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NEW TECHNIQUES TO ESTIMATE THE SOLUTION OF AUTONOMOUS SYSTEM

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Abstract. This research aims to solve the nonlinear model formulated in a system of differential equations with an initial value problem (IVP) represented in COVID-19 mathematical epidemiology model as an application using new approach: *Approximate Shrunken* are proposed to solve such model under investigation, which combines classic numerical method and numerical simulation techniques in an effective statistical form which is shrunken estimation formula. Two numerical simulation methods are used firstly to solve this model: Mean Monte Carlo Runge-Kutta and Mean Latin Hypercube Runge-Kutta Methods. Then two approximate simulation methods are proposed to solve the current study. The results of the proposed approximate shrunken methods and the numerical simulation methods are compared with the standard results of the numerical method which is Runge-Kutta 4th Method from the year 2021 to 2025, using the absolute error, through comparison, it becomes clear that the approximate proposed solution is better and closer to the standard solution than the solutions of other methods that used to solve this system. The results are tabulated and represented graphically, as well as a discussion to prove the efficiency of the proposed methods.

Keywords: coronavirus disease model; shrunken estimation methods; Runge-Kutta numerical methods.

2020 AMS Subject Classification: 92C60.

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1. INTRODUCTION

Throughout history, many epidemic appeared and posed a real threat to the world, as well as greatly affected economic and population growth, and caused trips to stop in some cities.

These epidemic may be contagious or transmitted in other ways. Among these disease are the Black Death, which spread widely in Europe, malaria, the plague in Africa, SARS in China 2002-2003, AIDS and cancer, etc. [1-3]. At the beginning of 2019, the Corona virus appeared, specifically in the Chinese city of Wuhan, and this epidemic is considered one of the most dangerous and fastest spreading epidemics, and it is of the SARS-CV type [4]. In the year 2020, on March 29, the epidemic spread significantly and rapidly throughout the world, which led to the suspension of flights through airports, land transport between countries, schools and universities, and most jobs with direct mixing [5-7]. The world Health Organization declared this epidemic to be a pandemic after it infected 199 countries around the world and caused the death of thousands of people [8]. The emergence of the epidemic coincided with the period of spring festivals and celebrations in Asia, and this helps to spread the epidemic due to the mixing of many people, especially flights with all countries of the world. This is considered one of the reasons for the spread of the virus to the rest of the world [9]. As a result of the lack of health facilities in some countries, including developing countries, and the severity and speed of the virus's spread, the virus turned into a global pandemic that caused the death of thousands of people around the world because they did not receive appropriate treatment is social distancing and adherence to health prevention ways and the directive of the World Health Organization [5, 9]. The mathematical system in our study is an epidemiological model formulated in the form of a system of first order nonlinear differential equations. These epidemiological models deals with rapidly spreading diseases that occupy large areas, and this epidemic model is considered as stochastic-deterministic models [1, 5, 10], see also [11, 12]. SIR epidemic model was also studied by Temimi-Ansari method, Daftardar-Jafari method, and Banach contraction method, [13]. The stochasticity in COVID -19 for SIR epidemic model was discussed in Iraq to die out the epidemic in [14], see also [11]. Shafeeq, et al., studied Bifurcation analysis of a vaccination mathematical model with application to COVID-19 pandemic in 2022 [15]. For the first time, LTAM was discussed to solve the nonlinear epidemic model, this method is combine Laplace transform with Tamimi and Ansari iterative method, [16]. Yaseen, et al., in 2023 discussed stability and Hopf bifurcation of an epidemiological model with effect of delay the awareness programs and vaccination [17].

Many ways can be solved the epidemiological models, like semi-analytic methods. Mahdi and Maha in 2020 discussed some like semi analytic methods for nonlinear Smoking Habit Model [18]. As well, the numerical method Runge-kutta for the 4th order ($RK4$), which is one of the reliable methods for solving differential equations of different orders high accuracy [19]. Some authors created modified numerical simulation approaches to get good results for epidemiology models, such as MMC_FD was discussed by Maha, et al. in 2019 the new approach that mixed two different methods which are the Mean Monte Carlo simulation technique and numerical iteration method which is finite difference method to sample randomly from a nonlinear epidemic model [20] and MLH_FD was studied by Mohammed, et al. in 2018 a non-conventional hybrid numerical approach with multi-dimensional random sampling for cocaine abuse in Spain [21]. Shatha and Maha discussed Runge-kutta numerical method for solving nonlinear Influenza Model in 2021. Emad and Maha studied nonlinear COVID-19 mathematical model using a reliable $RK4$ numerical method, in 2022 [22, 23]. Mahdi and Maha in 2019 discussed the modified numerical simulation technique for solving nonlinear epidemic models [24] which is Mean Monte Carlo Runge-Kutta method (MMC_RK) which is an efficient numerical simulation technique mixed two methods of different natures together that are Monte Carlo simulation process (MC) and Runge-Kutta numerical method (RK) that used to find the solution for system of equations. Shatha and Maha in 2022 studied the other numerical simulation method is Mean Latin Hypercube Runge-Kutta (MLH_RK) which is hybrid of a Latin Hypercube sampling (LHS) simulation method and a numerical Runge-Kutta (RK) method, to solve the influenza model, [25, 26]. MLH_RK is one of the reliable method to solve such systems. Emad and Maha discussed applying a suitable approximate-simulation techniques of an epidemic model with random parameters in 2022 [25, 26].

In this study, many methods are used, the first one, the numerical method Runge-kutta for the 4th order ($RK4$) is used for solving the system under study. Addition that, There are two numerical simulation echniques in our study, one of them by Mean Monte Carlo Runge-Kutta method (MMC_RK) and the other Mean Latin Hypercube Runge-Kutta method (MLH_RK). The new approaches which called *Approximate Shrunken Methods* denoted by (ASM_MMCRK and ASM_MLHRK) are applied on COVID-19 mathematical model. *Approximate Shrunken Methods* are hybrid between classic numerical method which is Runge-Kutta 4th (RK4) and numerical simulation techniques which are Mean Monte Carlo Runge-Kutta method (MMC_RK) or Mean Latin Hypercube Runge-Kutta method (MLH_RK) in the statistical form which is shrinkage estimation method. These a new proposed methods are more accurate and reliable than other numerical simulation methods in solving such nonlinear mathematical system that is used under study.

This research is divided into the following: in Section 2, the mathematical model of COVID-19 is presented; Section 3, obtains the deriving of the numerical method $RK4$ and Section 4, contains numerical simulation methods MMC_RK, MLH_RK and in Section 5, the new approach approximate shrunken methods (ASM_MLHRK and ASM_MMCRK) that used for solving the epidemic model under study. Section 6, contains the discussion and tables of the methods used, as well as their graphic representation. Finally, Section 7 explains the final conclusion of the research.

2. MATHEMATICAL MODEL OF COVID-19

This model is used successfully to study the people vaccinated against [27]. The population consists of five types of individuals C, W, E, L and Z represent susceptible, vaccinated, asymptomatic, symptomatic and the recovery respectively. They are functions of time. The governing equations for the epidemic under study by non-linear ordinary differential equation of first order [27].

$$C'(t) = M - \tau C - \frac{\alpha(1+\beta E)C}{N} - \mu C + \gamma Z,$$

$$W'(t) = \tau C - \frac{\rho\alpha(1+\beta E)W}{N} - \mu W\ ,$$

$$E'(t) = \frac{\alpha(1+\beta E)C}{N} + \frac{\rho\alpha(1+\beta E)W}{N} - \delta E - \mu E, \qquad (1)$$

$$L'(t) = \ \theta\delta E - \sigma L - \mu L$$

$$Z'(t) = (1-\theta)\delta E + \sigma L - \gamma Z - \mu Z$$

Where Tables 1 and 2 represent variables C, W, E, L, Z and parameters M, τ, α, β, μ, γ, ρ, δ, θ and σ respectively. The system (1) has the initial conditions for the system obtained from [27] as the following: $C(0) = 50000000$, $W(0) = 0$, $E(0) = 1000$, $L(0) = 100$ and $Z(0) = 50$ with the predicted parameters that are given in Table 2:

Table 1. Variables of COVID-19 model [27].

Variable	Definition
$C(t)$	People who are not infected but are vulnerable to not having immunity
$W(t)$	People vaccinated against coronavirus
$E(t)$	People infected with the virus without showing any symptoms
$L(t)$	Infected people and symptoms of infection are clear to them
$Z(t)$	People who have recovered from the virus and died as a result of infection

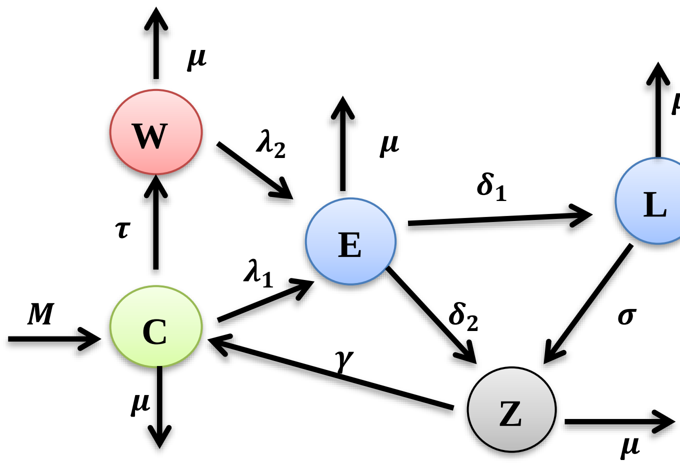


Figure 1. The state transformation process of individuals.

Table 2. Parameters of COVID-19 model [27].

Parameter	Definition	Value
α	The rate of transmission people infected with this virus	0.8883
β	Correction factor for the rate of movement of people without infection	0.45
μ	The normal rate of death	0.00003349 day
γ	The rate of invulnerability to infection	0.005
$1-\rho$	Vaccine efficacy and potency	0.8
$\frac{1}{\delta}$	The average period without symptoms of infection	7 day
θ	The proportion of people does not show the effects of the symptoms of the virus, but it develops into a state of infection	0.2
$1-\theta$	Proportion of asymptomatic individuals who recover	0.8
M	Birth rate in the community	1500/day
τ	Vaccination rate against the virus	0.01 day
$\frac{1}{\sigma}$	The average rate of people recovering from infection with the virus	10 days

3. NUMERICAL METHOD FOR SOLVING COVID-19 MODEL

$RK4$ is one of the numerical iterative methods with high accuracy. The nonlinear system (1) of the vaccine model against COVID-19 can be solved by $RK4$ with initial conditions: $C_0(t) = 50000000$, $W_0(t) = 0$, $E_0(t) = 1000$, $L_0(t) = 100$ and $Z_0(t) = 50$, with the predicted parameters in Table 2. The general form of $RK4$ is:

$$g_{i+1} = g_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{2}$$

where $RK4$ method on system (1) can be written as:

$$C_{i+1} = f(t_i, C_i, W_i, E_i, L_i, Z_i),$$

$$= C + \frac{1}{6}(kC_1 + 2kC_2 + 2kC_3 + kC_4) * h, \tag{3}$$

$$W_{i+1} = f(t_i, C_i, W_i, E_i, L_i, Z_i),$$

$$= \mathrm{W} + \frac{1}{6}(kW_1 + 2kW_2 + 2kW_3 + kW_4) * h, \tag{4}$$

$E_{i+1} = f(t_i, C_i, W_i, E_i, L_i, Z_i),$

$$= E_i + \frac{1}{6}(kE_1 + 2kE_2 + 2kE_3 + kE_4) * h, \quad (5)$$

$L_{i+1} = f(t_i, C_i, W_i, E_i, L_i, Z_i),$

$$= L_i + \frac{1}{6}(kL_1 + 2kL_2 + 2kL_3 + KL_4) * h, \quad (6)$$

$R_{i+1} = f(t_i, C_i, W_i, E_i, L_i, Z_i),$

$$= Z_i + \frac{1}{6}(kZ_1 + 2kZ_2 + 2kZ_3 + kZ_4) * h. \quad (7)$$

for all $i = 1,2, \dots, m$

Now, we must find KS_1, kV_1, kA_1, kI_1 and kR_1as follows:

$$kC_1 = f(t_i, C_i, W_i, E_i, L_i, Z_i), \quad (8)$$

$$kW_1 = f(t_i, C_i, W_i, E_i, L_i, Z_i), \quad (9)$$

$$kE_1 = f(t_i, C_i, W_i, E_i, L_i, Z_i), \quad (10)$$

$$kL_1 = f(t_i, C_i, W_i, E_i, L_i, Z_i), \quad (11)$$

$$kZ_1 = f(t_i, C_i, W_i, E_i, L_i, Z_i). \quad (12)$$

for all $i = 0,1, \dots, m$

Also, to find kC_2, kW_2, kE_2, kL_2 and kZ_2 as follows:

$$kC_2 = f_2\left(t_i \frac{1}{2}h, C_i + \frac{1}{2}h * kC_1, W_i + \frac{1}{2}h * kW_1, E_i + \frac{1}{2}h * kE_1, L_i + \frac{1}{2}h * kL_1, Z_i + \frac{1}{2}h * kZ_1\right), \quad (13)$$

$$kW_2 = f_2\left(t_i \frac{1}{2}h, C_i + \frac{1}{2}h * kC_1, W_i + \frac{1}{2}h * kW_1, E_i + \frac{1}{2}h * kE_1, L_i + \frac{1}{2}h * kL_1, Z_i + \frac{1}{2}h * kZ_1\right), \quad (14)$$

$$kE_2 = f_2\left(t_i \frac{1}{2}h, C_i + \frac{1}{2}h * kC_1, W_i + \frac{1}{2}h * kW_1, E_i + \frac{1}{2}h * kE_1, L_i + \frac{1}{2}h * kL_1, Z_i + \frac{1}{2}h * kZ_1\right), \quad (15)$$

$$kL_2 = f_2\left(t_i \frac{1}{2}h, C_i + \frac{1}{2}h * kC_1, W_i + \frac{1}{2}h * kW_1, E_i + \frac{1}{2}h * kE_1, L_i + \frac{1}{2}h * kL_1, Z_i + \frac{1}{2}h * kZ_1\right), \quad (16)$$

$$kZ_2 = f_2\left(t_i \frac{1}{2}h, C_i + \frac{1}{2}h * kC_1, W_i + \frac{1}{2}h * kW_1, E_i + \frac{1}{2}h * kE_1, L_i + \frac{1}{2}h * kL_1, Z_i + \frac{1}{2}h * kZ_1\right). \quad (17)$$

for all $i = 0,1, \dots, m.$

To get KC_3, kW_3, kE_3, kL_3 and kZ_3 as follows:

$$kC_3 = f_3\left(t_i \frac{1}{2}h, C_i + \frac{1}{2}h * kC_2, W_i + \frac{1}{2}h * kW_2, E_i + \frac{1}{2}h * kE_2, L_i + \frac{1}{2}h * kL_2, Z_i + \frac{1}{2}h * kZ_2\right), \quad (18)$$

$$kW_3 = f_3\left(t_i \frac{1}{2}h, C_i + \frac{1}{2}h * kC_2, W_i + \frac{1}{2}h * kW_2, E_i + \frac{1}{2}h * kE_2, L_i + \frac{1}{2}h * kL_2, Z_i + \frac{1}{2}h * kZ_2\right), \quad (19)$$

$$kE_3 = f_3\left(t_i \frac{1}{2}h, C_i + \frac{1}{2}h * kC_2, W_i + \frac{1}{2}h * kW_2, E_i + \frac{1}{2}h * kE_2, L_i + \frac{1}{2}h * kL_2, Z_i + \frac{1}{2}h * kZ_2\right), \quad (20)$$

$$kL_3 = f_3\left(t_i \frac{1}{2}h, C_i + \frac{1}{2}h * kC_2, W_i + \frac{1}{2}h * kW_2, E_i + \frac{1}{2}h * kE_2, L_i + \frac{1}{2}h * kL_2, Z_i + \frac{1}{2}h * kZ_2\right), \quad (21)$$

$$kZ_3 = f_3\left(t_i \frac{1}{2}h, C_i + \frac{1}{2}h * kC_2, W_i + \frac{1}{2}h * kW_2, E_i + \frac{1}{2}h * kE_2, L_i + \frac{1}{2}h * kL_2, Z_i + \frac{1}{2}h * kZ_2\right). \quad (22)$$

for all $i = 0,1, \dots, m$.

To obtain kC_4, kW_4, kE_4, kL_4 and kZ_4 as follows:

$$kC_4 = f_4(t_i + h, C_i + h * kC_3, W_i + h * kW_3, E_i + h * kE_3, L_i + h * kL_3, Z_i + h * kZ_3), \quad (23)$$

$$kW_4 = f_4(t_i + h, C_i + h * kC_3, W_i + h * kW_3, E_i + h * kE_3, L_i + h * kL_3, Z_i + h * kZ_3), \quad (24)$$

$$kE_4 = f_4(t_i + h, C_i + h * kC_3, W_i + h * kW_3, E_i + h * kE_3, L_i + h * kL_3, Z_i + h * kZ_3), \quad (25)$$

$$kL_4 = f_4(t_i + h, C_i + h * kC_3, W_i + h * kW_3, E_i + h * kE_3, L_i + h * kL_3, Z_i + h * kZ_3), \quad (26)$$

$$kZ_4 = f_4(t_i + h, C_i + h * kC_3, W_i + h * kW_3, E_i + h * kE_3, L_i + h * kL_3, Z_i + h * kZ_3). \quad (27)$$

for all $i = 0,1, \dots, m$.

For substituting Eqs. (8), (13), (18) and (23) in Eq. (3) to find the numerical solutions of C_i. Also putting Eqs. (9), (14), (19) and (24) in Eq. (4) for achieve the numerical solutions of W_i. in the same manner substitute Eqs. (10), (15), (20) and (25) in Eq. (5) to obtain the numerical solutions of E_i, substituting Eqs. (11), (16), (21) and (26) in Eq. (6) to find the numerical solutions of L_i. Finally, substituting Eqs. (12), (17), (22) and (27) in Eq. (7) for getting the numerical solutions of Z_i. All for each $i = 0,1, \dots, m$.

4. Approximate Simulation Methods for Solving COVID-19 Model

Some of the modified numerical simulation methods that are used in this study will be discussed in this section formulated in our model.

4.2 MEAN MONTE CARLO RUNGE-KUTTA (MMC_RK) METHOD

Mean Monte Carlo Runge-Kutta (MMC_RK) is an efficient numerical simulation method for solving such mathematical models. This method consists of mixing two different methods; one numerical is Runge-Kutta method 4th ($RK4$) and the other Monte Carlo simulation process (MC) is called the Mean Monte Carlo Runge-Kutta (MMC_RK), see [24]. MC estimates the model coefficients that are random variables while RK is used to solve the model numerically. The average of the last RK iteration results with each MC repetition is considered the estimated approximate solution for the model under study. The MMC_RK method is implemented by using MATLAB software, more details are shown in [24].

4.2 MEAN LATIN HYPERCUBE RUNGE-KUTTA (MLH_RK) METHOD

Mean Latin Hypercube Runge-Kutta (MLH_RK), see [25] is numerical simulation method that is a mixture of a simulation method which is Latin Hypercube Sampling (LHS) and a numerical method is Runge-Kutta (RK). It is considered one of the reliable methods for solving a system of nonlinear ordinary differential equations of the first order. LHS estimates the coefficients of the model that are considered random variable while RK is used to solve the model numerically. The average of the last RK iteration results with each LHS repetition is considered the estimated approximate solution for the model under study.

The process of MLH_RK is similar to MMC_RK which was talked about before. In addition to that MLH_RK is more accurate and faster than the MMC_RK method because it simulates model parameters at once whereas this integrated method is implemented using the MATLAB program, see [25].

5. APPROXIMATE SHRUNKEN METHODS

In this section, two new techniques are created to solve such models under study, especially epidemic models. These techniques have proven their efficiency and effectiveness in obtaining more accurate results than the modified numerical simulation methods in previous studies, and they are considered a new approach between statistics and numerical simulation.

5.1 APPROXIMATE SHRUNKEN METHOD (ASM_MMCRK)

Approximate Shrunken Method the form called ASM_MMCRK, is a new approach that is a hybrid between the classic approximate method which is RK4, and numerical simulation techniques which is MMC_RK in the shrinkage estimation statistical form. This newly proposed method is a more accurate and reliable method than other numerical simulation MMC_RK and MLH_RK methods in solving such nonlinear mathematical systems that are used under study. ASM_MMCRK gives alternative estimation values between statistical and approximate methods. This method has been calculated using the MATLAB program and as shown in the following algorithm:

- **Step 1**: The parameters of a model have been simulated by MC for *n* times.
- **Step 2**: One value is specified from Step 1 and transformed into a specific distribution, to replace Its value in the system, for each random parameter.
- **Step 3**: Solve the system *m*-times iterations numerically by RK to get the numerical solutions, the last iterative result is the final solution which is selected.
- **Step 4**: Repeat Steps 1 and 2 for *n* repetitions.
- **Step 5**: To find a solution of the system under study by MMC_RK, calculating the mean of final solutions from Step 4.
- **Step 6**: Using the proposed algorithm as follow:

$$\widehat{\boldsymbol{Sol}}_{\mathbf{ASM\_MMC}\boldsymbol{RK}} = \boldsymbol{W}(\boldsymbol{RK}\mathbf{4}) + (\mathbf{1} - \boldsymbol{W})(\boldsymbol{MMC\_RK})$$

Since the solution of the system for ASM_MMCRK is called $\widehat{Sol}_{\text{ASM\_MMCRK}}$ and w is weight function, where $0 \leq w \leq 1$.

5.2 APPROXIMATE SHRUNKEN METHOD (ASM_MLHRK)

Approximate Shrunken Method the form named ASM_MLHRK; is another proposed method which is mixture of a classical numerical method which is *RK4* and, and another numerical simulation techniques which is MLH_RK to produce a new algorithm in the statistical form which is shrinkage estimation form. This proposed algorithm is more accurate and efficient compared to other approximate simulation methods for solving such mathematical models. ASM_MLHRK is promising to create alternative estimation values between statistical and approximate methods. This method is implemented using the MATLAB program and as shown in the following algorithm:

- **Step 1**: All model parameters have been simulated by LHS for n times at ones.
- **Step 2**: For each random parameter, one value is specified and replaced in the system.
- **Step 3**: Solve the system *m*-times iterations numerically by RK to get the numerical solutions, the last iterative result is the final solution.
- **Step 4**: Repeat Steps 1 and 2 for *n* repetitions.
- **Step 5**: Calculate the mean of final solutions from Step 4, to find a solution of the system under study by MLH_RK.
- **Step 6**: Using the new algorithm as follow:

$$\boldsymbol{\widehat{Sol}_{\text{ASM\_MLH}RK} = W(RK4) + (1 - W)(MLH\_RK)}$$

Since the solution of the system for ASM_MLHRK is called $\widehat{Sol}_{\text{ASM\_MLHRK}}$ and w is weight function, where $0 \leq w \leq 1$.

6. RESULTS AND DISCUSSION

This section discusses the approximate simulation solutions of the epidemic model under the study of the people vaccinated against COVID-19. These results are discussed and analyzed in this section. Table 3 (a) and Table 3 (b) contain numerical simulation results through one year with step size $h = \{0.02, 0.08\}$ weekly and monthly for the period of the beginning of 2021 to the end of 2022 under study. Also, Table 5 (a) and Table 5 (b) contain the results of the numerical simulation solution for the groups $C(t), W(t), E(t), L(t)$ and $Z(t)$ of society, for a future period of time until 2025 in the interval [0,48].

Table 3 (a). Numerical simulation results of COVID-19 model through two years with iteration (24, 104)

Model Variables	Step Size, *h* (weekly & monthly)	RK4 (2 years)	MMC_RK (100 repetition)	MLH_RK (100 repetition)
$C(t)$	0.08 (monthly)	37085084.65590938	37085084.63344941	37085084.63874430
	0.02 (weekly)	37029322.77190823	37029322.75736202	37029322.76831016
$W(t)$	0.08 (monthly)	10276439.20625248	10276439.25109421	10276439.21572080
	0.02 (weekly)	10295713.93745516	10295713.91328052	10295713.91683632
$E(t)$	0.08 (monthly)	1613222.87881244	1613222.85395034	1613222.88396380
	0.02 (weekly)	1652523.89802588	1652523.87033773	1652523.89808829
$L(t)$	0.08 (monthly)	125188.11983965	125188.17256626	125188.13831513
	0.02 (weekly)	128652.85432294	128652.81162111	128652.89108403
$Z(t)$	0.08 (monthly)	775323.97875363	775323.94437549	775323.96720432
	0.02 (weekly)	782446.78337939	782446.72048965	782446.73254787

Table 3 (b). Approximate simulation results of COVID-19 model through two years with iteration (24, 104)

Model Variables	Step Size, *h* (weekly & monthly)	RK4 (2 years)	ASM_MMCRK (100 repetition)	ASM_MLHRK (100 repetition)
$C(t)$	0.08 (monthly)	37085084.65590938	37085084.66005684	37085084.65548420
	0.02 (weekly)	37029322.77190823	37029322.76902728	37029322.77998371
$W(t)$	0.08 (monthly)	10276439.20625248	10276439.21067146	10276439.20892619
	0.02 (weekly)	10295713.93745516	10295713.93889395	10295713.93671944
$E(t)$	0.08 (monthly)	1613222.87881244	1613222.87384391	1613222.87918716
	0.02 (weekly)	1652523.89802588	1652523.89806853	1652523.89803135
$L(t)$	0.08 (monthly)	125188.11983965	125188.12904200	125188.11836894
	0.02 (weekly)	128652.85432294	128652.83814106	128652.84746489
$Z(t)$	0.08 (monthly)	775323.97875363	775323.97175147	775323.97679611
	0.02 (weekly)	782446.78337939	782446.77772828	782446.79915969

The comparison between the new approach by approximate shrunken methods; ASM_MMCRK, ASM_MLHRK and the numerical simulation methods MMC_RK, MLH_RK compared with the numerical method $RK4$ are is achieved by absolute error criterion for $C(t)$,$W(t)$, $E(t)$, $L(t)$ and $Z(t)$ are shown numerically in Tables 4 and 6. Observe that the absolute error of the new methods ASM_MLHRK and ASM_MC, they are have error less than of the other numerical simulation methods MLH_RK and MMC_RK, for all groups $C(t)$, $W(t)$, $E(t)$, $L(t)$ and $Z(t)$ of population. ASM_MLHRK has the smallest value of absolute error and this means that the proposed method is more accurate and reliable than the other methods.

Prediction intervals that contain the minimum bound (5th percentile) and maximum bound (95th percentile) for MMC_RK and MLH_RK results in the future until 2025, MMC_RK and MLH_RK results are inside the predicted intervals, see Table 8.

Table 4. Absolute error for MMC_RK, MLH_RK, ASM_MMCRK and ASM_MLHRK compared with $RK4$ through two years

Model Variables	**Step Size, h (weekly & monthly)**	**MMC_RK (2 years)**	**MLH_RK (100 repetition)**	**ASM_MMCRK (100 repetition)**	**ASM_MLHRK (100 repetition)**
$\boldsymbol{C(t)}$	0.08 (monthly)	0.02245997	0.01716508	0.00414746	0.00042518
	0.02 (weakly)	0.01454621	0.00359807	0.00288095	0.00807548
$\boldsymbol{W(t)}$	0.08 (monthly)	0.04484173	0.00946832	0.00441898	0.00267371
	0.02 (weekly)	0.02417464	0.02061884	0.00143879	0.00073572
$\boldsymbol{E(t)}$	0.08 (monthly)	0.02486210	0.00515136	0.00496853	0.00037472
	0.02 (weekly)	0.02768815	0.00006241	0.00004265	0.00000547
$\boldsymbol{L(t)}$	0.08 (monthly)	0.05272661	0.01847548	0.00920235	0.00147071
	0.02 (weekly)	0.04270183	0.03676109	0.01618188	0.00685805
$\boldsymbol{Z(t)}$	0.08 (monthly)	0.03437814	0.01154931	0.00700216	0.00195752
	0.02 (weekly)	0.06288974	0.05083152	0.00565111	0.01578030

Table 5 (a). Expected numerical simulation results of COVID-19 model from 2021 to 2025 with iteration (48, 208)

Model Variables	**Step Size, *h* (weekly & monthly)**	**RK4 (4 years)**	**MMC_RK 1000 repetition (4 years)**	**MLH_RK 1000 repetition (4years)**
$C(t)$	0.08 (monthly)	7770078.03210272	7770078.03781565	7770078.03526455
	0.02 (weekly)	8063309.62351586	8063309.62957513	8063309.62726213
$W(t)$	0.08 (monthly)	10811867.78401260	10811867.79049931	10811867.78965045
	0.02 (weekly)	10952433.59246311	10952433.5953929	10952433.59486301
$E(t)$	0.08 (monthly)	8219721.15928793	8219721.15697412	8219721.15803524
	0.02 (weekly)	9624761.53721278	9624761.53483086	9624761.53505387
$L(t)$	0.08 (monthly)	2369666.36562734	2369666.36645547	2369666.36591673
	0.02 (weekly)	2441792.19283193	2441792.19387042	2441792.19356723
$Z(t)$	0.08 (monthly)	24873285.35075667	24873285.35312171	24873285.35107616
	0.02 (weekly)	24402059.21778145	24402059.22099882	24402059.21966456

Table 5 (a). Expected numerical simulation results of COVID-19 model from 2021 to 2025 with iteration (48, 208)

Model Variables	**Step Size, *h* (weekly & monthly)**	**RK4 (4 years)**	**ASM_MMCRK 1000 repetition (4 years)**	**ASM_MLHRK 1000 repetition (4 years)**
$C(t)$	0.08 (monthly)	7770078.03210272	7770078.03317467	7770078.03297931
	0.02 (weekly)	8063309.62351586	8063309.62219451	8063309.62227721
$W(t)$	0.08 (monthly)	10811867.78401260	10811867.78250540	10811867.78503365
	0.02 (weekly)	10952433.59246311	10952433.5917881	10952433.59302430
$E(t)$	0.08 (monthly)	8219721.15928793	8219721.15857161	8219721.15923087
	0.02 (weekly)	9624761.53721278	9624761.53523915	9624761.53602377
$L(t)$	0.08 (monthly)	2369666.36562734	2369666.36395723	2369666.36505389
	0.02 (weekly)	2441792.19283193	2441792.19138873	2441792.19309842
$Z(t)$	0.08 (monthly)	24873285.35075667	24873285.3505814	24873285.35089628
	0.02 (weekly)	24402059.21778145	24402059.2184932	24402059.21790663

Table 6. Absolute error for MMC_RK, MLH_RK ASM_MMCRK and ASM_MLHRK compared with $RK4$ from 2021 to 2025

Model Variables	Step Size, $\boldsymbol{h}$ (weekly & monthly)	MMC_RK 1000 rep. (48 years)	MLH_RK 1000 rep. (48 years)	ASM_MMCRK 1000 rep. (48 years)	ASM_MLHRK 1000 rep. (48 years)
$\boldsymbol{C}(\boldsymbol{t})$	0.08 (monthly)	0.00571293	0.00316183	0.00107195	0.00087659
	0.02 (weakly)	0.00605927	0.00374627	0.00132135	0.00123865
$\boldsymbol{W}(\boldsymbol{t})$	0.08 (monthly)	0.00648671	0.00563785	0.00150720	0.00102105
	0.02 (weekly)	0.00292979	0.00239990	0.00067501	0.00056119
$\boldsymbol{E}(\boldsymbol{t})$	0.08 (monthly)	0.00231381	0.00125269	0.00071632	0.00005706
	0.02 (weekly)	0.00238192	0.00215891	0.00197363	0.00118901
$\boldsymbol{L}(\boldsymbol{t})$	0.08 (monthly)	0.00082813	0.00028939	0.00167011	0.00057345
	0.02 (weekly)	0.00103849	0.00073530	0.00144320	0.00026649
$\boldsymbol{Z}(\boldsymbol{t})$	0.08 (monthly)	0.00236504	0.00031949	0.00017527	0.00013961
	0.02 (weekly)	0.00321737	0.00188311	0.00071175	0.00012518

Table 7 discusses the convergence of the new algorithm using the residual error between every two consecutive terms for a certain number of terms of the proposed algorithm, where we notice that the error decreases as we take a larger number of iterations, and this confirms us the convergence of the proposed method.

Table 7. Absolute residual error $|ASM_{i+1} - ASM_i|$, $i = 1,2, \dots, 300$ number of iteration for a new approach ASM_MLHRK through two years.

Model Variables	$\boldsymbol{C(t)}$	$\boldsymbol{W(t)}$	$\boldsymbol{E(t)}$	$\boldsymbol{L(t)}$	$\boldsymbol{Z(t)}$
Step Size, h	0.08 (monthly)	0.08 (monthly)	0.08 (monthly)	0.08 (monthly)	0.08 (monthly)
$ASM_{51} - ASM_{50}$	0.52369031	0.60578526	0.19603743	0.72967378	0.66135922
$ASM_{53} - ASM_{52}$	0.45525823	0.55720193	0.15037214	0.56213405	0.60132548
$ASM_{54} - ASM_{53}$	0.40427536	0.50126341	0.11023465	0.49320176	0.54213087
$ASM_{55} - ASM_{54}$	0.34214152	0.46172540	0.09365271	0.44120832	0.49213560
$ASM_{56} - ASM_{55}$	0.32057767	0.39170362	0.05973103	0.40253703	0.44021346
$ASM_{57} - ASM_{56}$	0.25987686	0.34122304	0.03728907	0.37132409	0.39256381
$ASM_{58} - ASM_{57}$	0.19675235	0.29871023	0.02198266	0.32073651	0.26410872
$ASM_{59} - ASM_{58}$	0.15517794	0.25861024	0.01862936	0.28430712	0.22064315
$ASM_{61} - ASM_{60}$	0.10808257	0.19330451	0.01460987	0.21987324	0.19340567

Figure 2 represents the curves of methods that are used to solve the mathematical model of COVID-19 through two years with 100 repetitions and $h = \{0.08, 0.02\}$ step size weekly and monthly through two years. Figure 2(a) is related to the people $C(t)$ who are not infected with COVID-19 but are susceptible to infection. A sudden drop in the curves for all the methods used in the study after the 17th month. It is noticeable that the sudden descent in the curve as a result of a large number of infections during the study period to still down after the end of 2022. Figure 2(b) observes the curves of group $w(t)$ with people who are vaccinated against COVID-19. It is noticeable that the curves rise gradually until the 15th month, after which they increase with greater upwards until the end of the study period. Figure 2(c) is associated with the group of people $E(t)$ who are carriers of the virus without showing symptoms of infection and Figure 2(d) represents the infected people $L(t)$ with the epidemic. It is noticeable that a gradual and slight increase in the curve until the 15th month for each $E(t)$ and $L(t)$, after which they increase with greater upwards until the 20th month, then a slight decline at the end of the study period as a result of the impact and effectiveness of the vaccine on society. Figure 2(e) the group of people $Z(t)$ who have been cured of the disease, as they have been removed from the list of positive cases. There is a gradual rise of the curves especially from the 15th month to the 20th month, then it continues to rise until the end of the study period.

Table 8. Prediction intervals (5th percentile, 95th percentile) for ASM_MLHRK, ASM_MMCRK, MMC_RK and MLH_RK solutions

MMCRK from 2021 to 2025 ($t \leq 48$)		
Subpopulation	**(100 repetitions)**	**(1000 repetitions)**
$C(t)$	(1472991.20458186 , 26159524.9835266)	(1510599.40241837 , 25916625.6778357)
$W(t)$	(4357631.7258009 , 18607769.0236236)	(4549965.74214325, 18886164.251091)
$E(t)$	(2526701.92714055 , 13325113.3421321)	(2912097.91400961 , 13082969.7854761)
$L(t)$	(353279.936509981 , 3365956.15432413)	(400525.323142184 , 3567618.95559503)
$Z(t)$	(2630616.06801583 , 43519655.9740401)	(3111352.69199348 , 42216391.3520102)
MLHRK from 2021 to 2025 ($t \leq 48$)		
Subpopulation	**(100 repetitions)**	**(1000 repetitions)**
$C(t)$	(1523965.12557087 , 25643584.7303688)	(1634788.2103348 , 26133045.0923457)
$W(t)$	(5813361.83870447 , 19109267.030847)	(6199376.94435020 , 20123578.321549)
$E(t)$	(2571877.70593455 , 10905917.4496805)	(2936843.03372511 , 11276310.2945021)
$L(t)$	(319748.001520959 , 3088896.04854105)	(455056.32607721 , 3250647.24053192)
$Z(t)$	(2702074.70798966 , 35981783.6342994)	(2930625.85032167, 36139466.05487231)
ASM_MMCRK from 2021 to 2025 ($t \leq 48$)		
Subpopulation	**(100 repetitions)**	**(1000 repetitions)**
$C(t)$	(7772861.29848899 , 8323947.83072361)	(79322610.40506681 , 8745311.02304821)
$W(t)$	(10813263.7892022 , 11089673.1151684)	(11099750.86093174 , 11505381.5103409)
$E(t)$	(8233745.20583935 , 11010506.5648508)	(8534066.18043477 , 11449213.2038117)
$L(t)$	(2370410.01055453, 2517652.03681914)	(2748023.48708571 , 2844610.103047832)
$Z(t)$	(23973749.9536893 , 24868765.0722051)	(24034629.2305118 , 25428155.3295012)
ASM_MLHRK from 2021 to 2025 ($t \leq 48$)		
Subpopulation	**(100 repetitions)**	**(1000 repetitions)**
$C(t)$	(7943100.05869372, 49788658.7942648)	(8033499.03984756 , 50233846.0845782)
$W(t)$	(56679.5422693364 , 11279228.911598)	(78438.40982011 , 12489312.83045317)
$E(t)$	(35072.8601023087 , 6781499.16035943)	(63124.520.3103267 , 7216083.2836046)
$L(t)$	(10120.6313225389, 2994205.63318524)	(25740.318703712 , 3125644.601328490)
$Z(t)$	(104905.624799627, 24866319.3351259)	(235657.37820936 , 25036217.21056321)

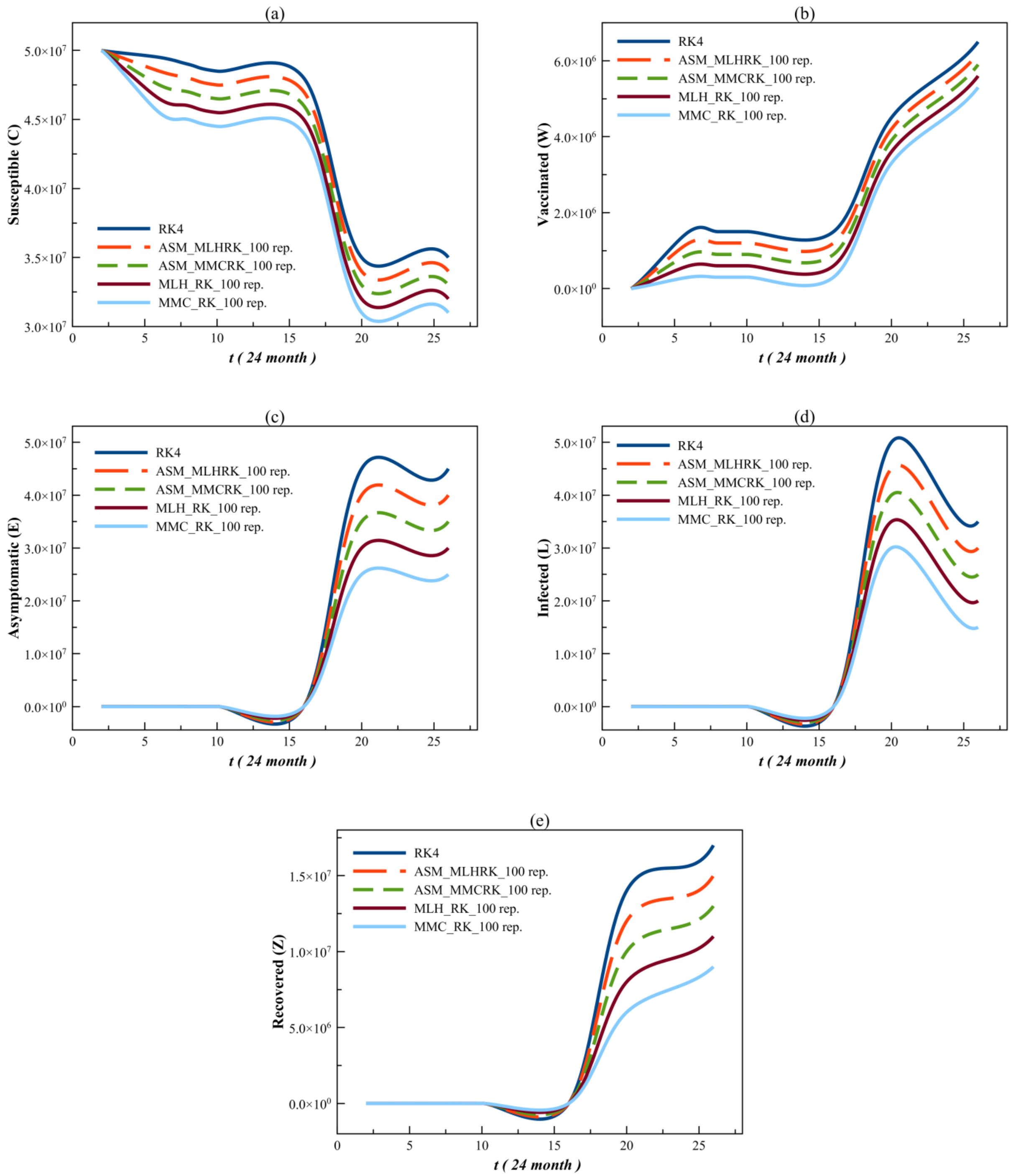
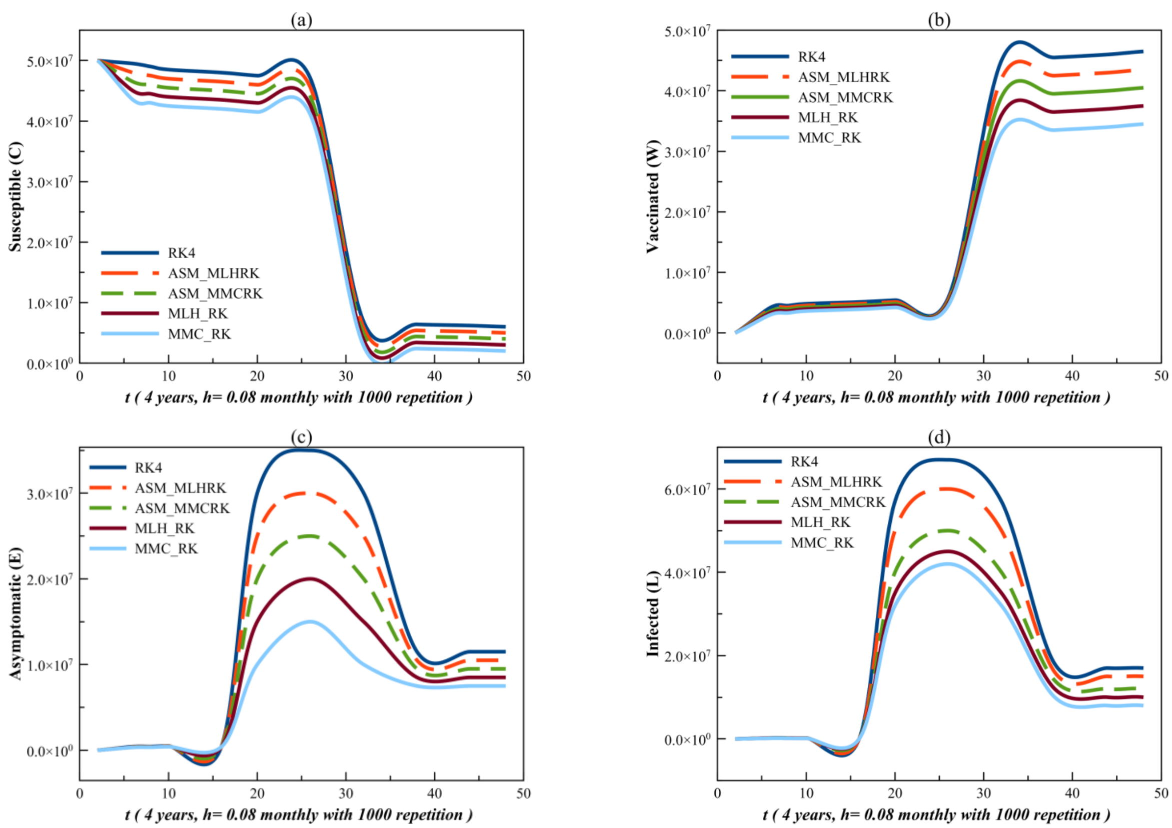


Figure 2. Curves of approximate simulation methods ASM_MLHRK, ASM_MMCRK, MMC_RK and MLH_RK compared with RK4 of (a) $C(t)$, (b) $W(t)$, (c) $E(t)$, (d) $L(t)$ and (e) $Z(t)$ through two years when $h = 0.08$ monthly with $m = 100$ repetitions.

Figure 3 describes the curve of the mathematical model of the COVID-19 epidemic through 4 years from 2021 to 2025 with 100 repetitions. Figure 3 explains the convergence between the approximate simulation methods ASM_MLHRK, ASM_MMCRK, MMC_RK and MLH_RK when $h = 0.08$ monthly with $m = 1000$ repetitions.

Figure 3 (a) represented the group of people who are not infected with COVID-19 $C(t)$. Note that the gradual descent in the curve of this group of people for all methods that are used in the first months of the study with step size $h = \{0.02, 0.08\}$ weekly and monthly through four years, and then it begins with a sudden decline due to a large number of infectious as a result of mixing and non-compliance with health prevention methods, after which it returns to stability in the last months of the study as a result of people's desire to receive the vaccine against COVID-19, also see how close the curves of proposed methods ASM_MLHRK and ASM_MMCRK with the curve of numerical method RK4 than the other numerical simulation methods. Figure 3 (b), this curve describes the group of people vaccinated against COVID-19 $W(t)$, and as we see there is a slight rise in the curve of this class in the first months of the study period for all methods ASM_MLHRK, ASM_MMCRK, RK4, MMC_RK and MLH_RK are used with $h = \{0.02, 0.08\}$ weekly and monthly through four years, then the curve begins to rise significantly in the middle of the study period to continue rising until the 40th month as a result of the increase in the number of vaccinated against this epidemic and the high level of awareness the health of the people will take after that, stability until the year 2025. We also notice very clearly how close the curves of new approaches ASM_MLHRK and ASM_MMCRK of the curve of $RK4$ are to the other approximate simulation methods. Figure 3 (c), this curve of this group $E(t)$, which represents people infected with the epidemic without showing symptoms of infection. Noticeable there is a rise in the highest level in the middle of the study period, specifically the 20th month, due to mixing and lack of commitment to health prevention methods, then the curve drops until the 40th month, then stabilizes until the end of the study period, also we notice the curve of proposed method ASM_MLHRK converge to the curve of numerical method RK4 than the other methods. Figure 3 (d), represents people infected with the epidemic without showing symptoms of infection $L(t)$, also note that there is an increase in the curve of this class for all methods ASM_MLHRK, ASM_MMCRK, RK, MMC_RK and MLH_RK are used under study, and to reach its highest level in the middle of the study period, specifically the 22^{th} month, due to mixing and lack of commitment to health prevention methods, then the curve drops until the 41^{th} month, then stabilizes until the end of the study period, also the curve of proposed method ASM_MLHRK and ASM_MMCRK converge to the curve of numerical method $RK4$ than the other methods. Figure 3 (e) the curves of this group of people who have been cured or died $Z(t)$ as a result of the epidemic, notice that there is a discrepancy in the level of rise and fall in the curve of this class and for all methods are used under study with $h = \{0.02, 0.08\}$ through four over a period of 48 months, where we notice the rise until the 15^{th} month and then returns to decline in 25^{th} month, after which it rises very much to settle at its highest rate in the last months of the study. Whereas, the curve of the proposed algorithm remains the closest to the curve of numerical method $RK4$ than the other numerical simulation methods.



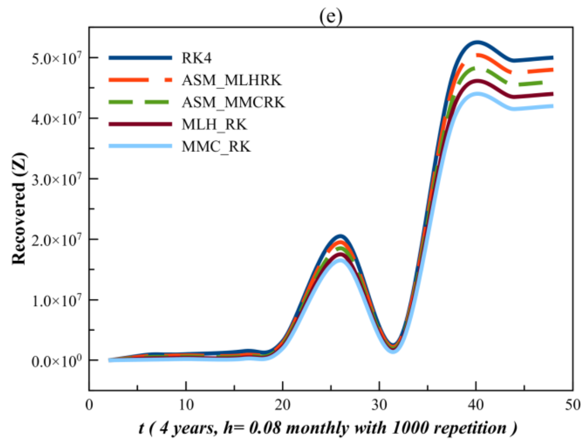


Figure 3. Curves of approximate simulation methods ASM_MLHRK, ASM_MMCRK, MMC_RK and MLH_RK compared with RK4 of (a) $C(t)$, (b) $W(t)$, (c) $E(t)$, (d) $L(t)$ and (e) $Z(t)$ from 2021 to 2025 when $h = 0.08$ monthly with $m = 1000$ repetitions.

7. CONCLUSION

The numerical simulation process is considered a more reliable method than the classical methods that depend on fixed period of time. Because the natural epidemic models have randomness in their coefficients. For this reason, these numerical simulation techniques is considered a more suitable method than the classical methods like $RK4$ that solve models depending on fixed parameters.

The mathematical model in our research is represented by the COVID-19 epidemic. It is formulated as a system of first-order nonlinear ordinary differential equations. The study period during 48 months from 2021 to 2025. Many methods are used for solving the model, including a numerical method, which is the Runge-Kutta method as a standard solution, and the other two modified numerical simulation methods MMC_RK and MLH_RK to solve this system. All the previous methods are formulated to create a new approach that is used for the first time, called the approximate shrunken approach represented in (ASM_MLHRK) and (ASM_MMCRK) methods.

In our study, the shrinkage estimation solution represents a good estimator for the solution of the system under study. This solution is considered a link between the traditional approach of solving systems represents in numerical methods that depend on fixed coefficients for the model and the concepts of the modified simulations approach for solving these systems when dependent on random coefficients.

The results for all proposed methods; ASM_MLHRK and ASM_MMCRK are more convergence to and close to the $RK4$ numerical result that represents a criterion solution in the current model than the other methods mentioned. Where note that the approximate shrunken method (ASM_MLHRK) is the most closely.

Studying the epidemic model under study gives an impression of the impact of this virus on society. The results show that the category $C(t)$ of people not infected with the epidemic began to decrease during the study period. While the category $W(t)$ is associated with vaccinated people, we notice an increase in this category as a result of the impact and effectiveness of the vaccine on society, also for the category $E(t)$ of infected people without showing symptoms, there is a gradual rise in this category of people for not adhering to health prevention methods, as well as not adhering to social distancing. However category $L(t)$ of infected people and the symptoms are clear to them, we notice a gradual decrease in all methods $RK4$, ASM_MLHRK, ASM_MMCRK, MMC_RK, and MLH_RK by educating people to take the vaccine against the virus. Finally, the category $Z(t)$ of people who have been cured or died due to disease, a clear increase for this class.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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