| $g_2$ | Reproductive rates of predators based on prey encountered |
| $\beta$ | The benefit coefficient of the predator's existence for the cleaner fish |
| $\delta$ | The impact coefficient of the cleaner fish on the predator's mortality. |
| $\lambda$ | The impact coefficient of the cleaner fish on the predator's growth. |
| $q_i$ | catchability rate |
| $E_i$ | fishing effort |

TABLE 1. The meaning of bioeconomic parameters

## 2. Positivity and Boundedness of Solution

### 2.1. Positivity of Solutions.

**Theorem 1.** *The set* $\left\{(x,y,z)\in\mathbb{R}^3 : x,y,z\geq 0\right\}$ *is positively invariant for system.*

**Proof.** Note that the plans x=0, y=0 and z=0 are invariant, indeed

$$\tfrac{\mathrm{d}}{\mathrm{d}t}x(t)\big|_{x=0}=0 \quad \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{y=0}y(t)=0 \quad \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{z=0}z(t)=0.$$

So, if we start with strictly positive initial points, the solutions do not exceed these plans and remain positive for any $t>0$. So the set $\{(x,y,z)\in\mathbb{R} : x,y,z\geq 0\}$ is positively invariant.

### 2.2. Boundedness of solutions.

**Theorem 2.** *The solutions of system (1) are bounded.*

**Proof.**

(i) We consider the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t)=R_1x\left(1-\frac{x}{K}\right)-g_1xy-q_1E_1x\leq R_1x\left(1-\frac{x}{K}\right)$$

By integrating, we have x $\leq K$.

(ii) For $y$, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}y(t) &= R_2 y\left(\frac{1}{1+\lambda z}-\frac{y}{K}\right)+g_2xy-\frac{my}{1+\delta z}-q_2E_2y \\
&\leq R_2 y\left(\frac{1}{1+\lambda z}-\frac{y}{K}\right)+g_2xy \\
&\leq R_2 y\left(1-\frac{y}{K}\right)+g_2xy \\
&\leq R_2 y\left(1-\frac{y}{K}\right)+g_2Ky \\
&\leq y\left(R_2+g_2K-\frac{R_2y}{K}\right) \\
&\leq (R_2+g_2K)\,y\left(1-\frac{R_2y}{K\,(R_2+g_2K)}\right)
\end{aligned}
$$

So $y$ is bounded.

(iii) For $z$, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}z(t) &= R_3 z\left(1-\frac{z}{K}\right)+\beta yz-q_3E_3z \\
&\leq R_3 z\left(1-\frac{z}{K}\right)+\beta yz
\end{aligned}
$$

y is bounded, so there is M such that y $\leq M$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}z(t) &\leq R_3 z\left(1-\frac{z}{K}\right)+\beta Mz \\
&\leq z\left(R_3+\beta M-\frac{R_3z}{K}\right) \\
&\leq (R_3+\beta M)\,z\left(1-\frac{R_3z}{(R_3+\beta M)\,K}\right)
\end{aligned}
$$

So $z$ is bounded.

## 3. Stability Analysis

### 3.1. Equilibrium points.

To search the equilibrium, we solve the three equations

$$\dot{x}(t)=0,\quad \dot{y}(t)=0 \quad et \quad \dot{z}(t)=0$$

The system has the coexisting equilibrium point $(x^*, y^*, z^*)$, where

$$\begin{cases} x^* = \dfrac{K}{R_1}\left(R_1 - q_1E_1 - g_1y^*\right) \\ \\ z^* = \dfrac{K}{R_3}\left(R_3 - q_3E_3 + \beta y^*\right) \end{cases}$$

and $y^*$ is the solution of the cubic equation

$$A_3(y^*)^3 + A_2(y^*)^2 + A_1y^* + A_0 = 0 \tag{2}$$

where

$$\begin{aligned}
A_3 &= -\frac{R_2\lambda\delta K^2\beta^2}{KR_3^2} + g_2\frac{R_2\lambda\delta K^2\beta^2 g_1K}{R_3^2R_1} \\
A_2 &= -\frac{R_2}{K}\left[(\lambda+\delta)\frac{K\beta}{R_3} + 2\lambda\delta\frac{K^2\beta(R_3-q_3E_3}{R_3^2}\right] - \frac{q_2E_2\lambda\delta K^2\beta^2}{R_3^2} \\
&\quad -g_2\left[-\frac{g_1K}{R_1}\left[(\lambda+\delta)\frac{K\beta}{R_3} + 2\lambda\delta\frac{K^2\beta(R_3-q_3E_3}{R_3^2}\right] + \frac{K(R_1-q_1E_1)}{R_1}\frac{\lambda\delta K^2\beta^2}{R_3^2}\right] \\
A_1 &= \frac{R_2\delta K\beta}{R_3} - \frac{R_2}{K}\left[1 + (\lambda+\delta)\frac{K(R_3-q_3E_3)}{R_3} + \lambda\delta\frac{K^2(R_3-q_3E_3)^2}{R_3^2)}\right] \\
&\quad +g_2\frac{g_1K}{R_1}\left[1 + (\lambda+\delta)\frac{K(R_3-q_3E_3)}{R_3} + \lambda\delta\frac{K^2(R_3-q_3E_3)^2}{R_3^2)}\right] \\
&\quad -g_2\frac{K(R_1-q_1E_1)}{R_1}\left[(\lambda+\delta)\frac{K\beta}{R_3} + 2\lambda\delta\frac{K^2\beta(R_3-q_3E_3}{R_3^2}\right] \\
&\quad -\frac{m\lambda K\beta}{R_3} - q_2E_2\left[(\lambda+\delta)\frac{K\beta}{R_3} + 2\lambda\delta\frac{K^2\beta(R_3-q_3E_3}{R_3^2}\right] \\
A_0 &= -(g_2+q_2E_2)\left[1 + (\lambda+\delta)\frac{K(R_3-q_3E_3)}{R_3} + \lambda\delta\frac{K^2(R_3-q_3E_3)^2}{R_3^2)}\right] \\
&\quad -\frac{m\lambda K(R_3-q_3E_3)}{R_3} - m + R_2 + \frac{R_2K\delta(R_3-q_3E_3)}{R_3}
\end{aligned}$$

The discriminant of Eq.(2) is written as

$$\Delta = -27A_3^2A_0^2 + 18A_3A_2A_1A_0 - 4A_3A_1^3 - 4A_2^3A_0 + A_1^2A_2^2$$

According to [7], we have the following theorem.

**Theorem 3.** *Now, if* $\Delta > 0$*, and*

- *if* $A_2 > 0, A_1 > 0, A_0 > 0$ *or* $A_2 < 0, A_1 > 0, A_0 > 0$ *or* $A_2 < 0, A_1 < 0, A_0 > 0$*, then Eq.(2) has* *a single positive root.*

- *if* $A_2 > 0, A_1 < 0 A_0 < 0$, *or* $A_2 < 0, A_1 < 0 A_0 > 0$, *Eq.(2) has two positive roots.*
- *if* $A_2 > 0, A_1 < 0$, *and* $A_0 > 0$ *Eq.(2) has three positive roots.*

## 3.2. **Stability.** The Jacobian matrix for our system is expressed as follows

$$J = \begin{pmatrix} R_1 - \frac{2R_1x}{K} - g_1y - q_1E_1 & -g_1x & 0 \\ g_2y & \frac{R_2}{1+\lambda z} - \frac{2R_2y}{K} + g_2y - \frac{m}{1+\delta z} - q_2E_2 & \frac{-R_2\lambda y}{(1+\lambda z)^2} + \frac{\delta my}{(1+\delta z)^2} \\ 0 & \beta z & R_3 - \frac{2R_3z}{K} + \beta y - q_3E_3 \end{pmatrix}$$

At the positive equilibrium point $(x^*, y^*, z^*)$, the Jacobian matrix will be in the following form

$$J^* = \begin{pmatrix} \frac{-R_1x^*}{K} & -g_1x^* & 0 \\ g_2y^* & -\frac{R_2y^*}{K} & \frac{-R_2\lambda y^*}{(1+\lambda z^*)^2} + \frac{\delta my^*}{(1+\delta z^*)^2} \\ 0 & \beta z^* & -\frac{R_3z^*}{K} \end{pmatrix}$$

The corresponding characteristic equation of $J^*$

$$X^3 + a_2X^2 + a_1X + a_0 = 0 \tag{3}$$

with

$$\begin{aligned} a_2 &= \frac{R_1x^*}{K} + \frac{R_2y^*}{K} + \frac{R_3z^*}{K} \\ a_1 &= \frac{R_1R_2x^*y^*}{K^2} + \frac{R_1R_3x^*z^*}{K^2} + \frac{R_3R_2z^*y^*}{K^2} + g_1g_2x^*y^* + \beta\left(\frac{-R_2\lambda y^*}{(1+\lambda z^*)^2} + \frac{\delta my^*}{(1+\delta z^*)^2}\right)z^* \\ a_0 &= \frac{R_1R_2R_3x^*y^*z^*}{K^3} + \frac{R_1x^*}{K}\beta\left(\frac{-R_2\lambda y^*}{(1+\lambda z^*)^2} + \frac{\delta my^*}{(1+\delta z^*)^2}\right)z^* + \frac{R_3g_1g_2x^*y^*z^*}{K} \end{aligned}$$

If $a_0 > 0$, $a_1 > 0$, $a_2 > 0$ and $a_1a_2 - a_0 > 0$, then the conditions of Routh-Hurwitz are verified and consequently the interior equilibrium point $(x^*, y^*, z^*)$ is locally asymptotically stable.

## 3.3. **Hopf bifurcation.** Now, we are studying the local Hopf bifurcation of $(x^*, y^*, z^*)$. Any of the parameters of the model may be a bifurcation parameter. We consider $\alpha$ (comprising $(R_1, R_2, R_3, K, g_1, g_2, \beta, \delta, \lambda, q_1, q_2, q_3, E_1, E_2, E_3)$) as the generic bifurcating parameter of the system.

**Theorem 4.** *The system (1) undergoes a Hopf bifurcation around the interior equilibrium point at the critical parameter value* $\alpha = \alpha^*$ *if*

- $a_1 a_2 = a_0$
- $a_1 > 0$
- $\dot{a}_0 - (\dot{a}_2 a_1 + a_2 \dot{a}_1) \neq 0$

**Proof:**

If $a_1 a_2 = a_0$ then we have

$$\begin{aligned} X^3 + a_2 X^2 + a_1 X + a_0 &= X^3 + a_2 X^2 + a_1 X + a_1 a_2 \\ &= X^2 (X + a_2) + a_1 (X + a_2) \\ &= \left(X^2 + a_1\right)(X + a_2) \end{aligned}$$

In this case, the equation admits three roots $X_1 = -a_2$, $X_2 = i\sqrt{a_1}$ and $X_3 = -i\sqrt{a_1}$. Therefore, we have a pair of purely imaginary eigenvalues.

Following the steps outlined in [12], we are now verifying the transversality condition.

$$\frac{\mathrm{d}X}{\mathrm{d}\alpha} = -\left.\frac{X^2 \dot{a}_2 + X\dot{a}_1 + \dot{a}_0}{3X^2 + 2Xa_2 + a_1}\right|_{X = i\sqrt{a_1}}$$

$$= \frac{\dot{a}_0 - (\dot{a}_2 a_1 + a_2 \dot{a}_1)}{2\left(a_2^2 + a_1\right)} + i\left[\frac{\sqrt{a_1}\,(a_1 \dot{a}_1 + a_2 \dot{a}_0 - a_2 \dot{a}_2 a_1)}{2a_1\left(a_2^2 + a_1\right)}\right].$$

Then

$$\frac{\mathrm{dRe}X}{\mathrm{d}\alpha}|_{\alpha=\alpha^*} \neq 0 \Leftrightarrow \frac{\dot{a}_0 - (\dot{a}_2 a_1 + a_2 \dot{a}_1)}{2\left(a_2^2 + a_1\right)} \neq 0$$

$$\Leftrightarrow \dot{a}_0 - (\dot{a}_2 a_1 + a_2 \dot{a}_1) \neq 0$$

## 4. Optimal Harvesting Policy

In this section, our objective is to determine an optimal harvesting policy by using Pontryagin's Principle [9]. To apply Pontryagin's Principle to our problem, we need to define the objective function which is written in the form

$$J = \int_0^\infty e^{-\delta t} \pi (x, y, z, E_1, E_2, E_3)\, dt$$

In this formulation, we consider $\pi$ as the net revenue, which represents the earnings obtained after subtracting costs, at any time t in the future. It is expressed as follows

$$\pi(x,y,z,E_1,E_2,E_3) = (p_1q_1x - c_1)E_1 + (p_2q_2y - c_2)E_2 + (p_3q_3y - c_3)E_3$$

On the other hand, $\delta$ represents the instantaneous annual rate of discount. This rate is used to convert the future value of revenues into an equivalent present value.

$$J = \int_0^\infty e^{-\delta t}\left[(p_1q_1x - c_1)E_1 + (p_2q_2y - c_2)E_2 + (p_3q_3y - c_3)E_3\right]dt$$

and the control variables $E_i$ satisfying

$$0 \leqslant E_i \leqslant E_i^{max}$$

The Hamiltonian equation is

$$\begin{aligned} H &= e^{-\delta t}\left[(p_1q_1x - c_1)E_1 + (p_2q_2y - c_2)E_2 + (p_3q_3y - c_3)E_3\right] + Q_1\left[R_1x\left(1 - \frac{x}{K}\right) - g_1xy - q_1E_1x\right] \\ &\quad + Q_2\left[R_2y\left(\frac{1}{1+\lambda z} - \frac{y}{K}\right) + g_2xy - \frac{my}{1+\delta z} - q_2E_2y\right] + Q_3\left[R_3z\left(1 - \frac{z}{K}\right) + \beta yz - q_3E_3z\right] \end{aligned}$$

We set

$$\frac{\partial H}{\partial E_1} = 0 \qquad \frac{\partial H}{\partial E_2} = 0 \qquad \frac{\partial H}{\partial E_3} = 0$$

as the necessary conditions for the control variables $E_1$, $E_2$ and $E_3$ to be optimal. Then we get

$$Q_1 = \frac{e^{-\delta t}(p_1q_1x - c_1)}{q_1x} \quad Q_2 = \frac{e^{-\delta t}(p_2q_2y - c_2)}{q_2y} \quad Q_3 = \frac{e^{-\delta t}(p_3q_3z - c_3)}{q_3z}$$

From the Pontryagin's maximum principle

$$\begin{aligned} \dot{Q}_1 &= -\frac{\partial H}{\partial x} = -e^{-\delta t}p_1q_1E_1 - Q_1\left[R_1 - \frac{2R_1x}{K} - g_1y - q_1E_1\right] - Q_2g_2y \\ \dot{Q}_2 &= -\frac{\partial H}{\partial y} = -e^{-\delta t}p_2q_2E_2 - Q_1g_1x - Q_2\left[\frac{R_2}{1+\lambda z} - \frac{2R_2y}{K} + g_2y - \frac{m}{1+\delta z} - q_2E_2\right] - Q_3\beta z \\ \dot{Q}_3 &= -\frac{\partial H}{\partial z} = -e^{-\delta t}p_3q_3E_3 - Q_2\left[\frac{-R_2\lambda y}{(1+\lambda z)^2} + \frac{\delta my}{(1+\delta z)^2}\right] - Q_3\left[R_3 - \frac{2R_3z}{K} + \beta y - q_3E_3\right] \end{aligned}$$

With the help of equilibrium equations

$$\begin{aligned}
\dot{Q}_1 &= -e^{-\delta t}p_1q_1E_1+Q_1\frac{R_1x}{K}-Q_2g_2y \\
\dot{Q}_2 &= -e^{-\delta t}p_2q_2E_2-Q_1g_1x+Q_2\frac{R_2y}{K}-Q_3\beta z \\
\dot{Q}_3 &= -e^{-\delta t}p_3q_3E_3-Q_2\left[\frac{-R_2\lambda y}{(1+\lambda z)^2}+\frac{\delta my}{(1+\delta z)^2}\right]+Q_3\frac{R_3z}{K}
\end{aligned}$$

by replacing $Q_1$, $Q_2$ and $Q_3$ with their expressions we get

$$\begin{aligned}
\dot{Q}_1 &= -e^{-\delta t}p_1q_1E_1+\frac{e^{-\delta t}(p_1q_1x-c_1)}{q_1x}\frac{R_1x}{K}-\frac{e^{-\delta t}(p_2q_2y-c_2)}{q_2y}g_2y \\
\dot{Q}_2 &= -e^{-\delta t}p_2q_2E_2-\frac{e^{-\delta t}(p_1q_1x-c_1)}{q_1x}g_1x+\frac{e^{-\delta t}(p_2q_2y-c_2)}{q_2y}\frac{R_2y}{K}-\frac{e^{-\delta t}(p_3q_3z-c_3)}{q_3z}\beta z \\
\dot{Q}_3 &= -e^{-\delta t}p_3q_3E_3-\frac{e^{-\delta t}(p_2q_2y-c_2)}{q_2y}\left[\frac{-R_2\lambda y}{(1+\lambda z)^2}+\frac{\delta my}{(1+\delta z)^2}\right]+\frac{e^{-\delta t}(p_3q_3z-c_3)}{q_3z}\frac{R_3z}{K}
\end{aligned}$$

After integration of the previous equations

$$\begin{aligned}
Q_1 &= \frac{e^{-\delta t}}{\delta}\left[p_1q_1E_1-\frac{(p_1q_1x-c_1)}{q_1}\frac{R_1}{K}+\frac{(p_2q_2y-c_2)}{q_2}g_2\right] \\
Q_2 &= \frac{e^{-\delta t}}{\delta}\left[p_2q_2E_2+\frac{(p_1q_1x-c_1)}{q_1}g_1-\frac{(p_2q_2y-c_2)}{q_2}\frac{R_2}{K}+\frac{(p_3q_3z-c_3)}{q_3}\beta\right] \\
Q_3 &= \frac{e^{-\delta t}}{\delta}\left[p_3q_3E_3+\frac{(p_2q_2y-c_2)}{q_2}\left[\frac{-R_2\lambda}{(1+\lambda z)^2}+\frac{\delta m}{(1+\delta z)^2}\right]-\frac{(p_3q_3z-c_3)}{q_3}\frac{R_3}{K}\right]
\end{aligned}$$

Then

$$\begin{aligned}
E_1 &= \frac{\delta(p_1q_1x-c_1)}{p_1q_1^2x}+\frac{R_1(p_1q_1x-c_1)}{q_1^2p_1K}-\frac{g_2(p_2q_2y-c_2)}{q_2p_1q_1} \\
E_2 &= \frac{\delta(p_2q_2y-c_2)}{p_2q_2^2y}-\frac{g_1(p_1q_1x-c_1)}{p_2q_2q_1}+\frac{R_2(p_2q_2y-c_2)}{p_2q_2^2K}-\frac{\beta(p_3q_3z-c_3)}{p_2q_2q_3} \\
E_3 &= \frac{\delta(p_3q_3z-c_3)}{p_3q_3^2z}-\frac{(p_2q_2y-c_2)}{p_3q_3q_2}\left[\frac{-R_2\lambda}{(1+\lambda z)^2}+\frac{\delta m}{(1+\delta z)^2}\right]+\frac{R_3(p_3q_3z-c_3)}{p_3q_3^2K}
\end{aligned}$$

## DISCUSSION

In this section, we will conduct numerical simulations to illustrate the theoretical results obtained in the previous sections. The simulations will help us visualize and gain a better understanding of the studied phenomena, using the values specified in the following table.

| Parameter | Values |
|---|---|
| $R_1$ | 0.438 |
| $R_2$ | 0.095 |
| $R_3$ | 0.357 |
| $K$ | 30 |
| $g_1$ | 0.027 |
| $g_2$ | 0.012 |
| $\beta$ | 0.018 |

| Parameter | Values |
|---|---|
| $\delta$ | 0.023 |
| $\lambda$ | 0.016 |
| $q_1$ | 0.002 |
| $q_2$ | 0.001 |
| $q_3$ | 0.015 |
| $E_1$ | 14 |
| $E_2$ | 26 |
| $E_3$ | 15 |

TABLE 2. Bioeconomic parameters and their values

By assigning the values mentioned in the previous table for the parameters, we observe that the system reaches a positive equilibrium point $(x^*, y^*, z^*)$. The values of x, y, and z correspond to the sizes of the cleaner fish population, predator species, and prey, respectively. The positive equilibrium signifies that all three populations are capable of coexisting and maintaining their respective sizes without drastic fluctuations or extinctions. We begin with the initial point (10, 10, 10) and observe that the solution of our system converges towards the interior equilibrium point. This indicates that the system is locally asymptotically stable. The stability of the equilibrium point indicates that the interactions between the species, influenced by the given parameter values, have reached a balanced state. This suggests that cleaner fish play a significant role in regulating population dynamics and maintaining the health and stability of the ecosystem.

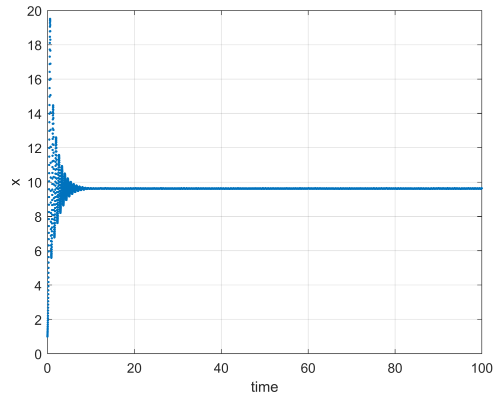


**Fig 1.** Biomass of prey around the interior equilibrium points.

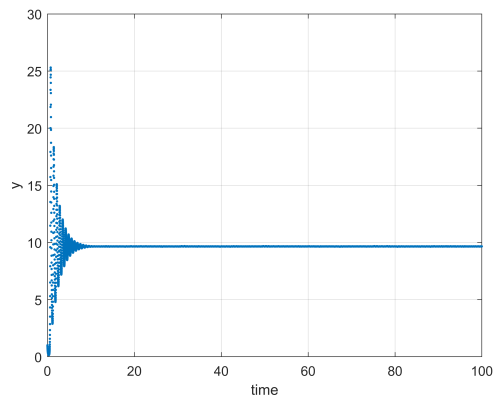


**Fig 2.** Biomass of predator around the interior equilibrium points.

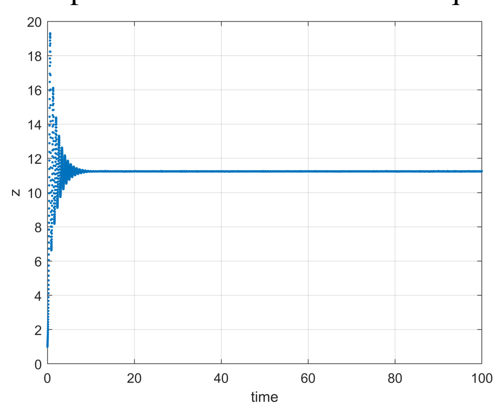


**Fig 3.** Biomass of cleaner fish around the interior equilibrium points.

In order to observe the impact of the cleaner fish on the other two species, we will use a numerical simulation by varying the parameters alpha and delta. By adjusting the $\lambda$ and $\delta$ in the simulation, we can study how changes in these parameters related to the cleaner fish affect the densities of both populations and the interactions within the ecosystem. When we choose a very large $\lambda$, which means that the cleaner fish eats the predator's eggs more than normal, we can observe that the system oscillates around the equilibrium point and no longer converges to this equilibrium. In other words, the significant increase in the cleaning activity of the fish disrupts the natural balance between the predator and its prey. These constant oscillations can lead to instability in the populations of predators and prey, disrupting the interactions between species and affecting the overall structure of the ecosystem. Additionally, this can also have an impact on other organisms that depend on these species for their own survival.

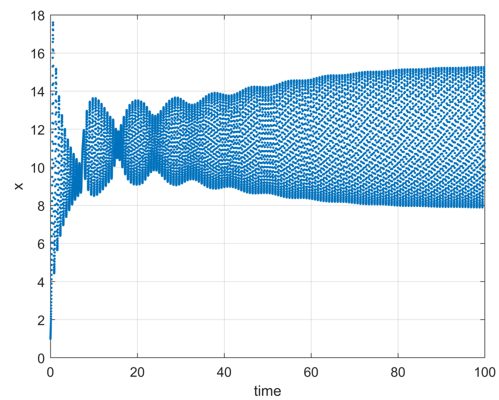


**Fig 4.** Biomass of prey for large $\lambda = 0.67$

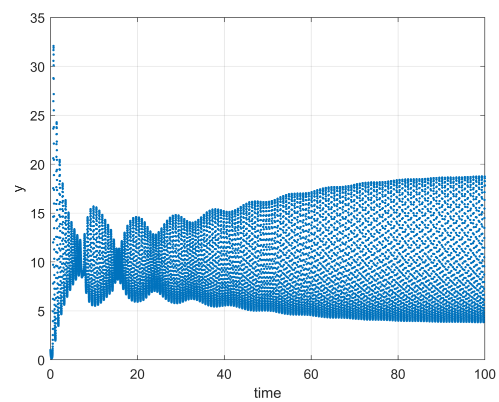


**Fig 5.** Biomass of predator for large $\lambda = 0.67$

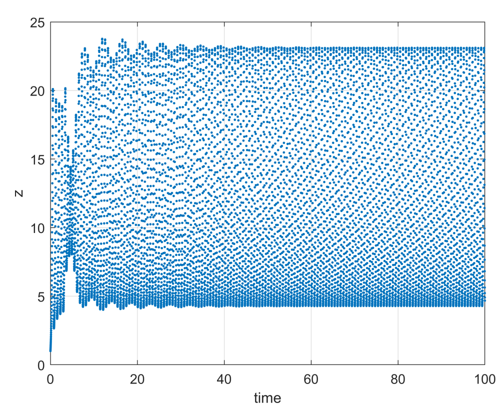


**Fig 6.** Biomass of cleaner fish for large $\lambda = 0.67$

By choosing a very small $\delta$, which means that the cleaner fish does not clean the predator fish regularly, we observe a very slight oscillation in the biomass of the predators and cleaner fish, while the oscillation is more pronounced in the prey. This leads to a decrease in the biomass of the three species, but overall, the system does not completely lose its stability, which is due to the fact that the predator-prey system is already stable in the absence of the cleaner fish. When the cleaner fish does not clean the predator fish regularly, it results in a slight oscillation in the biomass of the predators and cleaner fish, as they may be exposed to accumulated parasites or debris. However, due to the established and balanced predator-prey interactions, the predator population remains relatively stable despite these oscillations. Furthermore, the more pronounced oscillation in the biomass of the prey is due to a decrease in predation pressure from the predators. With reduced predation, the prey population tends to increase, leading to increased competition for available resources and an overall decrease in their biomass.

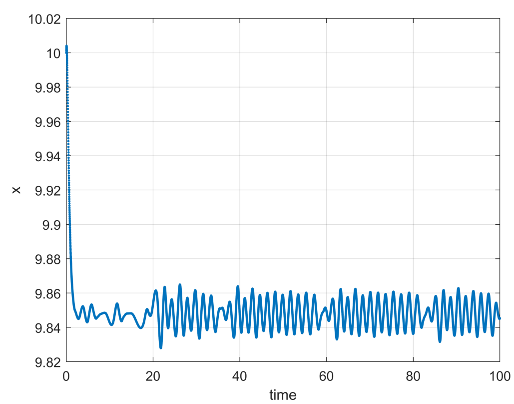


**Fig 7.** Biomass of prey for small $\delta = 0.0013$

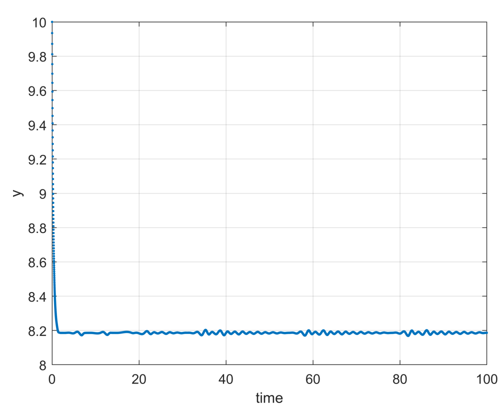


**Fig 8.** Biomass of predator for small $\delta = 0.0013$

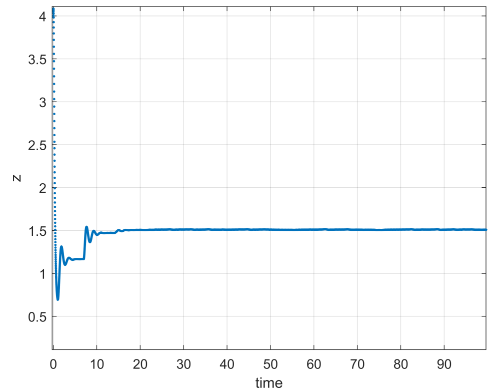


**Fig 9.** Biomass of cleaner fish for small $\delta = 0.0013$

Despite these oscillations and the decrease in the biomass of the three species, the system manages to maintain some level of stability. This is largely explained by the fact that the predator-prey system had already reached a natural equilibrium before the introduction of the cleaner fish. The interactions and regulations between the prey and predators are sufficiently established to maintain the overall stability of the system, even in the presence of a disturbance caused by reduced cleaning activity of the cleaner fish.

In summary, these results emphasize the importance of cleaner fish in regulating populations and maintaining ecosystem stability. The achieved positive equilibrium indicates that cleaner fish, predators, and prey can coexist without drastic fluctuations. Numerical simulations demonstrate that changes in the cleaning activity of cleaner fish can disrupt the natural balance, leading to oscillations and instability in predator and prey populations. However, despite these disturbances, the system manages to maintain a certain level of stability due to established predator-prey interactions. Thus, cleaner fish play a crucial role in population regulation and ecosystem health maintenance, but their activity needs to be balanced to avoid disruptions.

## Conclusion

This study falls within the scope of predicting the behavior of marine species in order to preserve marine biodiversity. The focus was on modeling the interaction between a prey, a predator, and a cleaner fish through a dynamic system. Consequently, a thorough analysis of the system was conducted to study its stability and identify potential bifurcations. The ultimate objective of this study was to determine an optimal fishing policy. To achieve this goal, we examined the system's equilibria, their stability, and also considered the crucial role of the cleaner fish in preserving the balance of the ecosystem. The most significant novelty of this work lies in recognizing the essential role of the cleaner fish in maintaining the equilibrium of the marine ecosystem. By understanding and modeling its impact on prey and predator populations, we can develop optimal fishing policies that consider the sustainability of marine resources and the preservation of biodiversity. Therefore, this study contributes to advancing knowledge of marine ecosystem dynamics and highlights the critical importance of considering interactions between different species when implementing marine resource management policies.

## CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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