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STABILITY ANALYSIS OF AN ORDER FRACTIONAL OF A NEW CORONA VIRUS DISEASE (COVID-19) MODEL

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Abstract. In this paper, we propose a fractional order SEIHR pandemic model of Covid-19 that describes the spreadability of Corona virus within a population. We prove the global existence, positivity, and boundedness of solutions. In addition, the local and the global stability of the disease-free equilibrium and the disease equilibrium are studied by using Routh-Hurwitz criteria and construction an appropriate Lyapunov functions. Finally, numerical simulations are performed to verify the theoretical results.

Keywords: mathematical modeling; Caputo fractional derivative; stability; disease Covid-19; Lyapunov function.
2020 AMS Subject Classification: 92C60.

1. Introduction

The WHO organization adds that Corona viruses are a group of viruses that are known to cause diseases ranging from common cold to more severe diseases, such as Middle East Respiratory Syndrome (MERS) and severe acute respiratory syndrome (SARS). The Corona virus

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(nCoV) is a new strain of virus that has not been previously discovered in humans. Detailed investigations have concluded that the SARS-CoV virus transmitted from civet cats to humans and that the Middle East respiratory syndrome (MERS-CoV) virus has passed from camels to humans. A new type of coronavirus has emerged in China and has been designated with several names related to the time and place of the epidemic spread in China such as the new coronavirus 2019–2020, the CoronaWuhan virus. The disease was lately branded (COVID-19) by theWorld Health Organization (WHO).

The World Health Organization announced on January 30, 2020, that Covid-19 is a public health emergency and on March 11, 2020 it was identified as a pandemic. Covid-19 symptoms are not specific and many cases have shown that the person may be asymptomatic. Most cases have two common symptoms, including dry cough and fever. Some conditions include symptoms including fatigue, muscle and joint pain, respiratory sputum production, sore throat, olfactory loss, headache or chills, and shortness of breath. Moreover, the growth of this infection can develop into acute respiratory distress syndrome, acute pneumonia and death. Covid-19 virus is highly prevalent among people who are in close contact with each other within about one meter. The common incubation period is from 1 to 14 days and still there is no specific treatment method to be adopted. Physical spacing was universally accepted as the most effective strategy for reducing disease severity and controlling it. Preventing human contact in work environments, schools and other open circuits is the goal of such preventive measures [1].
Data on the evolution of covid19 around the world can be found in [2][3].

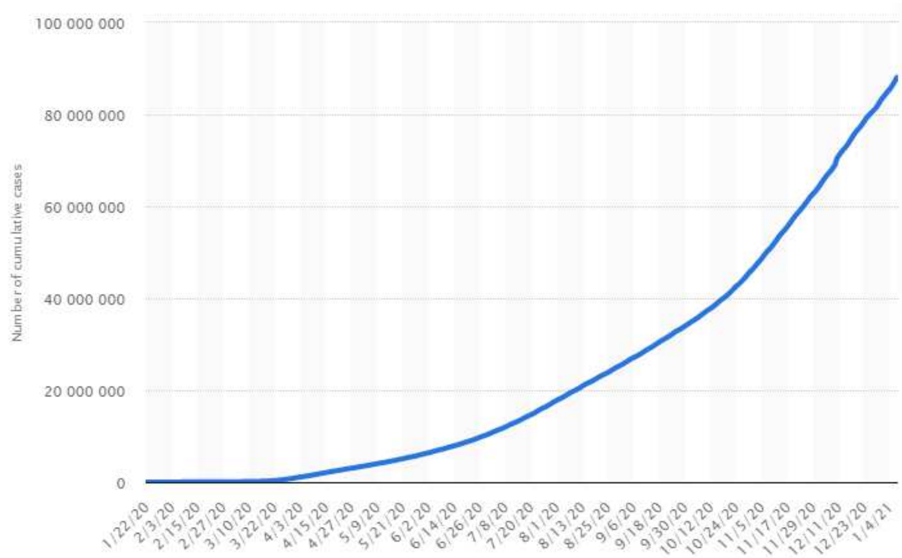


FIGURE 1. Number of cumulative cases of coronavirus (COVID-19) worldwide from January 22, 2020 to January 7, 2021, by day

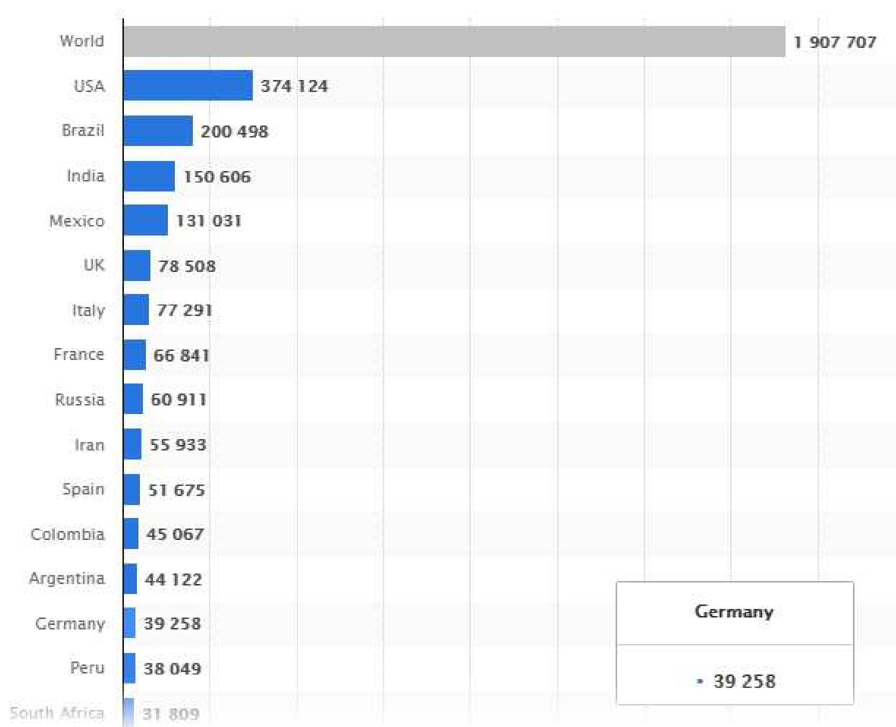


FIGURE 2. Number of novel coronavirus (COVID-19) deaths worldwide as of January 8, 2021, by country

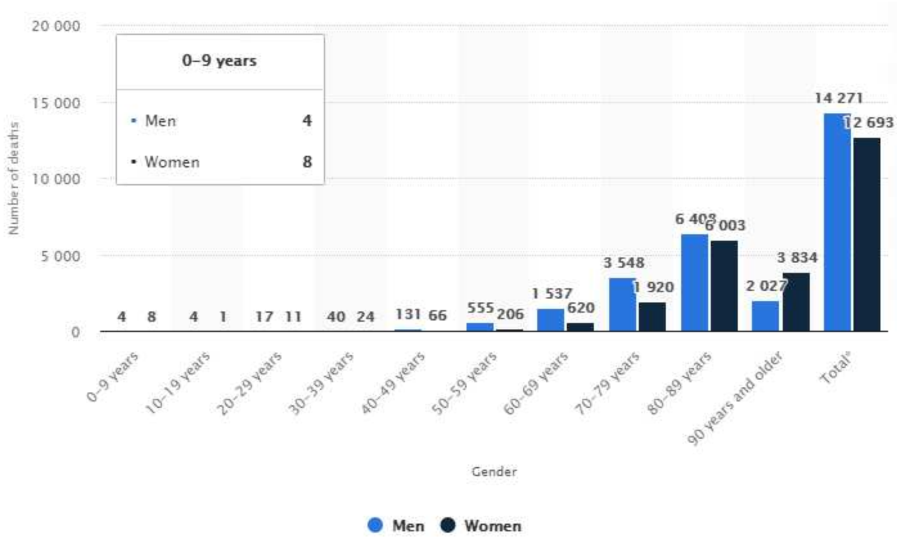


FIGURE 3. Number of coronavirus (COVID-19) deaths in Germany in 2020, by gender and age

Therefore, while the new Corona virus is transmitted very quickly and contains several different mutations, the strategic goals of the World Health Organization must be adhered to until the scientists and researchers reach an effective treatment [4,5,6,7].

There are several mathematical modelling studies that have emerged which aim at understanding the COVID-19 dynamics. For example, Khajji et al [8] have devised a new modelling approach based on a multi-regions discrete time model that consists of two groups, the human population and the animal population in different regions, It aimed to describe the spatial-temporal evolution of COVID-19 which emerges in different geographical regions and showing the influence of one region on another. Authors proposed several control strategies, including awareness campaigns in a given region, security campaigns, health measures to prevent the movement of individuals from one region to another and encouraging the individuals to join quarantine centers and the disposal of infected animals. Kouidere et al [9], proposed a mathematical model with five compartments to describe transmission process of the COVID-19 virus. The population is divided into potential people, infected people without symptoms, people with serious complications as well as those under health surveillance and quarantine and people who recovered from the disease. In addition, optimal control strategies were proposed, which consist of conducting awareness campaigns for citizens along with practical measures to reduce the spread of the virus including diagnosis of individuals, surveillance of airports and imposing quarantine on infected people.

For more examples of works that addresses the problem of modelling and controlling the COVID-19 transmission, we refer to [10, 11, 12, 13, 14, 15]. In classical mathematical models, ordinary differential equations were used with an integer order, to provide a meaningful demonstration and to understand the dynamics of biological systems [16,17,18,19]. These models depend on classical derivatives which have certain limits linked to the order of the differential equations considered. To overcome these restrictions, many authors have sought help from a recently emerging field of mathematics known as fractional calculus. In fractional calculus, the differential operators used are a non-integer or fractional order which have impacts on memory and are useful for demonstrating many natural phenomena, truths and facts related to nature with non-local dynamics and abnormal behavior, for example, their applicability in different fields of science and engineering [20,21]. In particular, fractional derivatives are used to describe the viscoelastic properties of many polymeric materials [22], in mechanics [23] and in decision-making problems [24]. The study of epidemiological processes dynamics involving memory effects is appropriate because such frameworks rely on the strength of memory which is constrained by the order of the fractional derivative operator.

One of the most important problems in the qualitative theory of fractional differential equations is stability theory. Following Lyapunov's seminal 1892 thesis, these two methods are expected to also work for fractional differential equations:

• Lyapunov's first method: the method of linearisation of the nonlinear equation along an orbit and the transfer of asymptotic stability from the linear to the nonlinear equation.

• Lyapunov's second method: the method of Lyapunov candidate functions, i.e. of scalar functions on the state space such that their energy decreases along orbits.

The stability analysis of differential systems of fractional order has recently received interest in this field. A.Mouaouine et al [25]. present the SIR epidemic model with nonlinear incidence rate using the Caputo fractional derivative. F.Zhang [26] presented the stability of nonlinear fractional differential systems with Caputo derivatives by using comparison method is studied. Pinto et al [27], proposed a fractional model for malaria transmission under control strategies, These epidemic dynamics were studied numerically for different orders of the fractional derivative, and of some parameters. M. R. Sidi Ammi et al [28] presented a fractional order SIRS model with non-linear incidence rate, they proved the existence of a unique positive solution to this model and they studied the stability analysis of the disease free equilibrium and positive fixed points.

Motivated by the fact that the local derivative does not take into account the time scale and does not accurately reproduce the non-local properties, dependent on the frequency and history of the epidemic [29] and in order to have a more realistic model that takes into account the time scale, we propose a model with fractional derivatives that provide a useful tool to model the actual dynamics of the COVID-19 pandemic. The main contributions of this work are the following:

1. Development of an epidemic model with fractional derivative equations to better understand the dynamics of the COVID-19 disease.

2. Proposing an extension of the classical SEIR model with an additional compartment, where (S), (E), (I) and (R) stand for Susceptible, Exposed, Infectious and Restored respectively.

3. The population is divided into five different classes, namely Susceptible S(t), Infected without symptoms E(t), Symptomatically Infected I(t), hospitalized individuals H(t), Recovered R(t).

4. study the stability behavior of system at a Covid-19 disease-free equilibrium point and Covid-19 disease present equilibrium point.

5. Provides numerical simulation .

We will propose a fractional-order mathematical model that defines and describes the spread of the new Corona virus (Covid-19). There exist several definitions of the fractional derivative operator: Riemann-Liouville, Caputo, Grunwald-Letnikov, etc [30,31,32,33]. The reasons to use Caputo fractional derivative in this work are: Firstly, the fractional derivative of a constant is zero and the second reason is that the initial conditions for the fractional order of the differential equations with the Caputo's derivatives are in the same form as for the integer-order differential equations[33,34,35,36,37,38]. Our principle objective is, firstly, to test the local stability of the model in both disease-free model and in endemic equilibrium; and secondly, to test the global stability of the model.

The paper is organized as follows: In section 2, we discuss some preliminaries. In section 3, the formulation of the model and some basic properties are derived. In section 4, equilibria of the proposed model are obtained and their stability is discussed. In section 5, the global stability of the equilibrium point is discussed. In section 6 we give some numerical simulations. Lastly, in section 7 we conclude the paper.

2. Preliminaries

We introduce the definition of the Caputo fractional derivative and we present some functions and useful properties that are used throughout this work [30, 31, 33, 35].

1) The Caputo fractional derivative of order $\alpha > 0$ of a function $f : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-x)^{n-\alpha-1} f^{(n)}(x)dx, \tag{1}$$

where $D = d/dt, n-1 < \alpha \leq n, n \in \mathbb{N}^*$ and where $\Gamma(\cdot)$ is the gamma function. In particular, when $0 < \alpha \leq 1$, we have

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f^{(1)}(x)}{(t-x)^\alpha} dx, \tag{2}$$

2) The Laplace transform of the Caputo fractional derivative is

$$\mathscr{L}[D^\alpha f(t)] = \lambda^\alpha F(\lambda) - \sum f^{(k)}(0)\lambda^{\alpha-k-1}, \tag{3}$$

with $F(\lambda)$ is the Laplace transform of $f(t)$.

3) Let $\alpha, \beta \geq 0$. The Mittag-Leffler function $E_{\alpha,\beta}$ of parameters α and β is defined as follows

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{4}$$

4) The Laplace transform of The Mittag-Leffler functions is

$$\mathscr{L}\left[t^{\beta-1} E_{\alpha,\beta}(\mp a t^\alpha)\right] = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha \pm a} \tag{5}$$

5) Let $\alpha, \beta \geq 0$ and $z \in \mathbb{C}$ The Mittag-Leffler functions satisfy the equality given by

$$E_{\alpha,\beta}(z) = z E_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}, \tag{6}$$

6) Let $f : \mathbb{R}^+ \to \mathbb{R}$ with $n \geq 1$. We consider the fractional order system

(7)
$$\begin{cases} D^{\alpha}x(t) = f(x) \\ x(t_0) = x_0 \end{cases}$$

with $0 < \alpha \leq 1$, $t_0 \in \mathbb{R}$. For the global existence of system (7) solution, we need the following lemma.

lemma 1. [12] We assume that f satisfies the following conditions

(i) $f(x)$ and $(\partial f/\partial x)(x)$ are continuous for all $x \in \mathbb{R}^n$.

(ii) $\| f(x) \| \leq \omega + \lambda \| x \|$ for all $x \in \mathbb{R}^n$, where ω and λ are two positive constants.

Then, the system (7) has a unique solution on $[t_0, +\infty[$.

3. A Mathematical Model and Basic Properties

3.1. A mathematical model.

We propose a continuous model SEIHR to describe the evolution of Covid-19 disease within a population. The population is divided into five compartments: Susceptible individuals exposed to have new Corona virus $S(t)$, Asymptomatic infected cases or cases with mild symptoms $E(t)$, Infected people with symptoms and carriers of the virus $I(t)$, Hospitalized cases $H(t)$, The recovered cases $R(t)$. The total number of the population at time t is given by $N(t) = S(t) + E(t) + I(t) + H(t) + R(t)$.

The graphical representation of the proposed model is shown in Figure (4).

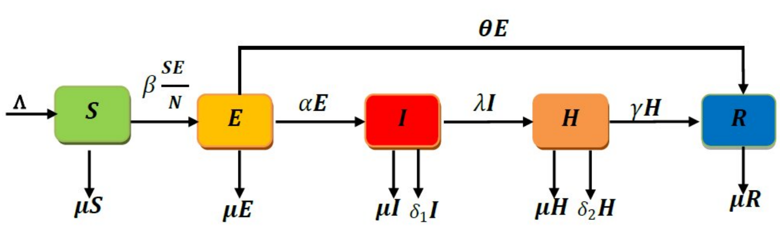


FIGURE 4. The flow between the five compartments SEIHR of Covid-19

We consider the following system of fractional order of differential equations

(8)
$$\begin{cases} D^{\alpha}S = \Lambda - \beta\frac{SE}{N} - \mu S \\ D^{\alpha}E = \beta\frac{SE}{N} - (\mu + \alpha + \theta)E \\ D^{\alpha}I = \alpha E - (\mu + \lambda + \delta_1)I \\ D^{\alpha}H = \lambda I - (\mu + \gamma + \delta_2)H \\ D^{\alpha}R = \gamma H + \theta E - \mu R \end{cases}$$

where $D^{\alpha}S$, $D^{\alpha}E$, $D^{\alpha}I$, $D^{\alpha}H$ and $D^{\alpha}R$ are the derivatives of $S(t), E(t), I(t), H(t)$ and $R(t)$ respectively, of arbitrary order α $(0 < \alpha \leq 1)$ in the sense of Caputo. $S(0) \geq 0, E(0) \geq 0, I(0) \geq 0, H(0) \geq 0$, and $R(0) \geq 0$ are the given initial states.

Also, Λ represents new birth rate in susceptible human population, β represents the transmission coefficient from susceptible individuals to asymptomatic infected cases or cases with mild symptoms due to the movement and contact that occur among them, μ represents the natural death rate in all compartments, α represents the rate of transmission of asymptomatic infected cases or cases with mild symptoms to infected individuals with symptoms, λ is the transmission coefficient of the infected people with symptoms to the hospitalized cases. We can see that there is no SI type term since it has been judged that an infected person is generally symptomatic and everyone avoids it. γ is the transmission coefficient of the hospitalized cases to the recovered cases, θ represents the rate of transmission of asymptomatic infected cases or cases with mild symptoms to recovered cases due the strong immunity of these individuals, δ_1 and δ_2 respectively represent the death rate of the infected individuals and the death rate of the hospitalized cases.

3.2. Basic properties of the model.

3.2.1. *Invariant Region.* It is necessary to prove that all solutions of the system (7) with positive initial data will remain positive for all times $t > 0$. This will be established by the following lemma.

lemma 2. The feasible region Ω defined by

$$\Omega = \left\{ (S(t), E(t), I(t), H(t), R(t)) \in \mathbb{R}_{+}^{5} : S(t) + E(t) + I(t) + H(t) + R(t) \leq N(0) + \frac{\Lambda}{\mu} \right\}$$

with initial conditions $S(0) \geq 0, E(0) \geq 0, I(0) \geq 0, H(0) \geq 0$, and $R(0) \geq 0$

is positively invariant with respect to system (7).

Proof. The fractional derivative of the total people, obtained by adding all the system equation (7) is

$$D^{\alpha}N(t) = \Lambda - \mu N(t) - \delta_1 I - \delta_2 H, \tag{9}$$

implies that

$$D^{\alpha}N(t) \leq \Lambda - \mu N(t).$$

Applying the Laplace transform in the previous inequality, we obtain

$$\lambda^{\alpha}\mathscr{L}(N(t))-\lambda^{\alpha-1}N(0)\leq\frac{\Lambda}{\lambda}-\mu\mathscr{L}(N(t)). \tag{10}$$

we write it as

$$\mathscr{L}(N(t))\leq\frac{\lambda^{\alpha-1}}{\lambda^{\alpha}+\mu}N(0)+\frac{\lambda^{\alpha-(\alpha+1)}}{\lambda^{\alpha}+\mu}\Lambda, \tag{11}$$

From (5) we have

$$N(t)\leq E_{\alpha,1}(-\mu t^{\alpha})N(0)+t^{\alpha}E_{\alpha,\alpha+1}(-\mu t^{\alpha})\Lambda. \tag{12}$$

Using (7) we deduce

$$N(t)\leq E_{\alpha,1}(-\mu t^{\alpha})N(0)+\frac{\Lambda}{\mu}(1-E_{\alpha,1}(-\mu t^{\alpha}), \tag{13}$$

Since

$$0\leq E_{\alpha,1}(-\mu t^{\alpha})\leq 1. \tag{14}$$

Then,

$$N(t)\leq N(0)+\frac{\Lambda}{\mu}, \tag{15}$$

where $N(0)$ is the initial value of the total number of people. Thus, the region Ω is a positively invariant set for the system (7).

The first three equations in system (7) are independent of the variables H and R. Hence, it is sufficient to consider the following reduced system

$$\begin{cases} D^{\alpha}S=\Lambda-\beta\frac{SE}{N}-\mu S\\ D^{\alpha}E=\beta\frac{SE}{N}-(\mu+\alpha+\theta)E\\ D^{\alpha}I=\alpha E-(\mu+\alpha+\delta_1)I\end{cases} \tag{16}$$

Theorem 1. The fractional order initial value system (7) has a unique global solution on Ω.

Proof. Let $X(t)=[S\ \ E\ \ I\,]^{T}$ and $D^{\alpha}X(t)=F(X(t))$ □

where F is the right side of the system (16).

We see that $F(X)$ and $\frac{\partial F}{\partial X}(X)$ satisfy the first condition of lemma (1)

In addition, the system (16) can be rewritten as follows

$$F(X(t)) = \Delta + (M_1 + EM_2)X(t)$$

Where

$$\Delta = \begin{bmatrix} \Lambda \\ 0 \\ 0 \end{bmatrix};\ M_1 = \begin{bmatrix} -\mu & 0 & 0 \\ 0 & -(\mu+\alpha+\theta) & 0 \\ 0 & \alpha & -(\mu+\lambda+\delta_1) \end{bmatrix} \text{ and } M_2 = \begin{bmatrix} -\frac{\beta}{N} & 0 & 0 \\ \frac{\beta}{N} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since

$$\begin{aligned} \|F(X(t))\| &\leq \|\Delta\| + (\|M_1\| + \mid E \mid \|M_2\|)\|X(t)\| \\ &\leq \|\Delta\| + (\|M_1\| + c\|M_2\|)\|X(t)\| \\ &\leq \omega + \lambda\|X(t)\| \end{aligned}$$

where

$$c = N(0) + \frac{\Lambda}{\mu},\ \omega = \|\Delta\| \text{ and } \lambda = \|M_1\| + c\|M_2\|$$

Then, system (7) has a unique global solution on Ω. □

3.2.2. *Positivity of solutions of the model.*

Theorem 2. If $S(0) \geq 0$, $E(0) \geq 0, I(0) \geq 0$, $H(0) \geq 0$ and $R(0) \geq 0$, then the solutions of system equation (7), $S(t)$, $E(t)$, $I(t)$, $H(t)$ and $R(t)$ are positive for all $t > 0$.

Proof. From the first equation of the system (7), we have

$$D^{\alpha}S \geq -\left(\beta\frac{E(t)}{N} + \mu\right)S(t) \tag{17}$$

Since $E(t)$ is bounded by a constant m, we have

$$D^{\alpha}S \geq -dS(t)$$

where d can be taken, for example $d = \beta\frac{m}{N} + \mu$.

By applying the Laplace transform in the previous inequality to obtain

$$\lambda^{\alpha}\mathscr{L}(S(t)) - \lambda^{\alpha-1}S(0) \geq -d\mathscr{L}(S(t)). \tag{18}$$

so

$$\mathscr{L}(S(t)) \geq \frac{\lambda^{\alpha-1}}{\lambda^{\alpha}+d} S(0). \tag{19}$$

From (5), we deduce

$$\mathscr{L}(S(t)) \geq \mathscr{L}(E_{\alpha,1}(-dt^{\alpha}))S(0). \tag{20}$$

Then

$$S(t) \geq E_{\alpha,1}(-dt^{\alpha})S(0). \tag{21}$$

Since $E_{\alpha,1}(-dt^{\alpha}) \geq 0$, therefore the solution $S(t)$ is positive.

Similarly, from the second and third equations of system (16), we prove that $E(t)$ and $I(t)$ are positive for all t≥0.This completes the proof.

□

4. Stability Analysis of the Model

4.1. Equilibrium Points. In this model, there are two equilibrium points, that is, Covid-19 disease-free equilibrium point and Covid-19 disease present equilibrium point. The equilibrium points are found by setting the right hand side of the system (16) equal to zero.

The Covid-19 disease-free equilibrium $E_{eq}^{0}\left(\frac{\Lambda}{\mu},0,0\right)$ is achieved in the absence of virus $(E = I = 0)$, while the Covid-19 disease present equilibrium $E_{eq}^{*}(S^{*},E^{*},I^{*})$ is achieved when the disease exists $(E \neq 0$ and $I \neq 0)$, where

$$S^{*} = \frac{N}{\beta}(\mu+\alpha+\theta), \tag{22}$$

$$E^{*} = \frac{\mu N}{\beta}(R_0 - 1), \tag{23}$$

$$I^{*} = \frac{\mu N \alpha}{\beta(\mu+\lambda+\delta_1)}(R_0 - 1), \tag{24}$$

$$R_0 = \frac{\Lambda\beta}{\mu N(\mu+\alpha+\theta)}. \tag{25}$$

R_0 is the basic reproduction number that measures the average number of the new infected individuals generated by a single infected individual in a population of susceptible individuals.

The value of R_0 will indicate whether the epidemic could occur or not. The reproduction basic number can be determined by using the next generation matrix method formulated in [39].

4.2. Local stability analysis. In this section, we analyze the local stability of the Covid-19 disease-free equilibrium.

Theorem 3. The Covid-19 disease-free equilibrium $E^0_{eq}\left(\frac{\Lambda}{\mu},0,0\right)$ of system (16) is asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

Proof. The Jacobian matrix at E_{eq} is

$$J(E_{eq}) = \begin{pmatrix} -\beta\frac{E}{N}-\mu & -\beta\frac{S}{N} & 0 \\ \beta\frac{E}{N} & \beta\frac{S}{N}-(\mu+\alpha+\theta) & 0 \\ 0 & \alpha & -(\mu+\lambda+\delta_1) \end{pmatrix} \tag{26}$$

The Jacobian matrix for the disease-free equilibrium is

$$J(E^0_{eq}) = \begin{pmatrix} -\mu & -(\mu+\alpha) & 0 \\ 0 & \beta\frac{\Lambda}{\mu N}-(\mu+\alpha+\theta) & 0 \\ 0 & \alpha & -(\mu+\lambda+\delta_1) \end{pmatrix} \tag{27}$$

The characteristic equation of this matrix is given by det $(J(E^0_{eq})-\lambda I_3)=0$, where I_3 is a square identity matrix of order 3.

Therefore, we see that the characteristic equation $\varphi(\zeta)$ of $J(E^0_{eq})$ are

$$\varphi(\zeta) = -(\mu+\zeta)\left[(\beta\frac{\Lambda}{\mu N}-(\mu+\alpha+\theta)-\zeta)(-(\mu+\lambda+\delta_1)-\zeta)\right], \tag{28}$$

where, eigenvalues of the characteristic equation of $J(E^0_{eq})$ are:

$$\begin{aligned} \zeta_1 &= -\mu \\ \zeta_2 &= (\mu+\alpha+\theta)(R_0-1) \\ \zeta_3 &= -(\mu+\lambda+\delta_1) \end{aligned} \tag{29}$$

And

$$R_0 = \frac{\Lambda\beta}{\mu N(\mu+\alpha+\theta)}. \tag{30}$$

Therefore, all the eigenvalues of the characteristic equation $J(E_{eq}^0)$ are clearly real and negative if $R_0 < 1$.

Thus, $|\arg(\zeta_i)| = \pi \geq \frac{\alpha\pi}{2}$ for $i = 1,2,3$

We conclude that the disease-free equilibrium is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$. □

Theorem 4. The Covid-19 disease present equilibrium E_{eq}^* is locally asymptotically stable if $R_0 > 1$ and unstable if $R_0 \leq 1$.

Proof. we present $E_{eq}^*(S^*, E^*, I^*)$ as the Covid-19 disease present equilibrium of system (16) and $S^* \neq 0, E^* \neq 0$, $I^* \neq 0$.

The Jacobian matrix at E_{eq}^* is

$$J(E_{eq}^*) = \begin{pmatrix} -\mu R_0 & -(\mu+\alpha+\theta) & 0 \\ \mu(R_0-1) & 0 & 0 \\ 0 & \alpha & -(\mu+\lambda+\delta_1) \end{pmatrix} \tag{31}$$

We see that the characteristic equation $\varphi(\zeta)$ of $J(E_{eq}^*)$ is

$$\varphi(\zeta) = \zeta^3 + a_1\zeta^2 + a_2\zeta + a_3, \tag{32}$$

where

$$a_1 = \mu R_0 + \mu + \lambda + \delta_1 > 0, \tag{33}$$

$$a_2 = \mu(R_0-1)(\mu+\alpha+\theta) + \mu R_0(\mu+\lambda+\delta_1) > 0, \tag{34}$$

$$a_3 = \mu(R_0-1)(\mu+\alpha+\theta)(\mu+\lambda+\delta_1) > 0, \tag{35}$$

By Routh- Hurwitz Criterion, the system (16) is locally asymptotically stable if $a_1 > 0$, $a_2 > 0$ $a_3 > 0$ and $a_1a_2 > a_3$.

Let be $D(\varphi)$ the discriminant of the polynomial $\varphi(\zeta)$ given by (32),when $n = 3$, we have

$$D(\varphi) = 4a_1^3a_3 - a_1^2a_2^2 - 18a_1a_2a_3 + 4a_2^3 + 27a_3^2 \text{ (See [40])}$$

According to [40], we can state the following theorem.

Theorem 5. We assume that $R_0 \geq 1$,

1) If $D(\varphi) > 0$ and $0 < \alpha \leq 1$ then E^* is locally asymptotically stable.

2) If $D(\varphi) < 0$ and $\alpha \leq \frac{2}{3}$ then E^* is locally asymptotically stable.

Thus, the present equilibrium E^*_{eq} of system (16) is locally asymptotically stable if $R_0 > 1$. □

4.3. Global stability analysis. To show that the system (16) is globally asymptotically stable, we use the Lyapunov function theory for both the Covid-19 disease free equilibrium and the Covid-19 disease present equilibrium. First, we present the global stability of the Covid-19 disease-free equilibrium E^0_{eq}.

4.3.1. *Global stabilty of the Covid-19 disease-free equilibrium.* .

Theorem 6: The Covid-19 disease free equilibruim E^0_{eq} is globally asymptotically stable if $R_0 \leq 1$ and unstable otherwise.

Proof. Let's consider the function

$$\begin{aligned} V &: \Gamma \to \mathbb{R} \\ V(S,E) &= \frac{1}{2}\left[(S-S_0)+E\right]^2 + \frac{\Lambda}{\mu\beta}(2\mu+\alpha+\theta)E \end{aligned} \tag{36}$$

where

$$\Gamma = \{(S,E) \in \Omega / S > 0, E > 0\}$$

The time fractional derivative of this function is

$$D^{\alpha}V(S,E) \leq -\mu(S-S_0)^2 - (\mu+\alpha+\theta)E^2 + \frac{\Lambda}{\mu\beta}(2\mu+\alpha+\theta)(\mu+\alpha+\theta)(R_0-1)E \leq 0 \tag{37}$$

Then, $D^{\alpha}V(S,E) \leq 0$ for $R_0 \leq 1$, consequently $V(S,E)$ is a Lyapunov function.

Note that $D^{\alpha}V(S,E) = 0$ if and only if $S = S_0$ and $E = 0$.

Hence, by Lasalle's invariance principle [41], E^0_{eq} is globally asymptotically stable on Ω. □

4.3.2. *Global stability of the Covid-19 disease present equilibrium.* The final result of the global stability of E^*_{eq} in this section is as follows.

Theorem 7. The Covid-19 disease disease present equilibrium point E^*_{eq} is globally asymptotically stable if $R_0 > 1$.

Proof. Let the function V

$$\begin{aligned} V \quad &: \quad \Gamma \rightarrow \ \mathbb{R} \\ V\left(S,E\right) \quad &= \quad S - S^{*}\ln(\frac{S}{S^{*}}) + E - E^{*}\ln(\frac{E}{E^{*}}) \end{aligned} \tag{38}$$

where

$$\Gamma = \{(S,E) \in \Omega / S > 0, E > 0\}$$

The time fractional derivative of this function is

$$D^{\alpha}V\left(S,E\right) = S^{*}D^{\alpha}\left(\frac{S}{S^{*}} - \ln(\frac{S}{S^{*}})\right) + E^{*}D^{\alpha}\left(\frac{E}{E^{*}} - \ln(\frac{E}{E^{*}})\right) \tag{39}$$

$$\leq S^{*}\left(1 - \frac{S}{S^{*}}\right)D^{\alpha}\left(\frac{S}{S^{*}}\right) + E^{*}\left(1 - \frac{E}{E^{*}}\right)D^{\alpha}\left(\frac{E}{E^{*}}\right) \tag{40}$$

$$\leq \left(1 - \frac{S}{S^{*}}\right)D^{\alpha}S + \left(1 - \frac{E}{E^{*}}\right)D^{\alpha}E \tag{41}$$

Using system (7) and the expressions the coordinates of equilibrium point E^{*}_{eq}, we have

$$D^{\alpha}V\left(S,E\right) \leq -\Lambda\frac{\left[S - S^{*}\right]^{2}}{SS^{*}} \leq 0 \tag{42}$$

Thus, $V\left(S,E\right)$ is a Lyapunov function.

Also, we obtain

$$D^{\alpha}V\left(S,E\right) = 0 \Leftrightarrow S = S^{*}. \tag{43}$$

In addition, it is clear that the largest invariant set of $\{(S,E) \in \Gamma / D^{\alpha}V\left(S,E\right) = 0\}$ in the singleton $\left\{E^{*}_{eq}\right\}$.

Hence, by LaSalle's invariance principle $[17]$ the Covid-19 disease present equilibrium point E^{*}_{eq} is globally asymptotically stable on Ω. $\square$

5. Numerical Simulations

In this section, we present some numerical solutions of model (7) for different values of the parameters to illustrate our results. By choosing $\Lambda = 100, \beta = 0.025, \alpha = 0.022, \lambda = 0.024, \gamma = 0.015, \mu = 0.065, \delta_{1} = 0.001, \delta_{2} = 0.004, \theta = 0.001, t_{f} = 250$ and different initial values for each variable of state, we have the Covid-19 disease free equilibrium $E^{0}_{eq} = (1.538 \times 10^{3}, 0, 0)$

and $R_0 = 0.336 < 1$. In this case, according to theorem (6), the Covid-19 disease free equilibrium E_{eq}^0 of the system (7) is globally asymptotically stable on Ω. (See Figure 5).

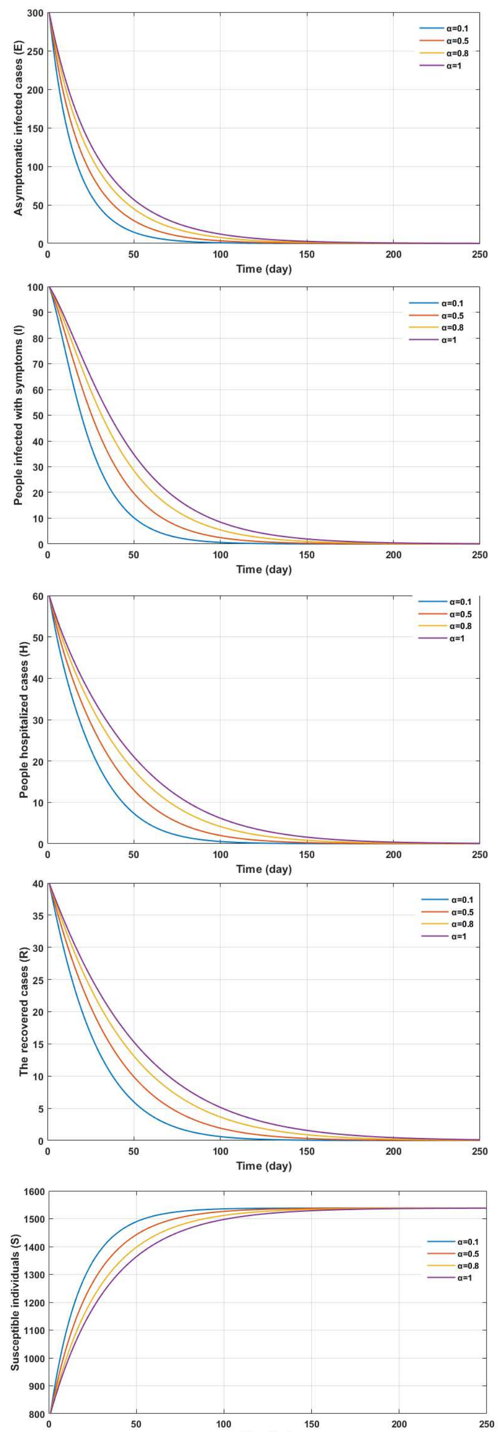


FIGURE 5. Stability of the free equilibrium E_{eq}^0 for different values of α

We present some numerical simulation concerning the global stability of the Covid-19 disease-free equilibrium E_{eq}^{0}. $(R_0 = 0.33$ and $R_0 < 1)$. In order to fix the value of α $(\alpha = 0.1)$ and using the different values of initial variables S_0, E_0, I_0, H_0 and R_0, we obtained the following graphic representations (See Figure 6)

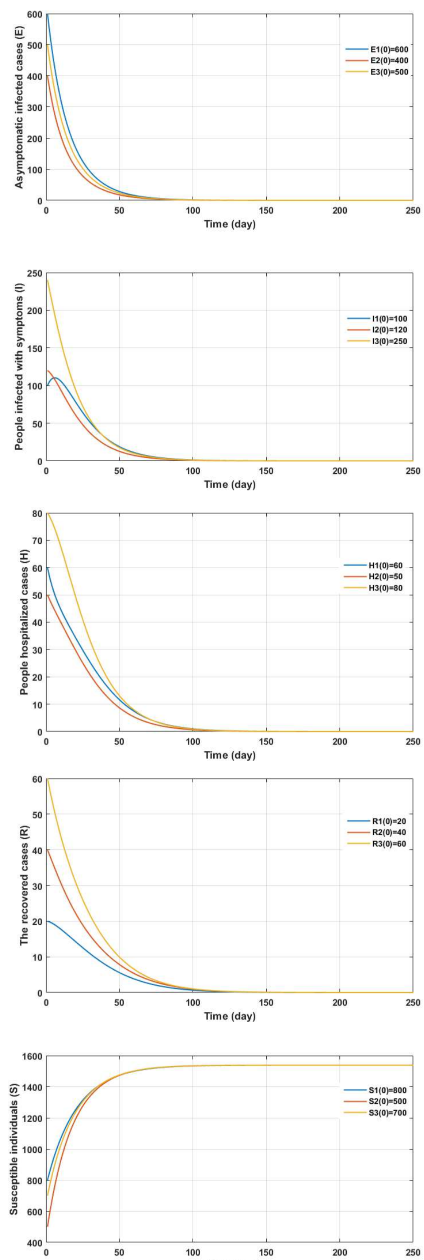
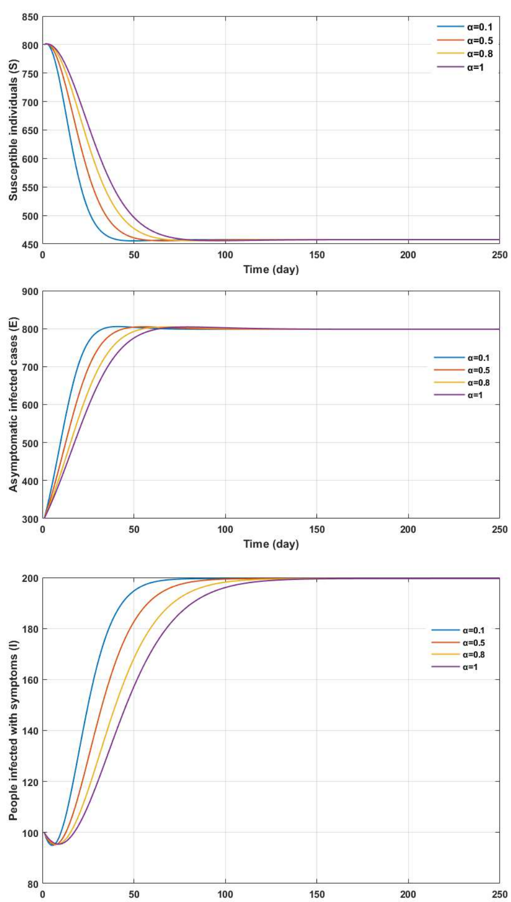


FIGURE 6. Stability of the free-equilibrium E_{eq}^{0} for different initial values for each variable of state.

From these figures, we observe that the number of potential individuals increases and approaches the number $S_0 = 1538$ (see Figure 6). Also, the number of the asymptomatic infected cases decreases and converges to zero (see Figure 6). The number of the infected people with symptoms and carriers of the virus increases at first, after that it decreases and approaches zero (see Figure 6). The number of the hospitalized cases decreases and approaches zero (see Figure 6).The number of the recovered cases decreases and approaches zero (see Figure 6). Therefore, the solution curves to the equilibrium $E^0_{eq}(S_0,0,0,0,0)$ when $R_0 < 1$. Hence, model (7) is globally asymptotically stable.

Also, for $\Lambda = 10^2, \beta = 0.25, \alpha = 0.022, \lambda = 0.024, \gamma = 0.015, \mu = 0.065, \delta_1 = 0.001, \delta_2 = 0.004, t_f = 250$. We have the Covid-19 disease equilibrium $E^*_{eq} = (4.58 \times 10^2, 7.98 \times 10^2, 1.99 \times 10^2)$ and $R_0 = 3.36 \succ 1$. In this case, according to theorem (7), the Covid-19 disease equilibrium E^*_{eq} of the system (7) is globally asymptotically stable on Ω. (See Figure (7))



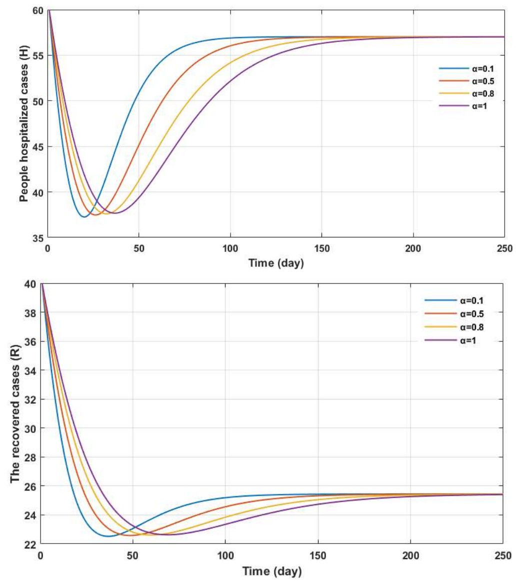
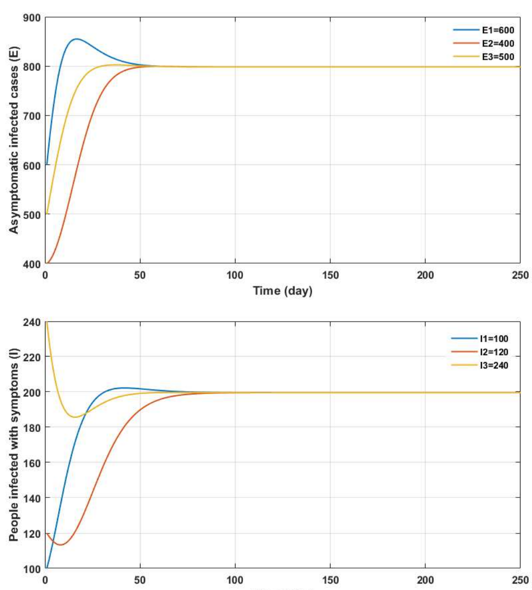


FIGURE 7. Stability of the present equilibrium E^*_{eq} for different values of α

Also, we present the numerical simulation of the global stability of the Covid-19 disease present equilibrium E^*_{eq}. In order to fix α $(\alpha = 0.1)$ and we use the same parameters and different initial values, ($R_0 = 3.36$ and $R_0 > 1$) and using the different values of initial variables S_0, E_0, I_0, H_0, and R_0, we obtained the graphic (See Figure (8)).



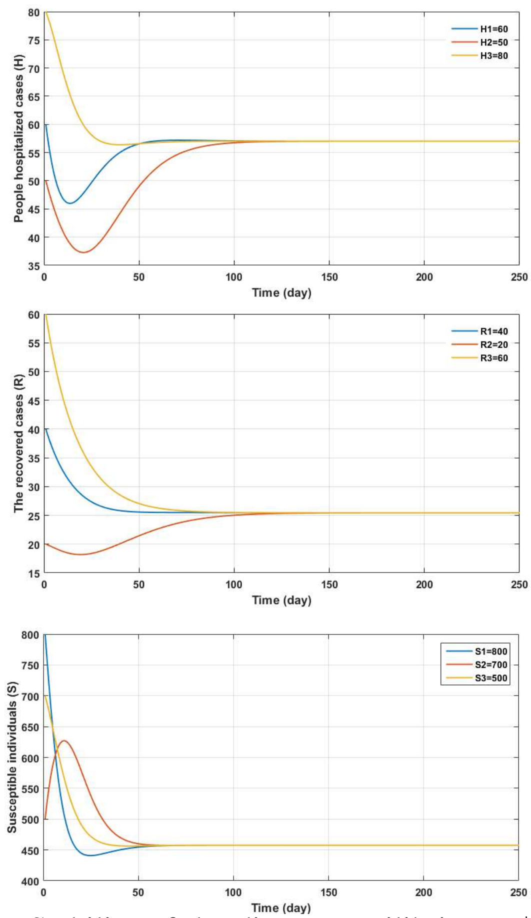


FIGURE 8. Stability of the disease-equilibrium E_{eq}^* for different initial values for each variable of state.

From these figures, we remark that the number of potential individuals increases at first, then it decreases slightly and approaches the value $S^* = 458$ (see Figure 8).Concerning the number of the asymptomatic infected cases or cases with mild symptoms, it decreases rapidly at first, then it increases slightly and converges the value $E^* = 798$ (see Figure 8). The number of the infected people with symptoms and carriers of the virus increases and converges the value $I^* = 199$ (see Figure8). Also, the number of the hospitalized cases decreases and converges the value $H^* = 57$ (see Figure 8). The number of the recovered cases decreases and converges the value $R^* = 25$ (see Figure 8). Therefore, the solution curves to the equilibrium $E_{eq}^*(S^*, E^*, I^*, H^*, R^*)$ when $R_0 > 1$. Hence, the model (7) is globally asymptotically stable.

The above figures, we show that the solutions of system (7) converge to the equilibrium points E_{eq}^0 and E_{eq}^* for different fractional value of α. In addition, the solutions converge rapidly to its steady state when the fractional value of α is very small.

6. Conclusion

In this work, we formulated and presented a fractional order of a mathematical model SEIHR of Covid-19 disease that describes the dynamics of citizens who were infected by this disease. We have found $R_0 = \frac{\Lambda\beta}{\mu N(\mu+\alpha+\theta)}$, as basic reproduction number of the system (7), which helps us to determine the dynamics of the system. We also studied and used the stability analysis theory for nonlinear systems to analyze the mathematical Covid-19 disease model and to study both the local and global stability of Covid-19 disease. Local asymptotic stability for the Covid-19 disease-free equilibrium E_{eq}^0 can be obtained if the threshold quantity $R_0 \leq 1$. On the other hand, if $R_0 > 1$, then the Covid-19 disease present equilibrium E_{eq}^* is locally asymptotically stable. A Lyapunov function was used to show that E_{eq}^0 is globally asymptotically stable if $R_0 \leq 1$. Also, a Lyapunov function was used to show that E_{eq}^* is globally asymptotically stable if $R_0 > 1$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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