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WHAT HAS GONE AROUND MAY COME BACK AROUND: REINFECTION IN THE EXTENDED STOCHASTIC MULTI-REGION CONTROL OF INFODEMICS AND SEASONAL CORONAVIRUSES

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Abstract. In this paper, we study an optimal control approach against seasonal coronaviruses by adding terms of reinfection to dynamics of the optimization constraint and which is mainly defined by a stochastic multi-region SIRS control differential system. In fact, in addition to the problem of infodemics spread, we take into account that protective immunity against such viruses is short-lasting as there is a risk of reinfection, while the immunization process control against the epidemic could be realized through any available actions either by following those who suggest long-term awareness in response of any surprising COVID-19-like in future or those who recommend some potentially effective medical intervention such as vaccines or antiviral drugs, while other strategists, especially in times of global epidemic emergencies, could not see any alternative approach to the closure policies in order to limit the movement of infected people. In front of all these different possibilities to intervene, we let our control functions open to define any of such considerations and we analyze some of their advantages on preventing the

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viruses spread through different numerical scenarios associated to a boolean variable whose values directly define the cases of uncontrolled and controlled regions but being interconnected by the factor of mobility.

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1. INTRODUCTION

1.1. The need to modeling in the context of seasonal coronaviruses. HCoV-229E, HCoV-NL63, HCoV-OC43, HCoV-HKU1, SARS-CoV, MERS-CoV, and SARS-CoV-2, are the most known human coronaviruses, and that attack the respiratory system and cause mild diseases in cases of the first four viruses, and severe diseases like COVID-19 and acute respiratory distress syndrome (ARDS) in cases of the last three ones. Protective immunity against coronaviruses is short-lasting as there is a risk of reinfection. In fact, the authors showed in [1] that reinfection occured for four seasonal coronavirues, namely for HCoV-229E, HCoV-NL63, HCoV-OC43 and HCoV-HKU1, most frequently at 12 months after infection. In the near future, Rezaei et al. in [2] are seeing that the incidence of this virus will depend more on the duration of immunity and that we have to accept that even vaccinated people up to the present, are susceptible to reinfection since there is still no proper vaccine that is able to guarantee a complete protection against this infection. Thus, if one would be interested to devise modeling framework to describe the infection dynamics caused by such viruses, it would be more preferably to rely on the logical form of an SIRS-type system as considered in [3, 4, 5, 6, 7, 8].

1.2. Role of media against infodemics. Since 2019, some countries started to launch awareness campaigns as multi-pronged effort to combat the spread of misinformation online, as well as enabling authorities concerned to block websites and accounts of influencers who spread fake news [9]. In actions against false information, awareness campaigns that use media coverage are very important and can reduce the contact of susceptible populations with infection. In fact, in such anti-epidemic prevention measures, the media helps to generate a psychological impact on the social conduct as explained in [10, 11, 12]. As a consequence, many researchers who contributed in the subject of epidemics modeling, introduced and discussed the effect of media in their models and showed how it can prevent the spread of diseases as in [13, 14, 15, 16, 17, 18]. More than this, in [19], the authors presented many forms of pandemic control models, but in

the end, they even concluded that any control policy would fail if there is not enough focus on the health educational system. As explained in [20], we note that infodemics are recently spreading very quickly in the virtual world, and the role of media becomes more essential than ever, for reporting and exhibiting truths about epidemics and importance of control interventions for the benefit of the population, in a convincing way so people can follow any necessary instruction and would not be influenced by rumors [21].

1.3. Contribution to existing control systems. Control methods have been applied to many models of various diseases, either relying on systems of differential equations [22, 23, 24, 25, 26, 27, 28, 29, 30, 31] or systems of difference equations [32, 33, 34, 35] and that have all been devised for the study of the disease dynamics under preventive or treatment policies. Recent meta-population epidemic deterministic models as in [36, 37, 38, 39, 40, 41, 42, 43] studied the regional spread of infection based on the framework of differential and difference equations. These last references focused on the importance of people mobility factor and discussed the effectiveness of closure policies between regions in the fight against of an infection that has the ability to spread in a large geographical territory. Here, we study the stochastic version proposed in [20] for those epidemic modeling approaches based on a susceptible-infected-removed-susceptible"again" (SIRS) multi-regions stochastic model in the presence of perturbed control with respect to immunization due to infodemics that can be diffused by some internet users, and we also consider the control closure strategies to be perturbed due to some escapes of infected travelers or because of the effects of some periods of reopening strategies. Many researchers have been interested in the study of spatial spread of infection under different perturbations using stochastic compartmental epidemic models as in [44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54]. We also also assume that contacts between susceptible and infected populations are unpredictable, but compared to [20], the main contribution here, is that optimal control problems subject to the devised SIRS stochastic system, depends on a boolean variable which defines the degree of importance given to a control strategy in a region. In this paper, we start from the idea of the optimal control approach studied in [37] in the case of a deterministic multi-regions discrete SIR model, then we redesign it using a stochastic framework. The boolean variable or index of importance is named ε_j, $j = 1, ..., p$ in the objective criterion to be optimized, and it is

defined as follows: if $\varepsilon_j = 1$, this means that a region Ω_j is controlled, and if $\varepsilon_j = 0$, this means that the region Ω_j is uncontrolled. Then, we state theorems that include solutions existence and necessary conditions of optimality in this stochastic case. Finally, we provide our numerical results with a discussion of the two values of ε_j.

2. THE MODEL WITH IMMUNIZATION

2.1. The stochastic model with no immunization. Let assume there are p geographical regions denoted Ω_j parts of the domain $\Omega = \bigcup_{j=1}^{p} \Omega_j$.

We define $N^{\Omega_j}(t)$ as the population size in domain Ω_j at time t. This represents the number of individuals who are physically present in Ω_j, both residents and travelers. Let the host population of Ω_j be grouped into three epidemiological compartments, let $S^{\Omega_j}(t)$, $I^{\Omega_j}(t)$ and $R^{\Omega_j}(t)$ be the number of individuals in the susceptible, infective, and removed compartments of Ω_j at time t, respectively. As defined in [20], the stochastic disease transmission in a given domain Ω_j at time t is modeled using a perturbed standard incidence which we present by

$$\sum_{k=1}^{p} \rho^{jk}(t) \frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)} S^{\Omega_j}(t)$$

where the stochastic disease transmission coefficient $\rho^{jk}(t)$ is the stochastic proportion of adequate contacts in domain Ω_j between a susceptible from Ω_j $(j = 1, ..., p)$ and an infective from another domain Ω_k at a time t, and which we define by

$$\rho^{jk}(t) = \beta_{jk} + \sigma_j \frac{dW^{\Omega_j}(t)}{dt}$$

where $\beta_{jk} > 0$ is the equivalent disease transmission coefficient to ρ^{jk} but in the deterministic case, which had been defined also as the proportion of adequate contacts in the study cases of [12, 18], σ_j $(j = 1, ..., p)$, are real constants and representing the intensities of fluctuations caused by media, and $\{W^{\Omega_j}(t)\}_{t\in[0,T]}$ is an independent random variable composed with continuous white noises independent to $\mathscr{F}_t \in \mathscr{F}$, and which is a standard Brownian motion supposed to be caused by to media coverage in region Ω_j. Let define a boolean variable ε_j ($\varepsilon_j = 1$ or $\varepsilon_j = 0$) associated to domain Ω_j, that will be called the importance index of Ω_j. ε_j is either

equaling to 1, in the case when it is important to control the region Ω_j, or $\varepsilon_j = 0$ otherwise. Let $I = \{1, ..., p\}$, and denote by $I_C \subset I$, the set of indexes of regions which are important to control (by immunization through vaccination or awareness), i.e. $I_C = \left\{ j \in I / \varepsilon_j = 1 \right\}$. The stochastic multi-regional discrete-time SIRS model associated to Ω_j with $\varepsilon_j = 0$ (no control is introduced yet in Ω_j) is then presented as follows

$$\begin{aligned} \dot{S}^{\Omega_j}(t) &= -\sum_{k=1}^{p} \rho^{jk}(t) \frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)} S^{\Omega_j}(t) + \left(N^{\Omega_j}(t) - S^{\Omega_j}(t)\right) d_j + e_j R^{\Omega_j}(t) \\ \dot{I}^{\Omega_j}(t) &= \sum_{k=1}^{p} \rho^{jk}(t) \frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)} S^{\Omega_j}(t) - \gamma_j I^{\Omega_j}(t) - d_j I^{\Omega_j}(t) \end{aligned} \tag{1}$$

$$\dot{R}^{\Omega_j}(t) = \gamma_j I^{\Omega_j}(t) - (d_j + e_j) R^{\Omega_j}(t) \tag{2}$$

where d_j is the birth and death rate, γ_j is the recovery rate and e_j is the losing removal individuals immunity rate. The biological background requires that all parameters be non-negative.

$N^{\Omega_j}(t) = S^{\Omega_j}(t) + I^{\Omega_j}(t) + R^{\Omega_j}(t)$ is the population size corresponding to domain Ω_j at time t. The population size remains constant for all $t \in [0, T]$, in fact

$$\dot{N}^{\Omega_j}(t) = \dot{S}^{\Omega_j}(t) + \dot{I}^{\Omega_j}(t) + \dot{R}^{\Omega_j}(t) = 0$$

Therefore, in function of the deterministic proportion of adequate contacts β_{jk} and the continuous Wiener process $W^{\Omega_j}(t)$, the stochastic system (1)-(2) becomes the Itô discrete multi-regions stochastic differential equations (SDEs) model

$$\begin{aligned} \dot{S}^{\Omega_j}(t) &= -\sum_{k=1}^{p} \beta_{jk} \frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)} S^{\Omega_j}(t) + \left(N^{\Omega_j}(t) - S^{\Omega_j}(t)\right) d_j + e_j R^{\Omega_j}(t) \\ &\quad -\sigma_j \sum_{k=1}^{p} \frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)} S^{\Omega_j}(t) \frac{dW^{\Omega_j}(t)}{dt} \end{aligned} \tag{3}$$

$$\begin{aligned} \dot{I}^{\Omega_j}(t) &= \sum_{k=1}^{p} \beta_{jk} \frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)} S^{\Omega_j}(t) - \gamma_j I^{\Omega_j}(t) - d_j I^{\Omega_j}(t) \\ &\quad +\sigma_j \sum_{k=1}^{p} \frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)} S^{\Omega_j}(t) \frac{dW^{\Omega_j}(t)}{dt} \end{aligned} \tag{4}$$

$$\dot{R}^{\Omega_j}(t) = \gamma_j I^{\Omega_j}(t) - (d_j + e_j) R^{\Omega_j}(t) \tag{5}$$

2.2. Presentation of the control model. Let introduce a stochastic control variable $\theta^{\Omega_j}(t)$ which characterizes the effectiveness of immunization in the above mentioned model (3-5) when the control variable function denoted by $u^{\Omega_j}(t)$, is perturbed with disturbances caused by infodemics and rumors, and also ideas from feelings of fear and misconception. Then for

a given domain Ω_j with $\varepsilon_j = 1$ (Ω_j is targeted by immunization), the model is given by the following equations

$$\begin{aligned} \dot{S}^{\Omega_j}(t) &= - \sum_{k=1}^{p} \rho^{jk}(t)\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) + \left(N^{\Omega_j}(t) - S^{\Omega_j}(t)\right)d_j \\ &\quad -\theta^{\Omega_j}(t)S^{\Omega_j}(t) + e_j R^{\Omega_j}(t) \end{aligned} \tag{6}$$

$$\begin{aligned} \dot{I}^{\Omega_j}(t) &= \sum_{k=1}^{p} \rho^{jk}(t)\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) - \gamma_j I^{\Omega_j}(t) - d_j I^{\Omega_j}(t) \\ \dot{R}^{\Omega_j}(t) &= \gamma_j I^{\Omega_j}(t) - (d_j + e_j)R^{\Omega_j}(t) + \theta^{\Omega_j}(t)S^{\Omega_j}(t) \end{aligned} \tag{7}$$

with

$$\theta^{\Omega_j}(t) = u^{\Omega_j}(t) + \delta_j \frac{dW^{\Omega_j}(t)}{dt}.$$

Thus, in a more general form, it refers to the stochastic control difference equation written at a time t as

$$\dot{x}^{\Omega}(t) = f(t, x^{\Omega}(t), u^{\Omega}(t)) + g(t, x^{\Omega}(t), u^{\Omega}(t))\frac{dW^{\Omega}(t)}{dt}$$

where at time t and for $j = 1, ..., p$

$$x^{\Omega}(t) = x^{\Omega_j}(t) = \begin{pmatrix} S^{\Omega_j}(t) \\ I^{\Omega_j}(t) \\ R^{\Omega_j}(t) \end{pmatrix}$$

$u^{\Omega}(t) = u^{\Omega_j}(t)$,

$f(t, x^{\Omega}(t), u^{\Omega}(t))$

$$= \begin{pmatrix} -\sum_{k=1}^{p} \beta_{jk}\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) + \left(N^{\Omega_j}(t) - S^{\Omega_j}(t)\right)d_j - u^{\Omega_j}(t)S^{\Omega_j}(t) + e_j R^{\Omega_j}(t) \\ \sum_{k=1}^{p} \beta_{jk}\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) - \gamma_j I^{\Omega_j}(t) - d_j I^{\Omega_j}(t) \\ \gamma_j I^{\Omega_j}(t) - (d_j + e_j)R^{\Omega_j}(t) + u^{\Omega_j}(t)S^{\Omega_j}(t) \end{pmatrix}$$

and

$$g(t, x^{\Omega}(t), u^{\Omega}(t)) = \begin{pmatrix} -\left(\sigma_j \sum_{k=1}^{p} \frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) + \delta_j S^{\Omega_j}(t)\right) \\ \sigma_j \sum_{k=1}^{p} \frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) \\ \delta_j S^{\Omega_j}(t) \end{pmatrix}$$

Our goal is to minimize the population of the infected group and the cost of immunization in all regions which are important to control. Our control functions taking values between $u_{min}^{\Omega_j}$ and $u_{max}^{\Omega_j}$, where $u_{min}^{\Omega_k}, u_{max}^{\Omega_k} \in]0,1[$, $\forall k = 1, ..., p$.

2.3. A stochastic optimal control approach.

2.3.1. *Optimal control characterization and necessary conditions.* Aiming to minimize the number of the infected people and maximize the ones in the removed category for a given region which is important to control, we try to find here, an optimal control for each region including in our optimization criterion, different indexes of importance ε_j, $j = 1, ..., p$. If $\varepsilon_j = 1$ it means that Ω_j is controlled, and if $\varepsilon_j = 0$ it means that Ω_j is uncontrolled. Then, we are interested by minimizing the functional

$$J(u) = \sum_{k=1}^{p} \varepsilon_k J_k(u^{\Omega_k}) \tag{8}$$

where $J_k(u^{\Omega_k})$ is given by $J_j(u^{\Omega_j}) = \mathbb{E}\left(\int_0^T f_0(t, x^{\Omega_j}(t), u^{\Omega_j}(t))dt\right)$
with

$$f_0(t, x^{\Omega_j}(t), u^{\Omega_j}(t)) = \left(\alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) + \frac{A_j}{2}(u^{\Omega_j}(t))^2\right)$$

where $A_j > 0$, $\alpha_j^I > 0$, $\alpha_j^R > 0$ are the weight constants of control, the infected and the removed in region Ω_j respectively.

In [37, 43], the authors have studied the special case of the minimization problem of the cost functional J when there was no stochasticity. Here, our goal is to minimize the number of infected people, minimize the systemic costs attempting to increase the number of removed people in each Ω_j while being under the stochastic perturbations explained above. In other words, we are seeking an optimal control $u^{\Omega_j *}$ such that

$$J(u^{\Omega_j *}) = \min\{J(u^{\Omega_j}) / u^{\Omega_j} \in U\}$$

where U_j is the control set defined by

$$U_j([0,T]) = \{u^{\Omega_j}(t) \ \mathscr{F}_t \textit{ progressively measurable} | u_{min}^{\Omega_j} \leqslant u^{\Omega_j}(t) \leqslant u_{max}^{\Omega_j}\}$$

for all $j \in I_C, t \in [0,T]$

Let define the Hamiltonian function H by

$$H(x^{\Omega},u^{\Omega},\mu^{\Omega},\nu^{\Omega}) = \sum_{k=1}^{p} \varepsilon_k \left(f_0(x^{\Omega},u^{\Omega}) + \langle f(x^{\Omega},u^{\Omega}),\mu^{\Omega}\rangle + tr\left[\nu^{\Omega^T} g(x^{\Omega},u^{\Omega})\right]\right)$$

At time $t \in [0,T]$ and for $j = 1,...,p$, it can be rewritten as

$$H(x^{\Omega_j}(t),u^{\Omega_j}(t),\mu^{\Omega_j}(t),\nu^{\Omega_j}(t)) = \sum_{k=1}^{p} \varepsilon_k \Big(f_0(x^{\Omega_j}(t),u^{\Omega_j}(t)) + \langle f(x^{\Omega_j}(t),u^{\Omega_j}(t)),\mu^{\Omega_j}(t)\rangle + \sum_{l=1}^{3} g^{l^T}(x^{\Omega_j}(t),u^{\Omega_j}(t))\nu^{\Omega^l_j}(t) \Big)$$

Here $.^T$ means the transposition, while in a domain Ω, $(\mu(t),\nu(t))$ is a pair of adjoint variables satisfying the following adjoint BSDE (Backward stochastic differential equation)

$$(9) \quad \begin{cases} d\mu^{\Omega}(t) &= -[f_x^T(t,x^{\Omega}(t),u^{\Omega}(t))\mu^{\Omega}(t) + \sum_{l=1}^{3} g^{l^T}_{x^{\Omega}}(t,x^{\Omega}(t),u^{\Omega}(t))\nu^{\Omega^l}(t) \\ & \quad + f_{0_{x^{\Omega}}}(t,x^{\Omega}(t),u^{\Omega}(t))]dt + \nu^{\Omega}(t)dW^{\Omega}(t), \\ \mu^{\Omega}(T) &= 0. \end{cases}$$

Using a stochastic version of Pontryagin's maximum principle [55], we characterize the optimal control u in the following theorem to find its analytical formulation.

Theorem 2.3.1. (Stochastic maximum principle and characterization of $u^{\Omega *}$)

If there exists an optimal pair $(x^{\Omega *},u^{\Omega *})$ and a pair of processes $(\mu(t),\nu(t))$ satisfying (9), then for $j = 1,...,p$, we have

$$H(t,x^{\Omega_j *}(t),u^{\Omega_j *}(t),\mu^{\Omega_j}(t),\nu^{\Omega_j}(t)) = \min_{u^{\Omega_j} \in U} H(t,x^{\Omega_j}(t),u^{\Omega_j *}(t),\mu^{\Omega_j}(t),\nu^{\Omega_j}(t)).$$

Moreover, we obtain the bounded stochastic control

$$u^{\Omega_j *} = min(max(u^{\Omega_j}_{min}, -\frac{(\mu_3^{\Omega_j}(t) - \mu_1^{\Omega_j}(t))S^{\Omega_j *}}{\varepsilon_j A_j}), u^{\Omega_j}_{max}),$$

solution of the FBSDEs (Forward-backward stochastic differential equations)

$$(10) \quad \begin{cases} dx^{\Omega_j}(t) &= f(t,x^{\Omega_j}(t),u^{\Omega_j}(t))dt + g(t,x^{\Omega_j}(t),u^{\Omega_j}(t))dW^{\Omega_j}(t) \\ d\mu^{\Omega_j}(t) &= -[f^T_{x^{\Omega_j}}(t,x^{\Omega_j}(t),u^{\Omega_j}(t))\mu^{\Omega_j}(t) + \sum_{l=1}^{3} g^{l^T}_{x^{\Omega_j}}(t,x^{\Omega_j}(t),u^{\Omega_j}(t))\nu^{\Omega^l_j}(t) \\ & \quad + f_{0_{x^{\Omega_j}}}(t,x^{\Omega_j}(t),u^{\Omega_j}(t))]dt + \nu^{\Omega_j}(t)dW^{\Omega_j}(t), \\ x^{\Omega_j}(0) &= (S_0^{\Omega_j},I_0^{\Omega_j},R_0^{\Omega_j}) \\ \mu^{\Omega_j}(T) &= 0. \end{cases}$$

Similar reasoning as in [20], can be followed for obtaining the result. Proofs for Solutions existence and sufficient conditions of optimality, can be found in the same mentioned reference.

2.3.2. *Numerical Results.*

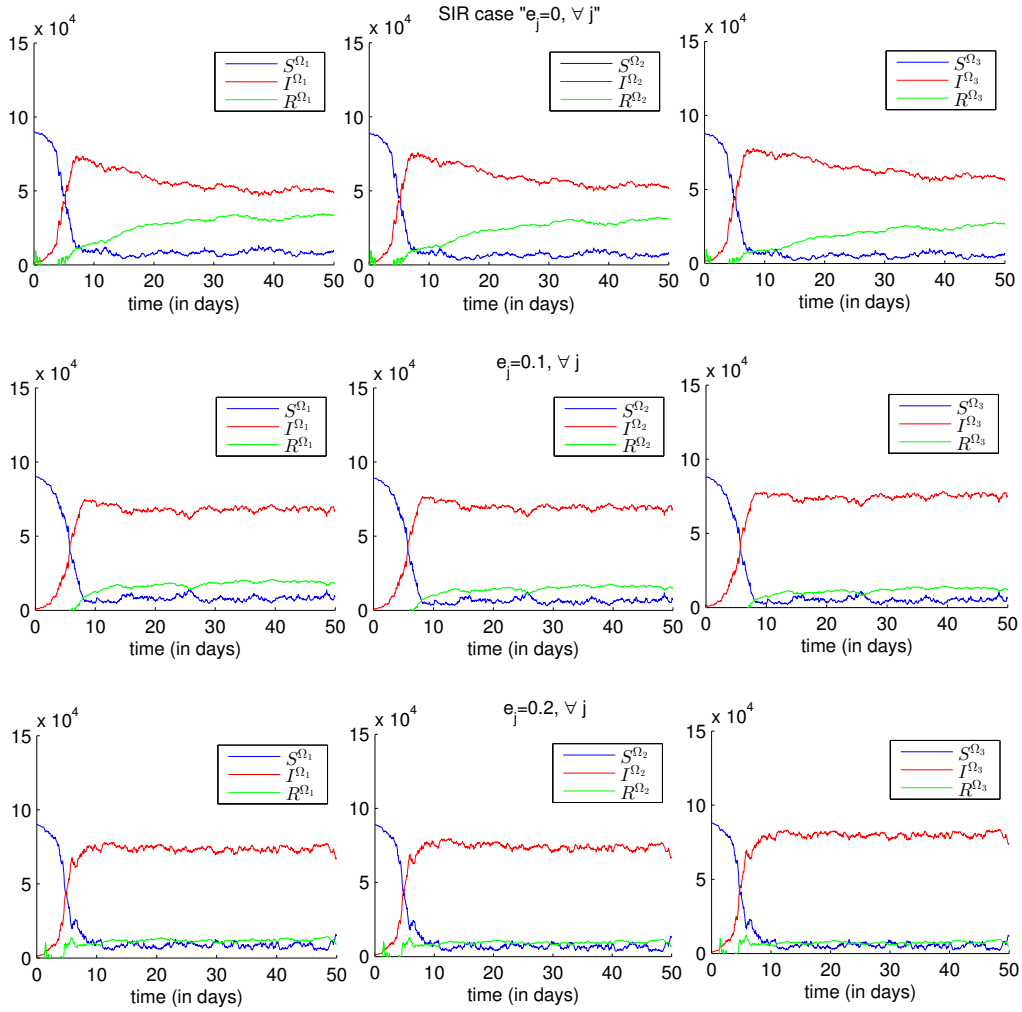

FIGURE 1. No control, i.e. $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$. $S^{\Omega_j} I^{\Omega_j} R^{\Omega_j}$, $j = 1,2,3$ stochastic dynamics without controls in cases of $e_j = 0,\ 0.1,\ 0.2$ in (a), (b) and (c) respectively. $S_0^{\Omega_1} = 90000$, $I_0^{\Omega_1} = 1200$, $S_0^{\Omega_2} = 89000$, $I_2^{\Omega_2} = 1100$, $S_0^{\Omega_3} = 88000$, $I_0^{\Omega_3} = 1000$, $R_0^{\Omega_j} = 0\ \forall\ j = 1,2,3$, $d_1 = 0.06$, $\gamma_1 = 0.04$, $\beta_1 = 0.5$, $d_2 = 0.05$, $\gamma_2 = 0.03$, $\beta_2 = 0.4$, $d_3 = 0.04$, $\gamma_3 = 0.02$, $\beta_3 = 0.1$

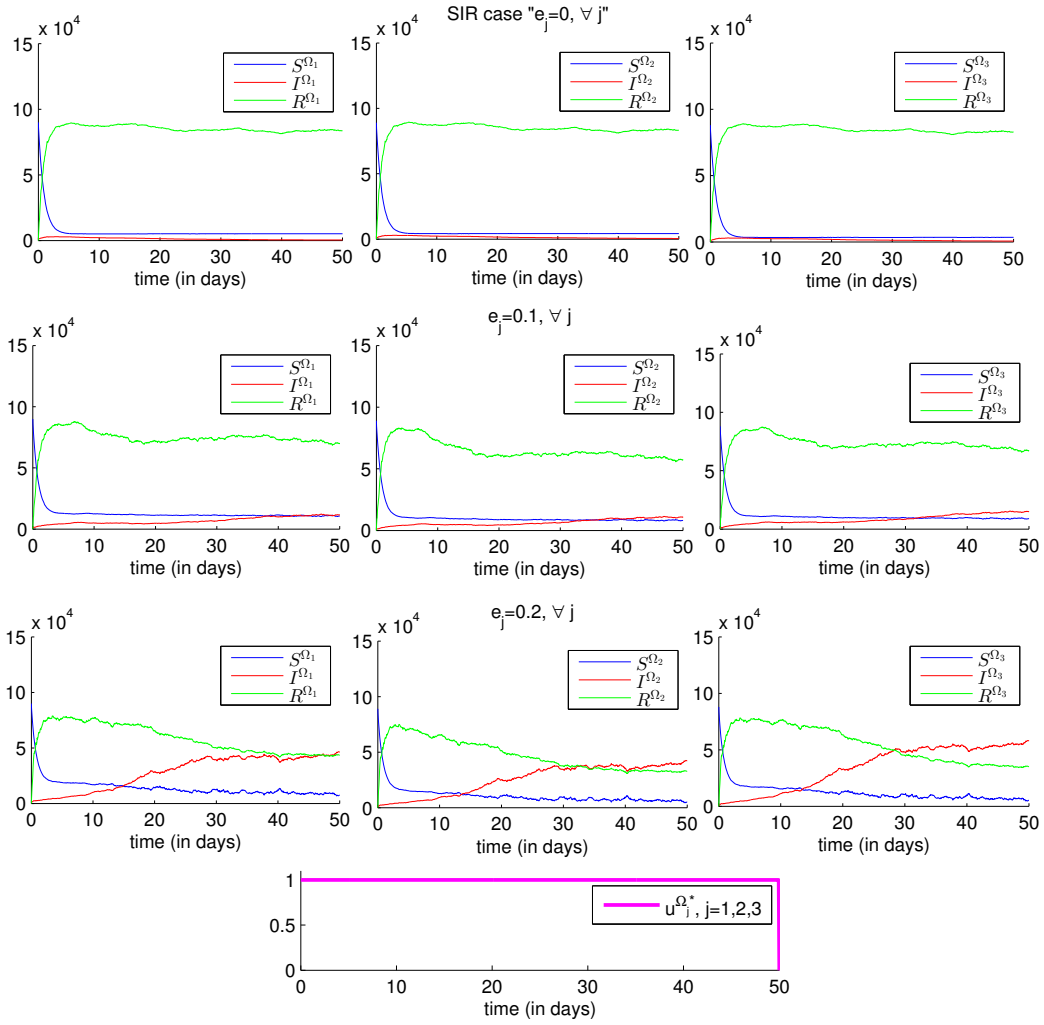

FIGURE 2. Control in all regions, i.e. $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$. $S^{\Omega_j} I^{\Omega_j} R^{\Omega_j}$, $j = 1,2,3$ stochastic dynamics in the presence of all optimal controls $u^{\Omega_1 *}, u^{\Omega_2 *}$ and $u^{\Omega_3 *}$ in cases of $e_j = 0,\ 0.1,\ 0.2$. $S_0^{\Omega_1} = 90000$, $I_0^{\Omega_1} = 1200$, $S_0^{\Omega_2} = 89000$, $I_2^{\Omega_2} = 1100$, $S_0^{\Omega_3} = 88000$, $I_0^{\Omega_3} = 1000$, $R_0^{\Omega_j} = 0\ \forall\ j = 1,2,3$, $d_1 = 0.06$, $\gamma_1 = 0.04$, $\beta_1 = 0.5$, $d_2 = 0.05$, $\gamma_2 = 0.03$, $\beta_2 = 0.4$, $d_3 = 0.04$, $\gamma_3 = 0.02$, $\beta_3 = 0.1$

In this part, we investigate the advantages of the immunization-based control in the prevention of the viruses spread, using the stochastic forward-backward sweep method that incorporate stochastic progressive-regressive schemes devised for the case of stochastic optimal control problem as in [20] and also detailed in [56]. In Figure 8.1., we compare between three cases that depend on the value of the losing removal individuals immunity rate, namely (a) $e_j = 0,\ j = 1,2,3$ which defines the SIR model case, (b) $e_j = 0.1$ and (c) $e_j = 0.2$, all under the

condition $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$ which defines the $S^{\Omega_j} I^{\Omega_j} R^{\Omega_j},\ j = 1,2,3$ stochastic dynamics when there is yet no control. As we can observe in this Figure, the number of infected people increases in the first days and stochastically stabilize in very important amount that takes higher values every time e_j is higher since when there is a fraction of removed people move again to the susceptible category, we will have more infected people. Simultaneously, the number of removed people in all three regions increases to a modest value because of the natural recovery but this becomes less important every time every time e_j is smaller since there are individuals who join the susceptible class. The Figure 8.2. depicts the $S^{\Omega_j} I^{\Omega_j} R^{\Omega_j},\ j = 1,2,3$ stochastic dynamics when immunization is followed in all regions meeting the condition $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$ with an investigation of the effectiveness of this policy in the three cases $e_j = 0,\ j = 1,2,3$, $e_j = 0.1$ and $e_j = 0.2$. Here, we can deduce that the immunization control strategy has succeeded in reducing the number of infected individuals and as we can see for the case $e_j = 0,\ j = 1,2,3$, I^{Ω_j} has increased just very little in the first 20 days and decreased towards zero values in almost remaining days. This number does not exceed the one third of its value in case of Figure 8.1. despite a slightly increase of its value in this Figure because of the importance given e_j. Simultaneously, R^{Ω_j} in all three regions increases exponentially after less than three days to an important value that is very close to initial condition $S_0^{\Omega_1}$ but this becomes less important every time every time e_j is smaller since there are individuals who join the susceptible class.

3. The Model With Immunization Plus Closure Policy

3.1. Presentation of the control model. Let $I = \{1, ..., p\}$ and $I_H \subset I$ the set of indexes of regions at high-risk and then, having the ability to spread the epidemic to other regions. Here, we study the case when a given region Ω_j is under immunization control u^{Ω_j} and at the same time under threat of infection coming from other regions. For this, we add to the immunization strategy, an other control denoted as $v^{j\Omega_k}$ to characterize the effectiveness of closure operations, in order to prevent infected of regions Ω_k, $k \in I_H$ to come to the controlled region Ω_j, where

$$\begin{cases} v^{j\Omega_k} \neq 0 & \forall k \in I_H \quad k \neq j \\ v^{j\Omega_k} = 0 & \text{elsewhere} \end{cases} \tag{11}$$

Then, the model (6)-(7) in the controlled region Ω_j is rewritten as follows

$$\dot{S}^{\Omega_j}(t) = -\sum_{k=1}^{p} \vartheta^{jk}(t)\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) + \left(N^{\Omega_j}(t) - S^{\Omega_j}(t)\right)d_j - \theta^{\Omega_j}(t)S^{\Omega_j}(t) + e_j R^{\Omega_j}(t) \tag{12}$$

$$\dot{I}^{\Omega_j}(t) = \sum_{k=1}^{p} \vartheta^{jk}(t)\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) - \gamma_j I^{\Omega_j}(t) - d_j I^{\Omega_j}(t)$$

$$\dot{R}^{\Omega_j}(t) = \gamma_j I^{\Omega_j}(t) - (d_j + e_j)R^{\Omega_j}(t) + \theta^{\Omega_j}(t)S^{\Omega_j}(t) \tag{13}$$

with the immunization control defined as

$$\theta^{\Omega_j}(t) = u^{\Omega_j}(t) + \delta_j \frac{dW^{\Omega_j}(t)}{dt}.$$

and the function $\vartheta^{jk}(t)$ defined as

$$\vartheta^{jk}(t) = \left(1 - v^{j\Omega_k}\right)\beta_{jk} + (1 - \varsigma_{jk})\sigma_j \frac{dW^{\Omega_j}(t)}{dt}$$

where σ_j and ς_{jk} $k \in I_H$, $(j = 1, ..., p)$ are real constants and representing the intensities of fluctuations caused by media and escapes of infected people in borders between Ω_k and Ω_j respectively.

3.2. A stochastic optimal control approach. Now, we consider the minimization problem of the following objective function

$$J(u, v) = \sum_{k=1}^{p} \varepsilon_k J_k(u^{\Omega_k}, v^{\Omega_k}) \tag{14}$$

but with a change of definition of $J_k(u^{\Omega_k}, v^{k\Omega})$ to
$J_j(u^{\Omega_j}, v^{j\Omega}) = \sum_{k \in I_H} \mathbb{E}\left(\int_0^T \left(\alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) + \frac{A_j}{2}(u^{\Omega_j}(t))^2 + \frac{B_j}{2}(v^{j\Omega_k}(t))^2\right)dt\right)$ where $B_j > 0$ is the weight constant of the new control, while $u^{\Omega_j} \in U_j$ and $v^{j\Omega} = \left(v^{j\Omega_k}\right)_{k \in I_H}$ belonging to the control set $V_j^{I_H}$ defined as

$$V_j^{I_H}([0,T]) = \{v^{j\Omega}(t)\ \mathscr{F}_t\text{-progressively measurable} | v_{min}^{\Omega_j} \leqslant v^{j\Omega_k}(t) \leqslant v_{max}^{\Omega_j}, k \in I_H\}$$

for all $j \in I_C, t \in [0,T]$.

The Hamiltonian in this case is defined as

$$H = \sum_{k=1}^{p} \varepsilon_k \sum_{l \in I_H} \left(\alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) + \frac{A_j}{2}(u^{\Omega_j}(t))^2 + \frac{B_j}{2}(v^{j\Omega_l}(t))^2 \right.$$
$$\left. + \mu^{\Omega_j^T}(t) f(t, x^{\Omega_j}(t), u^{\Omega_j}(t), v^{j\Omega_l}(t)) + \sum_{m=1}^{3} g^{m^T}(x^{\Omega_j}(t), u^{\Omega_j}(t), v^{j\Omega_l}(t)) \nu^{\Omega_j^l}(t)\right)$$

where the state function f are defined as

$$f(t,x^{\Omega}(t),u^{\Omega}(t),v^{j\Omega}(t))$$

$$= \begin{pmatrix} -\sum_{k=1}^{p}\left(1-v^{j\Omega_k}\right)\beta_{jk}\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t)+\left(N^{\Omega_j}(t)-S^{\Omega_j}(t)\right)d_j-u^{\Omega_j}(t)S^{\Omega_j}(t)+e_jR^{\Omega_j}(t) \\ \sum_{k=1}^{p}\left(1-v^{j\Omega_k}\right)\beta_{jk}\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t)-\gamma_jI^{\Omega_j}(t)-d_jI^{\Omega_j}(t) \\ \gamma_jI^{\Omega_j}(t)-(d_j+e_j)R^{\Omega_j}(t)+u^{\Omega_j}(t)S^{\Omega_j}(t) \end{pmatrix}$$

and the diffusion matrix g defined as

$$g(t,x^{\Omega}(t),u^{\Omega}(t),v^{j\Omega}(t)) = \begin{pmatrix} -\left(\sigma_j\sum_{k=1}^{p}(1-\varsigma_{jk})\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t)+\delta_jS^{\Omega_j}(t)\right) \\ \sigma_j\sum_{k=1}^{p}(1-\varsigma_{jk})\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) \\ \delta_jS^{\Omega_j}(t) \end{pmatrix}$$

while in a domain Ω, $(\mu(t),\nu(t)))$ is a pair of adjoint variables satisfying the following adjoint BSDE (Backward stochastic differential equation)

$$(15)\qquad \begin{cases} d\mu^{\Omega}(t) &= -[f_x^T(t,x^{\Omega}(t),u^{\Omega}(t),v^{j\Omega}(t))\mu^{\Omega}(t) \\ &\quad +\sum_{l=1}^{3}g_{x^{\Omega}}^{l^T}(t,x^{\Omega}(t),u^{\Omega}(t),v^{j\Omega}(t))\nu^{\Omega^l}(t) \\ &\quad +f_{0_{x^{\Omega}}}(t,x^{\Omega}(t),u^{\Omega}(t),v^{j\Omega}(t))]dt+\nu^{\Omega}(t)dW^{\Omega}(t), \\ \mu^{\Omega}(T) &= 0 \end{cases}$$

3.2.1. *Optimal control characterization and necessary conditions.* Using the stochastic Pontryagin's Maximum Principle as done for the first control strategy, we obtain the following optimal control characterization and necessary conditions.

Theorem 3.2.1. *If there exists an optimal pair* $(x^{\Omega*},u^{\Omega*},v^{j\Omega*})$ *and a pair of processes* $(\mu(t),\nu(t))$ *satisfying (15), then for* $j=1,...,p$, $k\in I_H$ *we have*

$$H(t,x^{\Omega_j*}(t),u^{\Omega_j*}(t),v^{j\Omega_k*}(t),\mu^{\Omega_j}(t),\nu^{\Omega_j}(t))$$

$$= \min_{(u^{\Omega_j},v^{j\Omega_k}(t))\in U_j\times V_j^{I_H}} H(t,x^{\Omega_j}(t),u^{\Omega_j*}(t),v^{j\Omega_k*}(t),\mu^{\Omega_j}(t),\nu^{\Omega_j}(t))$$

Moreover, we obtain the bounded stochastic control

$$u^{\Omega_j*} = min(max(u_{min}^{\Omega_j},-\frac{(\mu_3^{\Omega_j}(t)-\mu_1^{\Omega_j}(t))S^{\Omega_j*}}{\varepsilon_jA_j}),u_{max}^{\Omega_j}),$$

$$v^{j\Omega_k *} = min(max(u_{min}^{\Omega_j}, -\frac{(\mu_1^{\Omega_j}(t) - \mu_2^{\Omega_j}(t))\beta_{jk} I^{\Omega_k *} S^{\Omega_j *}}{\varepsilon_j B_j}), u_{max}^{\Omega_j}),$$

solutions of the FBSDEs (Forward-backward stochastic differential equations)

$$\begin{cases} dx^{\Omega_j}(t) &= f(t, x^{\Omega_j}(t), u^{\Omega_j}(t), v^{j\Omega_k}(t))dt + g(t, x^{\Omega_j}(t), u^{\Omega_j}(t), v^{j\Omega_k}(t))dW^{\Omega_j}(t) \\ d\mu^{\Omega_j}(t) &= -[f_{x^{\Omega_j}}^T(t, x^{\Omega_j}(t), u^{\Omega_j}(t), v^{j\Omega_k}(t))\mu^{\Omega_j}(t) \\ & \quad + \sum_{l=1}^{3} g_{x^{\Omega_j}}^{l^T}(t, x^{\Omega_j}(t), u^{\Omega_j}(t), v^{j\Omega_k}(t))\nu^{\Omega_j^l}(t) \\ & \quad + f_{0\,x^{\Omega_j}}(t, x^{\Omega_j}(t), u^{\Omega_j}(t), v^{j\Omega_k}(t))]dt + \nu^{\Omega_j}(t)dW^{\Omega_j}(t), \\ x^{\Omega_j}(0) &= (S_0^{\Omega_j}, I_0^{\Omega_j}, R_0^{\Omega_j}) \\ \mu^{\Omega_j}(T) &= 0. \end{cases}$$

3.2.2. *Numerical Results.* In Figure 8.3, we try to show the importance of the optimal closure strategy when it is followed in parallel with the optimal immunization policy discussed in Figure 8.2. This Figure depicts again the $S^{\Omega_j} I^{\Omega_j} R^{\Omega_j},\ j = 1,2,3$ stochastic dynamics when immunization plus closure policy are both followed in all regions and meeting the condition $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$ in the three cases $e_j = 0,\ j = 1,2,3$, $e_j = 0.1$ and $e_j = 0.2$. We can conclude that the closure control strategy has succeeded more in reducing the number of infected individuals and as we can see for the case $e_j = 0,\ j = 1,2,3$, I^{Ω_j} has increased in just less than 20 days and decreased towards zero values in almost remaining days. This number does not exceed less than the one third of its value in case of Figure 8.1. despite a very slightly increase of its value in this Figure because of the importance given e_j. We can see more the effect of travel-blocking control strategy in reducing I^{Ω_j} when we compare simulations between the second and third plots of Figure 8.2. and this present one. Simultaneously, R^{Ω_j} in all three regions increases exponentially after less than three days to an important value that is very close to initial condition $S_0^{\Omega_1}$ but this becomes less important every time every time e_j is smaller since there are individuals who join the susceptible class. In Figure 8.4, we investigate the effectiveness of both immunization and closure strategies when the degree of importance ε_j for different j is not equal in all regions. As a first example, we suppose that regions Ω_1 and Ω_2 are not under immunization control "case (a)", and we can deduce this has not been sufficient to prevent the epidemic in all regions. Despite that remark, we can say there is a significant and positive

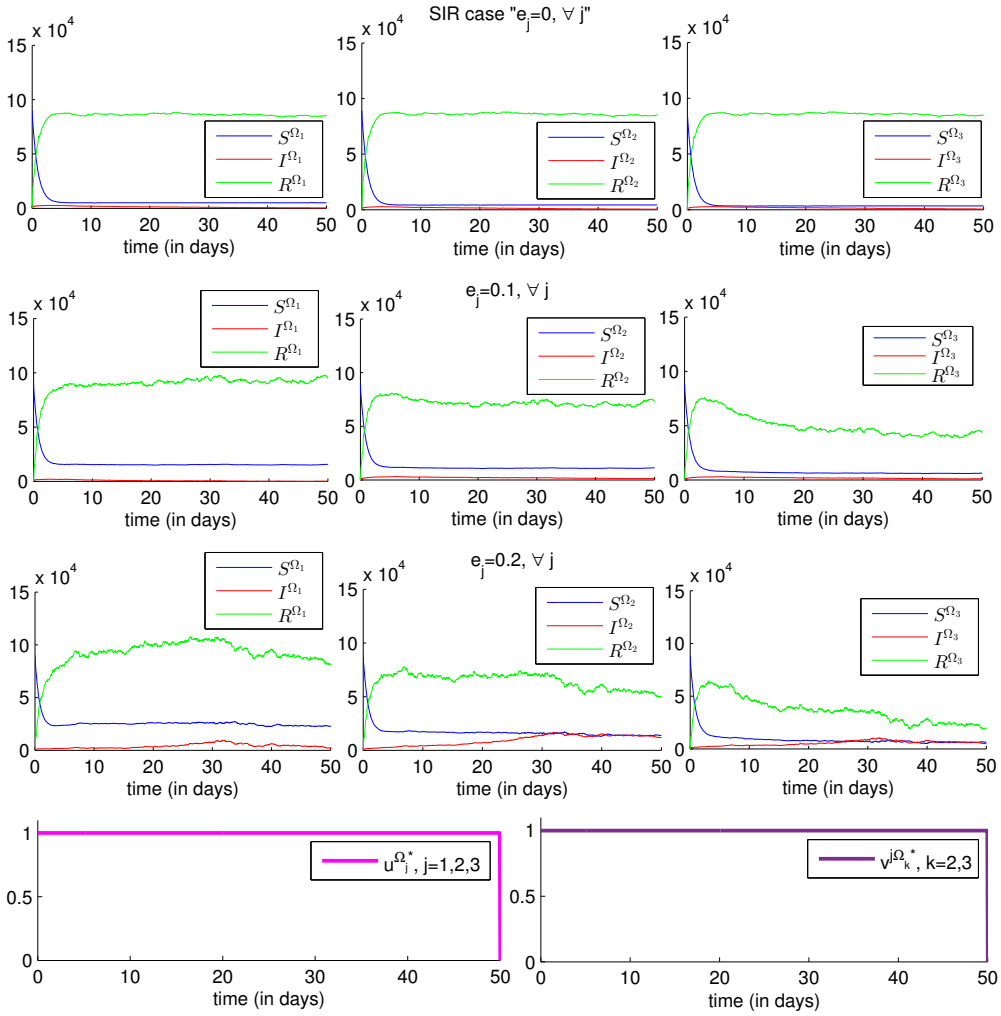

FIGURE 3. Control in all regions, i.e. $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$. $S^{\Omega_j} I^{\Omega_j} R^{\Omega_j}$, $j = 1,2,3$ dynamics with optimal controls with optimal controls $u^{\Omega_1 *}, u^{\Omega_2 *}$ and $u^{\Omega_3 *}$ plus $v^{1\Omega_k *}, k = 2,3$ in cases of $e_j = 0,\ 0.1,\ 0.2$. $S_0^{\Omega_1} = 90000$, $I_0^{\Omega_1} = 1200$, $S_0^{\Omega_2} = 89000$, $I_2^{\Omega_2} = 1100$, $S_0^{\Omega_3} = 88000$, $I_0^{\Omega_3} = 1000$, $R_0^{\Omega_j} = 0\ \forall\ j = 1,2,3$, $d_1 = 0.06$, $\gamma_1 = 0.04$, $\beta_1 = 0.5$, $d_2 = 0.05$, $\gamma_2 = 0.03$, $\beta_2 = 0.4$, $d_3 = 0.04$, $\gamma_3 = 0.02$, $\beta_3 = 0.1$.

change in this case where e_j is supposed to equal 0.2, with a slight reduction of the level of infection compared to the third case of the first figure, especially when we are close to the end because of the maximum value 1 taken by $v^{1\Omega_k *}, k = 2,3$ until the last 5 and 2 days respectively. As for the second example "case (b)" which means that immunization control has been followed in all regions except Ω_1, the number of removed people has increased significantly in Ω_2 and

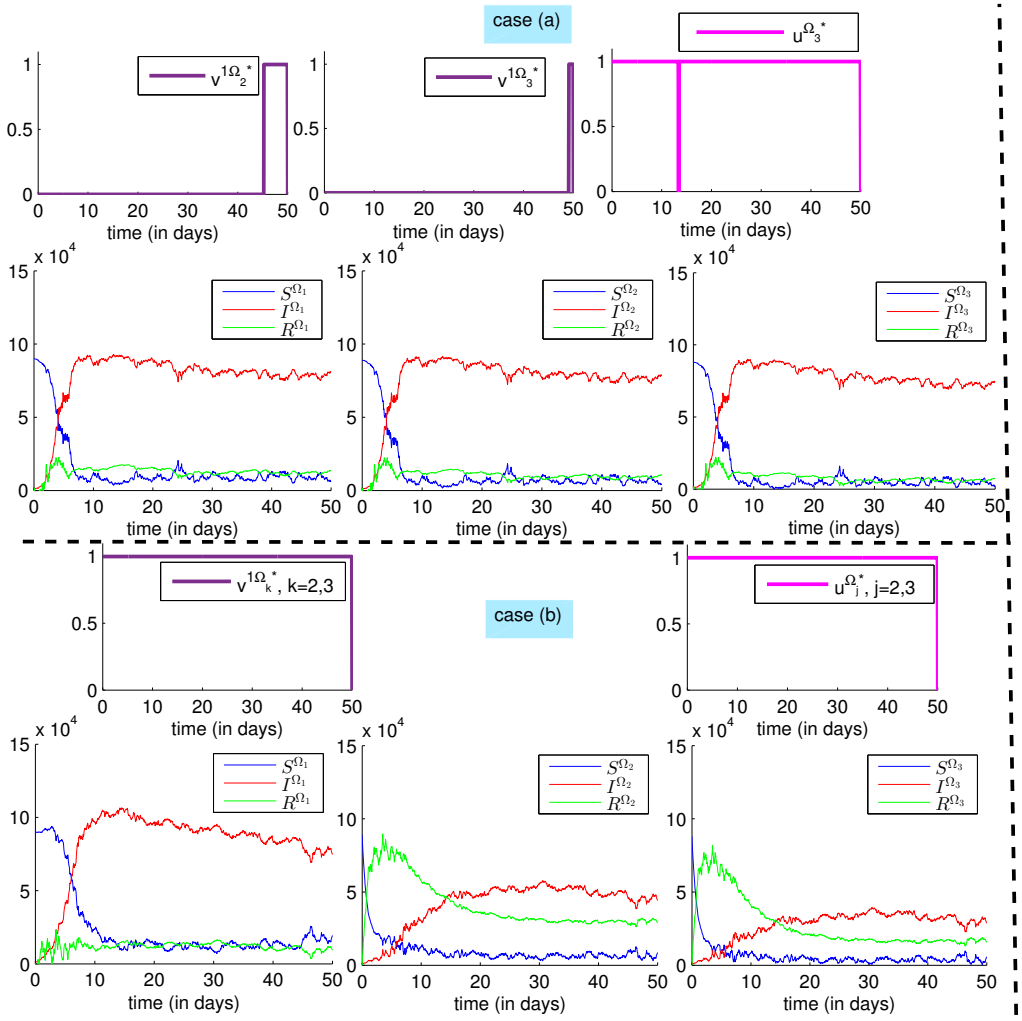

FIGURE 4. case (a): immunization control of region Ω_3 only; i.e. $\varepsilon_1 = \varepsilon_2 = 0$ and $\varepsilon_3 = 1$. case (b): immunization control in all regions except Ω_1; i.e. $\varepsilon_1 = 0$ and $\varepsilon_2 = \varepsilon_3 = 1$. $e_j = 0.2$, $\forall j$. $S_0^{\Omega_1} = 90000$, $I_0^{\Omega_1} = 1200$, $S_0^{\Omega_2} = 89000$, $I_2^{\Omega_2} = 1100$, $S_0^{\Omega_3} = 88000$, $I_0^{\Omega_3} = 1000$, $R_0^{\Omega_j} = 0 \ \forall \ j = 1,2,3$, $d_1 = 0.06$, $\gamma_1 = 0.04$, $\beta_1 = 0.5$, $d_2 = 0.05$, $\gamma_2 = 0.03$, $\beta_2 = 0.4$, $d_3 = 0.04$, $\gamma_3 = 0.02$, $\beta_3 = 0.1$.

Ω_3, while the number of infected people in these two regions is lower the case treated in the first figure. The main advantage of this last simulation, is that it is showing clearly how the immunization control strategy has an implicit effect on the number of infected people in region Ω_1 compared to the first case (a) in the same figure. This can not finally be taken as ideal cases compared to the case when all ε_j are supposed to equal 1 as in the previous figure, but this helps to prove the influence of one region on another.

4. Conclusion

In this paper, we introduced into a stochastic multi-regions SIRS epidemic model which describes the spread of infection that can affect people again even after their recovery, some control functions associated to immunization strategies followed in regions dependably on the value of a boolean variable which takes either 1 or 0 and defined as the degree of importance given to a region for controlling it. The effectiveness of the immunization policy has been investigated with additional control strategy which characterizes travel-blocking operations that aim to restrict the number of people coming from a region with a high-risk of infection. After all, we concluded that immunization plus closure policies, gave better and promising results than immunization alone, despite the fluctuations considered and that are related to media, infodemics and escapes.

Conflict of Interests

The authors declare that there is no conflict of interests.

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