Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2024, 2024:16
https://doi.org/10.28919/cmbn/8400
ISSN: 2052-2541

FRACTIONAL MODEL OF PREY-PREDATOR INTERACTIONS IN A RESERVED AREA WITH A COMPETITIVE AND TOXIC ENVIRONMENT

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Abstract. In this work, a mathematical approach is used to analyze the dynamics of a fractional-order predator-prey model with two effort functions in an environment that is both competitive and toxic. It is assumed that there are two major prey and predator groups; prey groups, moreover, occupy two different zones, one that is protected and the other not so. Susceptible predators are assumed to have access to both zones, infected predators do not have access to the reserved zone. Therefore, susceptible predators seek prey species in both the reserved and free zones. We first demonstrate the bounds of the solution. The existence is then verified. We will then turn to examining the local and the global stability using the Lyapunov method. As a final point, we will use numerical simulations to confirm our results and to verify the population's response to prey consumption in the reserved region.

Keywords: prey-predator; fractional-order; stability; toxicity; competition.

2020 AMS Subject Classification: 34A08, 34K37, 93A30.

1. Introduction

The question of fractional derivatives was evoked as early as 1695 by Leibnitz in a letter to L'Hospital, but when the latter asked him what the derivative of order one-half of the function x could be, Leibnitz replied that this led to a paradox from which useful consequences would one

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Received December 15, 2023

day be drawn. Many mathematicians have studied this question, in particular the mathematician Liouville who started the research on the subject, he knew the first fractional integration operator [43] and Riemann broke with this subject and developed what is now known as the Riemann definition, an unprecedented interest and development in this domain [42], at the end of the 1960's required a revision that led many authors, including Caputo, to find a new definition of the fractional derivation [18]. During the recent years, researchers used the fractional derivative theory in several disciplines such as biological modeling [48], medicine [21], physics [50]... but the application of these fractional derivatives to ecological modeling has been particularly productive. For example, B. Ghanbari et al. [22] proposed a model in which the population is divided into three sub-categories: Prey-mature, prey immature and predators. They assumed that predators attack mature prey with a Crowley-Martin type functional response. In another work, S. Djilali and B. Ghanbari [17] discussed the impact of an infective disease on species evolution, they proposed an eco-epidemiological model with a fractional order consisting of two categories of prey (infected and susceptible) and the predators that attack them. The coexistence of interacting biological species has been extensively studied by various other researchers using fractional order mathematical models [7, 26, 38, 28].

It is noticeable how many species became threatened by extinction due to several factors such as over-fishing, pollution, misuse, etc. Many measures have been taken to protect these species from these factors, including the creation of protected areas. The importance of reserved zones in predator-prey dynamics has indeed gotten a lot of attention by different researchers in the literature [8]. For example the predator-prey model dynamics with a reserved area were presented and examined by B. Dubey [12], Considering that the habitat is separated into two different regions (a free and a reserved zone). Predators are not permitted to access the protected zone, but must subsist on prey from the non-reserved zone. The interplay between predators and prey is also influenced by viruses. indeed, in an unreserved region, infected predators have a hard time tracking down preys that move in a herd. Under such circumstances, the predator can be seriously injured by the herd. As a result, whenever the predator tries to hunt the herd, it suffers. Many studies consider this case of interaction, such as, in [6], where in fear factors of susceptible and infectious predators are studied. They postulate that the attack rate of infected predators was lower than that of susceptible predators. D. K. Das et al. [16] advanced a model that takes into consideration the change in behavior of susceptible predators caused by the effect of infected predators, the Hopf bifurcation is discussed by considering the disease transmission rate as the bifurcation parameter. Then, they studied the optimal harvesting policy. In [30] the authors used a fractional prey-predator model to analyze two distinct susceptible and infected predator kinds in two areas: a free area where predators and prey can freely move, and a protected area where prey can live safely from predation.

Motivated by the previous works, we will analyze a population of prey and predators, but this time we will take into account that susceptible predators can move around freely inside the protected area and that the population of infected predators is weakened by disease, making it harder for them to hunt prey. An infected predator is therefore thought to be unable to enter the protected area.

As can be seen below, the organization of this article is as follows. The Definitions and Properties of the fractional order derivative essential to our study are given in the second section. Then after presenting our model in the third section, the fourth section will focus on the uniqueness and the existence of solutions, as well as their positivity. A discussion on the existence of equilibria follows. The Jacobian matrix and the traditional Lyapunov technique will then be used to study the local and global stability of equilibria. In the final section, we present numerical simulations to verify the theoretical results.

2. PRELIMINARIES

This section provides the Definitions and Properties of the fractional order derivative that are essential to our study.

Definition 2.0.1. [42] *For $0 < \alpha < 1$ and for a function $h : \mathbb{R}^+ \to \mathbb{R}$, the fractional order derivative in the Caputo sense is:*

$$D^\alpha h(y) = \frac{1}{\Gamma(1-\alpha)} \int_0^y \frac{h'(t)}{(x-t)^\alpha} dt.$$

Lemma 2.1. [41] *For $h \in C[a,b]$ and for the derivatives $D^\alpha h \in C[a,b]$ where $\alpha \in [0,1]$, $\exists \xi \in [a,b]$ so that:*

$$h(y) = h(a) + \frac{1}{\Gamma(\alpha)} D^{\alpha} h(\xi)(y-a)^{\alpha}, \quad \forall y \in [a,b].$$

Lemma 2.2. [5, 41] *For $h \in C[a,b]$ and for the derivatives $D^{\alpha}h \in C[a,b]$ where $\alpha \in [0,1]$.*

$$\text{If} \quad D^{\alpha}h(y) \leq 0 \quad \forall y \in [a,b], \quad h \text{ is a decreasing function}.$$

$$\text{If} \quad D^{\alpha}h(y) \geq 0 \quad \forall y \in [a,b], \quad h \text{ is an increasing function}.$$

Lemma 2.3. [29] *For $h(t) \in C[a,+\infty]$ where $D^{\alpha}h(t) + uh(t) \leq v$, $\alpha \in [0,1]$, $(u,v) \in \mathbb{R}^2$, $u \neq 0$ and $a \geq 0$ is the initial time. The solution verify:*

$$h(t) \leq (h(a) - \frac{v}{u}) E_{\alpha}[-u(t-a)^{\alpha}] + \frac{v}{u}$$

Lemma 2.4. [49] *Let $y^* \in \mathbb{R}^+$ and $\alpha \in [0,1]$. We have for the differentiable and continuous function $y(t) \in \mathbb{R}^+$:*

$$\text{for all time t¿0}, \quad D^{\alpha}(y(t) - y^* - y^* ln(\frac{y(t)}{y^*})) \leq (1 - \frac{y^*}{y(t)}) D^{\alpha} y(t)$$

3. Presentation of the Model

Susceptible predators consume prey in the free and reserved zones according to a Holling type II functional response [23]. Infected predator attack prey in the free area according to a Holling type I functional response [23]. Individuals in the free zone move to the protected area at the rate σ_1 defined in the table 1 . Individuals immigrate to the unreserved zone at the rate σ_2 (see table 1) and all compartments, S and I are respectively reduced by the fishing efforts rates q_1E_1, $q_2E_2 + \mu$ and $q_3E_3 + \eta$. Susceptible predators are also reduced by infection at the rate δS and both areas are reproducing at the birth rates r_1 and r_2 defined in the table 1 .
So in our model (3.1), we assume that:

(H_1) : In both zones, prey and susceptible predators are free to move around.

(H_2) : Predators infected with the virus never enter the reserved zone and capture prey since they need the energy to do so.

(H_3) : Prey species are consumed by susceptible predators in both the reserved and free zones.

(H_4) : Only the free zone is open to us for fishing.

The following diagram describes the transmission dynamics of our mathematical model:

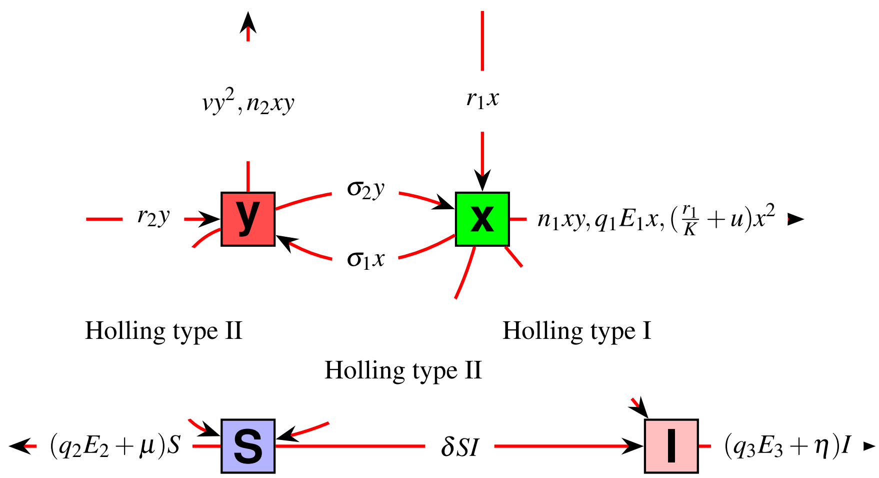

FIGURE 1. Diagram of the model (3.1)

According to the Figure 1 we would have the following equations:

$$
\begin{cases}
D^\alpha x = r_1x\left(1-\frac{x}{K}\right) - \sigma_1 x + \sigma_2 y - ux^2 - \frac{\beta x S}{\alpha' + x} - n_1xy - \gamma xI - q_1E_1x, \\
D^\alpha y = (r_2-\sigma_2)y + \sigma_1 x - \frac{cyS}{d+y} - vy^2 - n_2xy, \\
D^\alpha S = \frac{\alpha_1 cyS}{d+y} + \frac{\alpha_2 \beta x S}{\alpha' + x} - \delta SI - \mu S - q_2E_2S, \\
D^\alpha I = \alpha_2\gamma xI + \delta SI - \eta I - q_3E_3I.
\end{cases}
\tag{3.1}
$$

D^α is the fractional derivative of Caputo, defined in the Definition 2.0.1, with $0 < \alpha \leq 1$.

Supposing that the parameters are all strictly positive, with initial conditions

$$y(0) > 0, \quad x(0) > 0, \quad I(0) > 0 \quad \text{and} \quad S(0) > 0. \tag{3.2}$$

We define the variables and the parameters in the following table:

Variables	Description
x	Densities of biomass in the free zone
y	Densities of biomass in the reserved zone
S	How many predators are susceptible
I	How many predators have the disease
Parameters	**Description**
E_1	Harvesting effort of prey in the free zone
E_2	Harvesting effort of susceptible predators
E_3	Harvesting effort of contaminated predators
r_1, r_2	Fish population growth rates in both reserved and unprotected zones
q_1, q_2	The catchability coefficient of predator species
σ_1, σ_2	Migration's rate from a free zone to one that is protected and vice versa
n_1, n_2	The parameters of competition between x and y
γ	The ill predator intra-specific strength against the prey
δ	The rate at which the disease is spread
β	The rate of prey search by a susceptible predator in the free zone
c	The rate of prey search by a susceptible predator in the reserved zone
μ	Susceptible predator's death rate
η	Infected predator's mortality rate
d, α'	Rate saturation when prey is being attacked by susceptible predators
α_1, α_2	The rate of a predator converting because of prey
ux^2, vy^2	The reserved and non-reserved zones' reduction parameters
v, u	The corresponding toxicity coefficients
$\frac{\beta x S}{\alpha' + x}$	The functional response when a susceptible predator is feeding prey

TABLE 1. Variables and parameters descriptions

In order to avoid a decrease in population that could lead to extinction, we consider that

$$r_1 - \sigma_1 - q_1E_1 > 0, \quad r_2 - \sigma_2 > 0 \quad \text{and} \quad \alpha_1 c - \alpha_2\beta - \mu - q_2E_2 > 0. \tag{3.3}$$

Indeed, if fish populations do not migrate between reserved and non-reserved region ($\sigma_1 = \sigma_2 = 0$), for $r_1 - \sigma_1 - q_1E_1 < 0$ we conclude that $D^\alpha x < 0$ and if $r_2 - \sigma_2 < 0$, we have $D^\alpha y < 0$. In the other hand if $\alpha_1 c - \alpha_2\beta - \mu - q_2E_2 < 0$, then $D^\alpha S < 0$.

4. The Model's Mathematical Analysis

We'll look at the uniqueness and existence of solutions, as well as the positivity and bounds offered by the model (3.1) in this part. There will be a discussion about the existence of equilibriums.

4.1. Solution's positivity and boundedness. To demonstrate the biological validity of our model, we show in this section that the solutions are bounded and that the variables are always positive at all times.

Lemma 4.1. *Considering initial conditions 3.2, the solutions of the system 3.1 are positives at all times.*

Proof.

$$\begin{aligned}
&\text{for} \quad x = 0, \quad D^\alpha x = \sigma_2 y \quad \forall y \geq 0,\\
&\text{for} \quad y = 0, \quad D^\alpha x = \sigma_1 x \quad \forall x \geq 0,\\
&\text{for} \quad S = 0, \quad D^\alpha x = 0,\\
&\text{for} \quad I = 0, \quad D^\alpha I = 0.
\end{aligned}$$

So, from lemmas 2.1 and 2.2 the solutions $x(t)$, $y(t)$, $S(t)$ and $I(t)$ are non-negatives at all times. □

Lemma 4.2. *if* $\alpha_1 \leq \alpha_2$ *and* $\eta + q_3E_3 \leq \mu + q_2E_2$,
$W' = \left\{(x,y,S,I) \in \mathbb{R}_+^4 / x + y + \frac{S}{\alpha_2} + \frac{I}{\alpha_2} \leq \frac{\Lambda}{\eta + q_3E_3}\right\}$ *is a region of attraction, where:*

$$\Lambda = \frac{(r_2 + \eta + q_3E_3)^2}{4v} + \frac{K}{4(r_1 + Ku)}(r_1 + \eta - q_1E_1 + q_3E_3)^2.$$

Proof. We set $X = x + y + \frac{S}{\alpha_2} + \frac{I}{\alpha_2}$,

$$\begin{aligned} D^\alpha X &= (r_1 - q_1E_1)x - (\tfrac{r_1}{K} + u)x^2 - vy^2 + r_2y + (\tfrac{\alpha_1}{\alpha_2} - 1)\tfrac{cyS}{d+y} - \tfrac{(\mu+q_2E_2)}{\alpha_2}S - (n_1+n_2)xy \\ &\quad - \tfrac{\eta+q_3E_3}{\alpha_2}I, \end{aligned}$$

so,

$$\begin{aligned} D^\alpha X + (\eta + q_3E_3)X &= (r_1 - q_1E_1 + \eta + q_3E_3)x - (\tfrac{r_1}{K} + u)x^2 - vy^2 + (r_2 + \eta + q_3E_3)y \\ &\quad + (\tfrac{\alpha_1}{\alpha_2} - 1)\tfrac{cyS}{d+y} + \tfrac{(\eta+q_3E_3-\mu-q_2E_2)}{\alpha_2}S - (n_1+n_2)xy, \end{aligned}$$

we get:

$$D^\alpha X + (\eta + q_3E_3)X \leq K\frac{(r_1 + \eta - q_1E_1 + q_3E_3)^2}{4(r_1 + Ku)} + \frac{(r_2 + \eta + q_3E_3)^2}{4v} = \Lambda.$$

Using the fractional inequality theory in Lemma 2.3, we arrive at:

$$X(t) \leq X(0)E_\alpha(-(\eta + q_3E_3)t^\alpha) + \frac{\Lambda}{\eta + q_3E_3}\left(1 - E_\alpha(-(\eta + q_3E_3)t^\alpha)\right),$$

where $E_\alpha(u) = \sum_{i=0}^{\infty} \frac{u^i}{\Gamma(\alpha i + 1)}$ (Mittag-Leffler function [42]), $\Gamma(u) = \int_0^\infty t^{u-1}e^{-t}\,\mathrm{d}t$ (Euler's Gamma function) and $0 < E_\alpha(-(q_3E_3 + \eta)t^\alpha) \leq 1$.
For $t \to \infty$, we get $0 < X(t) \leq X(0) + \frac{\Lambda}{\eta+q_3E_3}$, proving this Lemma. □

4.2. Solutions' existence and uniqueness. We can express the system (3.1) as follows

$$D^\alpha Y = G(Y), \tag{4.1}$$

with $Y = (x, y, S, I)^t$ and

$$G(Y) := \begin{pmatrix} r_1x\left(1 - \frac{x}{K}\right) - \sigma_1x + \sigma_2y - ux^2 - \frac{\beta xS}{\alpha'+x} - q_1E_1x - n_1xy - \gamma xI \\ (r_2 - \sigma_2)y + \sigma_1x - vy^2 - n_2xy - \frac{cyS}{d+y} \\ \frac{\alpha_1 cyS}{d+y} + \frac{\alpha_2\beta xS}{\alpha'+x} - \delta SI - \mu S - q_2E_2S \\ \delta SI + \alpha_2\gamma xI - q_3E_3I - \eta I \end{pmatrix} := \begin{pmatrix} G_1(Y) \\ G_2(Y) \\ G_3(Y) \\ G_4(Y) \end{pmatrix}.$$

Lemma 4.3. *In the region* $W = \left\{(x,y,S,I) \in \mathbb{R}_+^4 / max\{x,y,S,I\} \leq C, C > 0\right\}$, *there is a unique solution of the system (3.1) where G satisfies Lipschitz's condition* [31, 42].

Proof. To show the existence and uniqueness of solutions of the system (3.1), let $Y, Y' \in W$

$$
\begin{aligned}
\left\|G(Y)-G(Y')\right\|_1 &= \sum_{i=1}^{4}|G_i(Y)-G_i(Y')|,\\
&= |(x-x')(r_1-\sigma_1-q_1E_1)-(x^2-x'^2)(u+\tfrac{r_1}{K})+\sigma_2(y-y')\\
&\quad+\tfrac{-\beta(\alpha'(xS-x'S')+xx'(S-S'))}{(\alpha'+x)(\alpha'+x')}-n_1(xy-x'y')-\gamma(xI-x'I')|\\
&\quad+|(y-y')(r_2-\sigma_2)+\sigma_1(x-x')-v(y^2-y'^2)-n_2(xy-x'y')\\
&\quad-c\tfrac{d(yS-y'S')+yy'(S-S')}{(d+y)(d+y')}|+|\alpha_1 c\tfrac{d(yS-y'S')+yy'(S-S')}{(d+y)(d+y')}\\
&\quad+\alpha_2\beta\tfrac{\alpha'(xS-x'S')+xx'(S-S')}{(\alpha'+x)(\alpha'+x')}-\delta(SI-S'I')-(\mu+q_2E_2)(S-S')|\\
&\quad+|\delta(SI-S'I')+\alpha_2\gamma(xI-x'I')-(\eta+q_3E_3)(I-I')|,\\
&\leq L\|Y-Y'\|_1,
\end{aligned}
$$

where

$$
\begin{aligned}
L &= \max((r_1-q_1E_1+M(n_1+n_2+\gamma(1+\alpha_2)+\beta(1+\alpha_2)+2(\tfrac{r_1}{K}+u))),(r_2+M(n_1\\
&\quad+n_2+2v+c(1+\alpha_1))),(\beta(1+\alpha_2)+\mu+q_2E_2+2\delta M+\beta M(1+\alpha_2)+c(1+\alpha_1)\\
&\quad+Mc(1+\alpha_1)),(\alpha_2\gamma M+\eta+q_3E_3+\gamma M+2\delta M)).
\end{aligned}
$$

Thus, G satisfies Lipschitz's condition [31, 42]. □

4.3. Equilibriums points. To get the equilibriums points, we solve $G_i(x,y,S,I)=0$ where $i=1,\ldots,4$ (see (4.1)). Our model (3.1) admits five positive equilibriums points.

-The trivial equilibrium:

$P_0(0,0,0,0)$ is the trivial equilibrium where the population is assumed to be zero.

-The equilibrium point with no predators:

The equilibrium $P_1(x_1,y_1,0,0)$ where predators don't exist is obtained by solving $G_i(x_1,y_1,0,0)=0$ for $i=1,2$. x is the non-negative solution of the following equation:

$$a_3x^3+a_2x^2+a_1x+a_0=0, \tag{4.2}$$

where:

$$
\begin{aligned}
a_3 &= (u+\tfrac{r_1}{K})\left(n_1n_2-v(u+\tfrac{r_1}{K})\right),\\
a_2 &= \tfrac{2v(r_1+Ku)(r_1-\sigma_1-q_1E_1)}{K}-n_2\sigma_2(\tfrac{r_1}{K}+u)-n_1n_2(r_1-\sigma_1-q_1E_1)\\
&\quad-n_1(r_2-\sigma_2)(u+\tfrac{r_1}{K})+\sigma_1 n_1^2,
\end{aligned}
\tag{4.3}
$$

$$\begin{aligned} a_1 &= -v(r_1-\sigma_1-q_1E_1)^2+(r_1-\sigma_1-q_1E_1)\left(n_2\sigma_2+n_1(r_2-\sigma_2)\right) \\ &\quad -2\sigma_1\sigma_2n_1+(r_2-\sigma_2)\sigma_2(u+\tfrac{r_1}{K}), \\ a_0 &= -\sigma_2(r_2-\sigma_2)(r_1-\sigma_1-q_1E_1)+\sigma_1\sigma_2^2. \end{aligned}$$

Using the condition of Descartes criteria [10], hold to the following different cases:

Coefficients	a_0	a_1	a_2	a_3
Choice 1	+	+	+	-
Choice 2	-	+	+	+
Choice 3	-	-	-	+
Choice 4	+	-	-	-
Choice 5	+	+	-	-
Choice 6	-	-	+	+

TABLE 2. Coefficients sign in the different cases respecting the criteria of Descartes.

According to the table 4 , if $a_0>0$ and $a_1>0$ so we have,
$(r_2-\sigma_2)(r_1-\sigma_1-q_1E_1)<\sigma_1\sigma_2$ and
$(r_1-\sigma_1-q_1E_1)\left(n_2\sigma_2+n_1(r_2-\sigma_2)\right)+(r_2-\sigma_2)\sigma_2(u+\frac{r_1}{K})>2\sigma_1\sigma_2n_1+v(r_1-\sigma_1-q_1E_1)^2$.
Then,

$$E_1>\frac{1}{q_1}\max\left(r_1-\frac{\sigma_1\sigma_2}{r_2-\sigma_2}-\sigma_1, r_1-\frac{\sqrt{\Delta}+n_2\sigma_2+n_1(r_2-\sigma_2)}{2v}-\sigma_1\right),$$

where $\Delta=(n_2\sigma_2+n_1(r_2-\sigma_2))^2+4v((u+\frac{r_1}{K})(r_2-\sigma_2)\sigma_2-2\sigma_1\sigma_2n_1)$.

In the other hand we suppose that $n_1n_2<v\left(u+\frac{r_1}{K}\right)$ to get, $a_3<0$.

Remark 4.4. *For a_2 whatever its sign, the criteria of Descartes is verified that is if $a_2<0$ we have the case 5 in table 4 and if $a_2>0$ the case 1 is verified.*

Then

$$y_1=\frac{x_1}{\sigma_2-n_1x_1}\left(\left(\frac{r_1+Ku}{K}\right)x_1-(r_1-\sigma_1-q_1E_1)\right)>0,$$

if

$$\min(\frac{(r_1-\sigma_1-q_1E_1)K}{r_1+Ku},\frac{\sigma_2}{n_1}) < x_1 < \max(\frac{(r_1-\sigma_1-q_1E_1)K}{r_1+Ku},\frac{\sigma_2}{n_1})$$

Remark 4.5. *We can also suppose that a_3, a_2¿0 and $a_0 < 0$, so whatever the sign of a_1, the criteria of Descartes is verified that if $a_1 < 0$ we have the case 6 in Table 4 and if $a_2 > 0$ the case 2 is verified.*

-The equilibrium point with no susceptible predators:

For $P_2(x_2,y_2,0,I_2)$ where susceptible predators are assumed to be absent, i.e. $G_i(x_2,y_2,0,I_2)=0$ for $i=1,2,4$, we get a non-negative solution:

$$\begin{aligned} x_2 &= \frac{\eta+q_3E_3}{\alpha_2\gamma}, \\ y_2 &= \frac{r_2-\sigma_2-n_2x_2+\sqrt{(r_2-\sigma_2-n_2x_2)^2+4\sigma_1x_2v}}{2v}, \\ I_2 &= \frac{1}{\gamma x_2}\Big((r_1-\sigma_1-n_1y_2-q_1E_1)x_2-(\frac{r_1}{K}+u)x_2^2+\sigma_2y_2\Big)>0, \end{aligned} \tag{4.4}$$

$$\text{if}\quad 0<x_2<\frac{r_1-\sigma_1-q_1E_1-n_1y_2+\sqrt{(r_1-\sigma_1-n_1y_2-q_1E_1)^2+4\sigma_2(u+\frac{r_1}{K})y_2}}{2(\frac{r_1}{K}+u)}. \tag{4.5}$$

-The equilibrium point with the absence of infected predators:

The equilibrium $P_3(x_3,y_3,S_3,0)$ where infected predators don't exist we solve $G_i(x_3,y_3,S_3,0)=0$ for $i=1,2,3$ and since $\alpha_1c-\alpha_2\beta-\mu-q_2E_2>0$, from conditions (3.3) we get:

$y_3=\frac{(\alpha_1cd)(\alpha'+x_3)}{(\alpha_1c-\mu-q_2*E_2)(\alpha'+x_3)+\alpha_2\beta x_3}>0$,

$S_3=\frac{-vy_3^2+(r_2-\sigma_2-n_2x_3)y_3+\sigma_1x_3}{(d+y_3)cy_3}>0$ if $y_3<\frac{(r_2-\sigma_2-n_2x_3)+\sqrt{(r_2-\sigma_2-n_2x_3)^2+4v\sigma_1x_3}}{2v}$

and x is the solution of the equation:

$$a_6x^6+a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0=0,$$

where,

$$\begin{aligned} a_6 &= c^2d\alpha_1\alpha_2^2k\beta^2u+c^2d\alpha_1(\mu+E_2q_2-\alpha_1c)^2(ku+r_1)+c^2d\alpha_1\alpha_2^2\beta^2r_1 \\ &\quad -2c^2d\alpha_1\alpha_2\beta(\mu+E_2q_2-\alpha_1c)(ku+r_1), \end{aligned}$$

$$
\begin{aligned}
a_5 &= -d\alpha_2^3k\sigma_1\beta^4+cd^2\alpha_1\alpha_2^2kn_2\beta^3-cd\alpha_1\alpha_2^2k\sigma_1\beta^3-c^2d\alpha_1\alpha_2^2l\beta^2+c^2d^2\alpha_1^2\alpha_2kn_2\beta^2\\
&\quad +dk\beta(\mu+E_2q_2-\alpha_1c)(3\alpha_2^2\sigma_1\beta^2-2cd\alpha_1\alpha_2n_2\beta+2c^2\alpha_1\alpha_2-c^2d\alpha_1^2n_2)\\
&\quad +d(\mu+E_2q_2-\alpha_1c)^2(-3\alpha_2k\sigma_1\beta^2-c^2\alpha_1k+cd\alpha_1kn_2\beta-c\alpha_1\beta)+c^2d\alpha_1^3k\alpha^{\prime}\beta^2u\\
&\quad +3\alpha^{\prime 3}c^2d\alpha_1ku-4\alpha^{\prime}c^2d\alpha_1\alpha_2k\alpha^{\prime}\beta u-c^3d^2\alpha_1^2\alpha_2\beta n_1+\alpha^{\prime}c^3d^2\alpha_1^2n_1\\
&\quad +c^2d\alpha_1\alpha_2^2\alpha^{\prime}\beta^2r_1+3c^2d\alpha_1\alpha^{\prime 3}r_1-4c^2d\alpha_1\alpha_2\alpha^{\prime 3}\beta r_1+(\mu+E_2q_2-\alpha_1c)^3dk\sigma_1\beta,\\
a_4 &= cd^2\alpha_1\alpha_2^2\beta^3(\sigma_2k+kn_2\alpha^{\prime}-kr_2)+3d\alpha_1^2k\alpha^{\prime 2}\sigma_1\beta^3-cd\alpha_1\alpha_2^2k\alpha^{\prime}\sigma_1\beta^3-2c^2d\alpha_1\alpha_2\alpha^{\prime 3}\beta r_1\\
&\quad +bc^2d\alpha_1\alpha_2^2\beta^2+c^2d^2\alpha_1^2\alpha_2\sigma_2k\beta^2-k\alpha^{\prime}\beta^2(2cd^2\alpha_1\alpha_2\sigma_2-c^2d\alpha_1\alpha_2^2+2c^2d^2\alpha_1^2\alpha_2n_2)\\
&\quad -4cd^2\alpha_1\alpha_2kn_2\alpha^{\prime 2}\beta^2-c^2d^2\alpha_1^2\alpha_2kr_2\beta^2+2\alpha^{\prime}cd^2\alpha_1\alpha_2kr_2\beta^2-6d\alpha_2k\alpha^{\prime 3}\sigma_1\beta^2\\
&\quad +\alpha^{\prime 2}bc^2d\alpha_1+4\alpha^{\prime 2}cd\alpha_1\alpha_2k\sigma_1\beta^2-\alpha^{\prime}c^3d^2\alpha_1^2\sigma_2-3c^2d\alpha_1l\alpha^{\prime 3}-2\alpha^{\prime}bc^2d\alpha_1\alpha_2\beta\\
&\quad +c^3d^2\alpha_1^2\alpha_2\sigma_2\beta-\alpha^{\prime}c^2d^2\alpha_1^2\sigma_2k\beta+\alpha^{\prime 2}cd^2\alpha_1\sigma_2k\beta+4c^2d\alpha_1\alpha_2l\alpha^{\prime 2}\beta\\
&\quad -3c^2d^2\alpha_1^2kn_2\alpha^{\prime 2}\beta+3cd^2\alpha_1kn_2\alpha^{\prime 3}\beta+\alpha^{\prime}c^2d^2\alpha_1^2kr_2\beta-\alpha^{\prime 2}cd^2\alpha_1kr_2\beta+3dk\alpha^{\prime 4}\sigma_1\beta\\
&\quad -3cd\alpha_1k\alpha^{\prime 3}\sigma_1\beta+3c^2d\alpha_1k\alpha^{\prime 4}u-2c^2d\alpha_1\alpha_2k\alpha^{\prime 3}\beta u+c^2d^3\alpha_1^2\alpha_2k\beta^2vx^4\\
&\quad -\alpha^{\prime}c^2d^3\alpha_1^2k\beta v+c^3d^3\alpha_1\alpha_1^2k\beta v+4\alpha^{\prime}c^3d^2\alpha_1^2n_1-3c^3d^2\alpha_1^2\alpha_2\alpha^{\prime}\beta n_1+3c^2d\alpha_1\alpha^{\prime 4}r_1,\\
a_3 &= -cd^2\alpha_1\alpha_2\beta^2k\alpha^{\prime}(4\sigma_2\alpha^{\prime}-\alpha_2\sigma 2\beta+\alpha_2r_2\beta+2n_1\alpha^{\prime 2}-4\alpha^{\prime}r_2-c\alpha_1n_2\alpha^{\prime}-2\alpha_1\sigma_2\\
&\quad +2c\alpha_1r_2)-3c^2d\alpha_1\alpha^{\prime 4}+bc^2d\alpha_1\alpha_2^2\alpha^{\prime}\beta^2-3d\alpha_2k\alpha^{\prime 4}\beta^2+2cd\alpha_1\alpha_2k\alpha^{\prime 3}\sigma_1\beta^2\\
&\quad +3bc^2d\alpha_1\alpha^{\prime 3}-4c^3d^2\alpha_1^2\sigma_2\alpha^{\prime 2}+2c^2d\alpha_1\alpha_2k\alpha^{\prime 3}\beta-3c^2d^2\alpha_1^2kn_2\alpha^{\prime 3}\beta\\
&\quad +3cd^2\alpha_1kn_2\alpha^{\prime 4}\beta+3c^2d^2\alpha_1^2k\alpha^{\prime 2}r_2\beta-4bc^2d\alpha_1\alpha_2\alpha^{\prime 3}\beta+3c^3d^2\alpha_1^2\alpha_2\sigma_2\alpha^{\prime}\beta\\
&\quad -3c^2d^2\alpha_1^2\sigma_2k\alpha^{\prime 2}\beta+3cd^2\alpha_1\alpha_2k\alpha^{\prime 3}\beta+3dk\alpha^{\prime 5}\sigma_1\beta-3cd^2\alpha_1k\alpha^{\prime 3}r_2\beta\\
&\quad -3cd\alpha_1k\alpha^{\prime 4}\sigma_1\beta+c^2d\alpha_1k\alpha^{\prime 5}u+2c^2d^3\alpha_1^2\alpha_2k\alpha^{\prime}\beta^2v-3c^2d^3\alpha_1^2k\alpha^{\prime 2}\beta v\\
&\quad +3c^3d^3\alpha_1\alpha_1^2k\alpha^{\prime}\beta v+6c^3d^2\alpha_1^2\alpha^{\prime 3}n_1-3c^3d^2\alpha_1^2\alpha_2\alpha^{\prime 2}\beta n_1+a^2c^2d\alpha_1\alpha^{\prime 3}r_1,\\
a_2 &= -a^2c^2d\alpha_1\alpha^{\prime 3}+3bc^2d\alpha_1\alpha^{\prime 4}-6c^3d^2\alpha_1^2\sigma_2\alpha^{\prime 3}+c^2d^2\alpha_1^2\alpha_2\sigma_2k\alpha^{\prime 2}\beta^2-c^3d^2\alpha_1^2\alpha_1\alpha^{\prime 3}\beta n_1\\
&\quad -2cd^2\alpha_1\alpha_2\sigma_2k\alpha^{\prime 3}\beta^2-c^2d^2\alpha_1^2\alpha_2k\alpha^{\prime 2}r_2\beta^2+2cd^2\alpha_1\alpha_2k\alpha^{\prime 3}r_2\beta^2-c^2d^2\alpha_1^2kn_2\alpha^{\prime 4}\beta\\
&\quad +a^2cd^2\alpha_1kn_2\alpha^{\prime 3}\beta-2bc^2d\alpha_1\alpha_2\alpha^{\prime 3}\beta+3c^3d^2\alpha_1^2\alpha_2\sigma_2\alpha^{\prime 2}\beta x^2-3c^2d^2\alpha_1^2\sigma_2k\alpha^{\prime 3}\beta\\
&\quad +3cd^2\alpha_1\sigma_2k\alpha^{\prime 4}\beta+3c^2d^2\alpha_1^2k\alpha^{\prime 3}r_2\beta-3cd^2\alpha_1k\alpha^{\prime 2}r_2\beta+dk\alpha^{\prime 6}\sigma_1\beta-cd\alpha_1k\alpha^{\prime 5}\sigma_1\beta\\
&\quad +c^2d^3\alpha_1^2\alpha_2k\alpha^{\prime 2}\beta^2v-3c^2d^3\alpha_1^2k\alpha^{\prime 3}\beta v+3c^3d^3\alpha_1\alpha_1^2k\alpha^{\prime 2}\beta v+4c^3d^2\alpha_1^2\alpha^{\prime 4}n_1,\\
a_1 &= c^2d\alpha_1\alpha^{\prime 5}-4c^3d^2\alpha_1^2\sigma_2\alpha^{\prime 4}+c^3d^2\alpha_1^2\alpha_2\sigma_2-c^2d^2+\alpha^{\prime 2}cd^2+\alpha^{\prime}c^2d^2\alpha_1^2k\alpha^{\prime 3}r_2\beta\\
&\quad -cd^2\alpha_1k\alpha^{\prime 5}r_2\beta-c^2d^3\alpha_1^2k\alpha^{\prime 3}\beta v+c^3d^3\alpha_1\alpha_1^2k\alpha^{\prime 4}\beta v+ac^3d^2\alpha_1^2\alpha^{\prime 4},\\
a_0 &= -c^3d^2\alpha_1^2\sigma_2\alpha^{\prime 5}.
\end{aligned}
$$

As $a_0 < 0$ we get from Descartes criteria [10], the following diferents cases:

Coefficients	a_0	a_1	a_2	a_3	a_4	a_5	a_6
Choice 1	-	+	+	+	+	+	+
Choice 2	-	-	+	+	+	+	+
Choice 3	-	-	-	+	+	+	+
Choice 4	-	-	-	-	+	+	+
Choice 5	-	-	-	-	-	+	+
Choice 6	-	-	-	-	-	-	+

TABLE 3. Coefficients sign in the different cases respecting the criteria of Descartes.

So from the first case we impose that $a_i > 0$ for i=1,...,6.

-The endemic equilibrium point:

Using $G_i(x_4, y_4, S_4, I_4) = 0$ for $i = 1, \ldots, 4$, the endemic equilibrium point $P_4(x_4, y_4, S_4, I_4)$ is:

$$\begin{aligned} x_4 &= \frac{vy^2+cy(q_3E_3+\eta)-(r_2-\sigma_2)y(d+y)\delta}{(\sigma_1-n_2y)(d+y)\delta+cy\alpha_2\gamma}, \\ S_4 &= \frac{\eta+q_3E_3-\alpha_2\gamma x_4}{\delta} > 0 \quad \text{if} \quad x_4 < \frac{\eta+q_3E_3}{\alpha_2\gamma}, \\ I_4 &= \frac{1}{\delta}\left(\frac{\alpha_2\beta x_4}{\alpha'+x_4} - (\mu+q_2E_2) + \frac{\alpha_1 c y_4}{(d+y_4)}\right) > 0. \end{aligned} \tag{4.6}$$

Using the last expressions in the following equation:

$$r_1x\left(1-\frac{x}{K}\right) - \sigma_1 x + \sigma_2 y - ux^2 - \frac{\beta xS}{\alpha'+x} - q_1E_1x - n_1xy - \gamma xI = 0,$$

we get

$$a_7y^7 + a_6y^6 + a_5y^5 + a_4y^4 + a_3y^3 + a_3y^3 + a_1y + a_0 = 0,$$

where,

$$\begin{aligned} a_7 &= \alpha^{'}eo^3 + h^2oq + ho^2w, \\ a_6 &= \alpha^{'}eo^3 + eho^2 + gho^2 + 3\alpha^{'}eio^2 + \alpha^{'}hmo^2 + fh^2o + h^2mo + h^3p + h^2iq + 2hloq \\ &\quad + lo^2w + 2hiow, \end{aligned}$$

$$\begin{aligned}
a_5 &= dgho^2+3\alpha' eio^2+elo^2+glo^2+\alpha' lmo^2+fh^2i+h^2im+dfh^2o+3\alpha' ei^2o+2ehio\\
&\quad +2ghio+2fhlo+2\alpha' himo+2hlmo+dh^3p+3h^2lp+2hilq+l^2oq+3\alpha' deo^2s\\
&\quad +dh^2qs+hi^2w+2ilow+2dho\sigma_1 w,\\
a_4 &= \alpha' ei^3+ehi^2+ghi^2+dglo^2+dfh^2i+2fhil+\alpha' hi^2m+2hilm+3\alpha' ei^2o+fl^2o\\
&\quad +2dghio+2dfhlo+2eilo+2gilo+l^2mo+2\alpha' ilmo+3hl^2p+3dh^2lp+il^2q\\
&\quad +dfh^2s+3\alpha' deo^2s+dh^2m\sigma_1+2dehos+2dghos+6\alpha' deios+2\alpha' dhmos\\
&\quad +2dhlqs+i^2lw+2dhi\sigma_1 w+2dlo\sigma_1 w,\\
a_3 &= \alpha' ei^3+dghi^2+fil^2+3\alpha' d^2eos^2+ei^2l+gi^2l+2dfhil+il^2m+\alpha' i^2lm+dfl^2o\\
&\quad +l^3p+2dgilo+3dhl^2p+d^2fh^2\sigma_1+3\alpha' dei^2\sigma_1+2dehi\sigma_1+2dghi\sigma_1+2dfhl\sigma_1\\
&\quad +2\alpha' dhim\sigma_1+2dhlm\sigma_1+2d^2gho\sigma_1+6\alpha' deio\sigma_1+2delo\sigma_1+2dglo\sigma_1\\
&\quad +2\alpha' dlmo\sigma_1+dl^2q\sigma_1+d^2hs^2w+2dil\sigma_1 w,\\
a_2 &= dfil^2+d^2eh\sigma_1^2+d^2gh\sigma_1^2+3\alpha' d^2ei\sigma_1^2+\alpha' d^2hm\sigma_1^2+3\alpha' d^2eo\sigma_1^2+dgi^2l+dl^3p\\
&\quad +3\alpha' dei^2\sigma_1+dfl^2\sigma_1+2d^2ghi\sigma_1+2d^2fhl\sigma_1+2deil\sigma_1+2dgil\sigma_1+dl^2m\sigma_1\\
&\quad +2\alpha' dilm\sigma_1+2d^2glo\sigma_1+d^2l\sigma_1^2w,\\
a_1 &= \alpha' d^3e\sigma_1^3+d^3gh\sigma_1^2+3\alpha' d^2ei\sigma_1^2+d^2el\sigma_1^2+d^2gl\sigma_1^2+\alpha' d^2lm\sigma_1^2+d^2fl^2\sigma_1\\
&\quad +2d^2gil\sigma_1,\\
a_0 &= -\alpha' d^4K\sigma_2\delta.
\end{aligned}$$

For,

$$\begin{aligned}
f &= r_1\delta K-\alpha' r_1\delta-\sigma_1\delta K-\alpha' u\delta K+\beta\alpha_2\gamma K-q_1\delta E_1K-\gamma\alpha_2\beta K-(\mu+q_2E_2)\gamma K,\\
g &= r_1\delta\alpha' K-\sigma_1\delta\alpha' K-\beta(q_3E_3+\eta)K-q_1\delta E_1\alpha' K-(\mu+q_2E_2)\alpha' K,\\
p &= -(r_1+uK)\delta,\\
m &= -n_1\delta Kd-\gamma K\alpha_1 c,\\
w &= -\sigma_2\delta K-n_1\delta\alpha K,\\
e &= -K\sigma_2\delta d,\\
q &= n_1\delta K,\\
h &= v-(r_2-\sigma_2)\delta,\\
l &= c(q_3E_3+\eta)-(r_2-\sigma_2)d\delta,\\
o &= -n_2\delta,\\
i &= \sigma_1\delta-n_2d\delta+c\alpha_2\gamma.
\end{aligned}$$

As $a_0 < 0$ we get from Descartes criteria [10], the following diferents cases:

Coefficients	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7
Choice 1	-	+	+	+	+	+	+	+
Choice 2	-	-	+	+	+	+	+	+
Choice 3	-	-	-	+	+	+	+	+
Choice 4	-	-	-	-	+	+	+	+
Choice 5	-	-	-	-	-	+	+	+
Choice 6	-	-	-	-	-	-	+	+
Choice 7	-	-	-	-	-	-	-	+

TABLE 4. Coefficients sign in the different cases respecting the criteria of Descartes.

Then we impose that $a_j > 0$ for j=1, …,7.

5. STABILITY ANALYSIS

In this part, the Jacobian matrix and the traditional Lyapunov technique will be utilized to determine the local and global stability of the equilibriums.

5.1. Local stability.

The Jacobian matrix is:

$$J(X) = \begin{pmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & 0 \\ J_{31} & J_{32} & J_{33} & J_{34} \\ J_{41} & 0 & J_{43} & J_{44} \end{pmatrix}, \tag{5.1}$$

where

$$J_{11} = r_1 - \sigma_1 - q_1E_1 - 2(\tfrac{r_1}{k} + u)x + n_1y(\gamma_1 I + \tfrac{\beta S\alpha_\prime}{(\alpha_\prime + x)^2}), \quad J_{12} = \sigma_2 - n_1x,$$
$$J_{13} = \tfrac{-\beta x}{\alpha_\prime + x}, \quad J_{14} = -\gamma x, \quad J_{21} = \sigma_1 - n_2y, \quad J_{22} = r_2 - \sigma_2 - 2vy - n_2x - \tfrac{cSd}{(d+y)^2},$$
$$J_{23} = \tfrac{cy}{(d+y)}, \quad J_{31} = \tfrac{\alpha_2\beta S\alpha_\prime}{(\alpha_\prime + x)^2}, \quad J_{32} = \tfrac{\alpha_1 cSd}{d+y}, \quad J_{33} = \tfrac{\alpha_2\beta x}{\alpha' + x} - \delta I - \mu - q_2E_2 + \tfrac{\alpha_1 cy}{d+y},$$
$$J_{34} = -\delta S, \quad J_{41} = \alpha_2\gamma I, \quad J_{43} = \delta I, \quad J_{44} = \delta S + \alpha_2\gamma x - q_3E_3 - \eta.$$

Proposition 5.1. *The trivial equilibrium $P_0(0,0,0,0)$ is unstable.*

Proof. From (5.1), the $J(0,0,0,0)$ characteristic equation is:

$$(q_3E_3+\eta\lambda)(q_2E_2+\mu+\lambda)(\lambda^2-b\lambda+c)=0,$$

with $b=(r_1-\sigma_1+r_2-\sigma_2-q_1E_1)$ and $c=(r_2-\sigma_2)(r_1-\sigma_1-q_1E_1)-\sigma_2\sigma_1$. Therefore the eigenvalues λ_1 and λ_2 of $J(0,0,0,0)$ are negatives, however $\lambda_3+\lambda_4=-q_1E_1-\sigma_1-\sigma_2+r_2+r_1$ is positive. Then one of the eigenvalues doesn't verify the condition of Matignon [35]. So, $P_0(0,0,0,0)$ is note stable. □

Proposition 5.2. *The system (3.1) equilibrium point $P_1(x_1,y_1,0,0)$ is locally asymptotically stable if* $x_1<\min\left(\frac{\eta+q_3E_3}{\alpha_2\gamma},\frac{\alpha'(\mu+q_2E_2)(d+y_1)-\alpha'\alpha_1cy_1}{\alpha_2\beta(d+y_1)-(\mu+q_2E_2)(d+y_1)+\alpha_1cy_1}\right)$ *and* $|\arg(\lambda_{3,4})|>\frac{\alpha\pi}{2}$.

Proof. From (5.1) the $J(x_1,y_1,0,0)$ characteristic equation is:

$$(\lambda^2+S\lambda+P)(\lambda-\alpha_2\gamma x_1+\eta+q_3E_3)(\lambda-\frac{\alpha_2\beta x_1}{\alpha'+x_1}+\mu+q_2E_2-\frac{\alpha_1cy_1}{d+y_1})=0.$$

Where

$$\begin{aligned} S &= \left(\sigma_2\frac{y_1}{x_1}+(u+\frac{r_1}{K})x_1+\sigma_1\frac{x_1}{y_1}+vy_1\right), \\ P &= \left(\sigma_2\frac{y_1}{x_1}+(u+\frac{r_1}{K})x_1\right)\left(\sigma_1\frac{x_1}{y_1}+vy_1\right)-(\sigma_2-n_1x_1)(\sigma_1-n_2y_1). \end{aligned}$$

Therefore, the first and second eigenvalues are:

$$\begin{aligned} \lambda_1 &= \alpha_2\gamma x_1-(\eta+q_3E_3)<0 \text{ if } x_1<\tfrac{\eta+q_3E_3}{\alpha_2\gamma}, \text{ then } |\arg(\lambda_1)|=\pi>\tfrac{\alpha\pi}{2}. \\ \lambda_2 &= \tfrac{\alpha_2\beta x_1}{\alpha'+x_1}-(\mu+q_2E_2)+\tfrac{\alpha_1cy_1}{d+y_1}<0 \text{ if } x_1<\tfrac{\alpha'(\mu+q_2E_2)(d+y_1)-\alpha'\alpha_1cy_1}{\alpha_2\beta(d+y_1)-(\mu+q_2E_2)(d+y_1)+\alpha_1cy_1}, \text{ then} \\ & \quad |\arg(\lambda_2)|=\pi>\tfrac{\alpha\pi}{2}. \end{aligned}$$

For $\lambda^2+S\lambda+P=0$ if $\Delta=S^2-4P>0$, λ_3 and λ_4 are purely real and negative.
If $\Delta<0$, λ_3 and λ_4 are complex number. So if $|\arg(\lambda_{3,4})|=\tan^{-1}\left(\frac{\sqrt{-\Delta}}{s}\right)>\frac{\alpha\pi}{2}$, P_1 is locally asymptotically stable. □

Proposition 5.3. *The equilibrium point $P_2(x_2,y_2,0,I_2)$ is locally asymptotically stable if* $x_2<\frac{\alpha'((\delta I_2+\mu+q_2E_2)(d+y_2)-\alpha_1cy_2)}{(\alpha_2\beta+\alpha_1cy_2-(\delta I_2+\mu+q_2E_2))(d+y_2)}$ *and the conditions of (5.2) are satisfied.*

Proof. From the Jacobian matrix $J(x_2,y_2,0,I_2)$, the characteristic equation at P_2 is:

$$\left(\lambda-(\frac{\alpha_2\beta x_2}{\alpha'+x_2}+\frac{\alpha_1cy_2}{d+y_2}-(\delta I_2+\mu+q_2E_2))\right)(\lambda^3+e_2\lambda^2+e_1\lambda+e_0)=0,$$

where:

$$
\begin{aligned}
e_2 &= -(r_1-\sigma_1-q_1E_1-2\left(\tfrac{r_1+Ku}{K}\right)x_2-n_1y_2-\gamma I_2+r_2-\sigma_2-2vy_2-n_2x_2),\\
e_1 &= (r_1-\sigma_1-q_1E_1-2\left(\tfrac{r_1+Ku}{K}\right)x_2-n_1y_2-\gamma I_2)(r_2-\sigma_2-2vy_2-n_2x_2)\\
&\quad -(\sigma_1-n_2y_2)(\sigma_2-n_1x_2)+\alpha_2\gamma^2 I_2x_2,\\
e_0 &= -V\alpha_2\gamma^2 I_2x_2,
\end{aligned}
$$

So $\lambda_1=\frac{\alpha_2\beta x_2}{\alpha'+x_2}+\frac{\alpha_1 cy_2}{d+y_2}-(\delta I_2+\mu+q_2E_2)<0$ if $x_2<\frac{\alpha'((\delta I_2+\mu+q_2E_2)(d+y_2)-\alpha_1 cy_2)}{(\alpha_2\beta+\alpha_1 cy_2-(\delta I_2+\mu+q_2E_2))(d+y_2)}$, then $|\arg(\lambda_1)|=\pi>\frac{\alpha\pi}{2}$.

The discriminant of $P(\lambda)=\lambda^3+e_2\lambda^2+e_1\lambda+e_0$ is defined in this form [2]:

$$\Delta(P)=18e_1e_2e_0+(e_2e_1)^2-4e_1^3-4e_2^3e_0-27e_0^2.$$

And then we get the following cases:

(5.2)

For $\Delta(P)>0$, P_2 is asymptotically stable if $e_0, e_1, e_2>0$ and $e_2e_1-e_0>0, \forall\alpha\in[0,1[$,

for $\Delta(P)<0$ and $e_0, e_1, e_2>0$, P_2 is asymptotically stable if $\alpha<\frac{2}{3}$ and $e_2e_1-e_0>0$,

for $\Delta(P)<0$, $e_0, e_1, e_2>0$ and $e_2e_1=e_0$, P_2 is asymptotically stable $\forall\alpha\in[0,1[$.

□

Proposition 5.4. *The equilibrium point $P_3(x_3,y_3,S_3,0)$ is locally asymptotically stable if $\delta S_3+\alpha_2\gamma x_3<\eta+q_3E_3$ and (5.3) are verified.*

Proof. The $J(x_3,y_3,S_3,0)$ characteristic equation is:

$$(\lambda-(\delta S_3+\alpha_2\gamma x_3-\eta-q_3E_3))\,(\lambda^3+f_2\lambda^2+f_1\lambda+f_0)=0,$$

where:

$$
\begin{aligned}
f_2 &= -(U+V+W),\\
f_1 &= UV-(\sigma_1-n_2y_3)(\sigma_2-n_1x_3)+\tfrac{\alpha_2\beta^2\alpha' x_3S_3}{(\alpha'+x_3)^3}+WV-WU+\tfrac{W^2y_3\alpha_1S_3d}{(d+y)^2},\\
f_0 &= -V\tfrac{\alpha_2\beta^2\alpha' x_3S_3}{(\alpha'+x_3)^3}-UVW+W(\sigma_1-n_2y_3)(\sigma_2-n_1x_3)+\tfrac{c^2\alpha_1dU^2y_3S_3}{(d+y_3)^2}+\tfrac{c(\sigma_2-n_1x_3)\alpha_2\beta\alpha' S_3y_3}{(d+y_3)(\alpha'+x_3)^2}\\
&\quad -\tfrac{\beta(\sigma_1-n_2y_3)\alpha_1 cdS_3x_3}{(d+y_3)(\alpha'+x_3)},\\
U &= r_1-\sigma_1-q_1E_1-2\tfrac{r_1+Ku}{K}x_3-n_1y_3-\tfrac{\beta\alpha' S_3}{(\alpha'+x_3)^2},\\
V &= r_2-\sigma_2-2vy_3-n_2x_3-\tfrac{cS_3d}{(d+y_3)^2},\\
W &= \tfrac{\alpha_1 cy_3}{d+y_3}+\tfrac{\alpha_2\beta x_3}{\alpha'+x_3}-\mu-q_2E_2.
\end{aligned}
$$

The first eigenvalues $\lambda_1=\delta S_3+\alpha_2\gamma x_3-\eta-q_3E_3<0$ if $\delta S_3+\alpha_2\gamma x_3<\eta+q_3E$, then $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$.

The discriminant of the polynomial $Q(\lambda) = \lambda^3 + f_2\lambda^2 + f_1\lambda + f_0$ is defined in this form [2]:

$$\Delta(Q) = 18 f_1 f_2 f_0 + (f_2 f_1)^2 - 4f_1^3 - 4f_2^3 f_0 - 27 f_0^2.$$

Supposing that $f_j > 0$ for $j = 0, \dots, 3$, then:

$$\begin{array}{lll} & (1) & \text{For } \Delta(Q) > 0,\ P_3 \text{ is asymptotically stable if } f_2 f_1 - f_0 > 0, \forall \alpha \in [0,1[, \\ (5.3) & (2) & \text{for } \Delta(Q) < 0\ \ P_3 \text{ is asymptotically stable if } \alpha < \frac{2}{3} \text{ and } f_2 f_1 - f_0 > 0, \\ & (3) & \text{for } \Delta(Q) < 0 \text{ and }\ f_2 f_1 = f_0,\ P_3 \text{ is asymptotically stable } \forall \alpha \in [0,1[. \end{array}$$

□

Proposition 5.5. *The equilibrium $P_4(x_4, y_4, S_4, I_4)$ is locally asymptotically stable if $\phi_0, \phi_1, \phi_2, \phi_3 > 0$ and $\phi_2\phi_3 - \phi_1 > \frac{\phi_0\phi_3^2}{\phi_1}$.*

Proof. From the Jacobian matrix $J(x_4, y_4, S_4, I_4)$, the P_4 characteristic equation is:

$$R(\lambda) = \lambda^4 + \Phi_3\lambda^3 + \Phi_2\lambda^2 + \Phi_1\lambda + \Phi_0,$$

where:

$$\begin{aligned}
\Phi_3 &= -(a_6 + f_6), \\
\Phi_2 &= a_6 f_6 - b_6 e_6 - c_6 g_6 - d_6 i_6 - h_6 j_6, \\
\Phi_1 &= a_6 h_6 j_6 + c_6 f_6 g_6 + d_6 f_6 i_6 + f_6 h_6 j_6 - i_6 c_6 h_6 - d_6 g_6 j_6, \\
\Phi_0 &= j_6 b_6 e_6 h_6 + i_6 c_6 f_6 h_6 + d_6 f_6 g_6 j_6 - j_6 a_6 f_6 h_6, \\
a_6 &= r_1 - \sigma_1 - q_1 E_1 - 2(\tfrac{r_1}{K} + u)x_4 - n_1 y_4 - \gamma I_4 - \tfrac{\beta\alpha' S_4}{(\alpha'+x_4)^2} - \gamma I_4, \\
b_6 &= \sigma_2 - n_1 x_4, \\
c_6 &= \tfrac{-\beta x_4}{\alpha'+x_4}, \\
d_6 &= -\gamma x_4, \\
e_6 &= \sigma_1 - n_2 y_4, \\
f_6 &= r_2 - \sigma_2 - 2v y_4 - n_2 x_4, \\
g_6 &= \tfrac{\alpha_2\beta\alpha' S_4}{(\alpha'+x_4)^2}, \\
h_6 &= -\delta S_4, \\
i_6 &= \alpha_2\gamma I_4, \\
j_6 &= \delta I_4.
\end{aligned}$$

Let $\Delta(R)$ the discriminant, where:

$$\begin{aligned}\Delta(R) &= 256\phi_0^3 - 192\phi_3\phi_1\phi_0^2 - 128\phi_2^2\phi_0 + 144\phi_2\phi_1^2\phi_0 - 27\phi_1^4 + 144\phi_3^2\phi_2\phi_0^2 \\ &\quad -6\phi_3^2\phi_1^2\phi_0 - 80\phi_3\phi_2^2\phi_1\phi_0 + 18\phi_3\phi_2\phi_1^3 + 16\phi_2^4\phi_0 - 4\phi_2^3\phi_1^2 - 27\phi_3^4\phi_0^2 \\ &\quad +18\phi_3^3\phi_2\phi_1\phi_0 - 4\phi_3^3\phi_1^3 - 4\phi_3^2\phi_2^3\phi_0 + (\phi_3\phi_2\phi_1)^2.\end{aligned} \tag{5.4}$$

Using [2] results and supposing that $\phi_j > 0$ for $j = 0, \dots, 3$,

(5.5)

(1) For $\Delta(R) > 0$ and $\phi_2\phi_3 - \phi_1 > \frac{\phi_0\phi_3^2}{\phi_1}$, P_4 is locally asymptotically stable for all $\alpha \in [0,1[$.

(2) For $\Delta(R) < 0$ and $\alpha < \frac{1}{3}$, P_4 is locally asymptotically stable.

(3) For $\Delta(R) < 0$, and $\phi_2 = \frac{\phi_3\phi_0}{\phi_1} + \frac{\phi_1}{\phi_3}$, P_4 is locally asymptotically stable for all $\alpha \in [0,1[$.

□

5.2. Global stability.

In this part, the global stability of the equilibriums will be examined using the Lyapunov method.

Proposition 5.6. *The equilibrium $P_1(x_1, y_1, 0, 0)$ is globally asymptotically stable if* $\frac{\sigma_2\alpha_2\alpha\prime}{x_1(\alpha\prime+x_1)} + \frac{\sigma_1\alpha_1 d}{y_1(d+y_1)} \le \frac{\alpha_2\alpha'}{\alpha'+x_1}n_1 + \frac{\alpha_1 d}{d+y_1}n_2 \le 2min(\frac{\alpha_1 dv}{d+y_1}, \frac{\alpha_2\alpha'(r_1+ku)}{k(\alpha'+x_1)})$.

Proof. We consider $V_1(x,y,S,I)$, the Lyapunov function:

$V_1(x,y,S,I) = \left(\frac{\alpha_2\alpha'}{\alpha'+x_1}\right)\left(x - x_1 - x_1\ln\left(\frac{x}{x_1}\right)\right) + \left(\frac{\alpha_1 d}{d+y_1}\right)\left(y - y_1 - y_1\ln\left(\frac{y}{y_1}\right)\right) + S + \left(\frac{\alpha'}{\alpha'+x_1}\right)I$,

we set $a = \frac{\alpha_2\alpha'}{\alpha'+x_1}$, $b = \frac{\alpha_1 d}{d+y_1}$ and $w = \frac{\alpha'}{\alpha'+x_1}$. From Lemma 2.4 we get:

$D^\alpha V_1 < a\frac{x-x_1}{x}D^\alpha x + b\frac{y-y_1}{y}D^\alpha y + D^\alpha S + wD^\alpha I$. Using (3.1), we remplace $D^\alpha x$, $D^\alpha y$, $D^\alpha S$ and $D^\alpha I$ with there expression to get:

$$\begin{aligned}D^\alpha V_1 &< a(x-x_1)(r_1\left(1-\tfrac{x}{K}\right) - \sigma_1 + \sigma_2\tfrac{y}{x} - ux - \tfrac{\beta S}{\alpha'+x} - q_1E_1 - n_1y - \gamma I) \\ &\quad +b(y-y_1)(r_2 - \sigma_2 + \sigma_1\tfrac{x}{y} - \tfrac{cs}{d+y} - vy - n_2x) + (\tfrac{\alpha_1 cy}{d+y} + \tfrac{\alpha_2\beta x}{\alpha'+x} - \delta I - \mu - q_2E_2)S \\ &\quad +w(\delta S + \alpha_2\gamma x - q_3E_3 - \eta)I,\end{aligned}$$

since $w = \frac{\alpha'}{\alpha'+x_1} \le 1$, we get $w\delta SI - \delta SI < 0$. Then:

$$\begin{aligned}D^\alpha V_1 &< -a\left(\tfrac{r_1+Ku}{K}\right)(x-x_1)^2 - vb(y-y_1)^2 - (an_1+bn_2)(x-x_1)(y-y_1) \\ &\quad +a\sigma_2(\tfrac{y}{x} - \tfrac{y_1}{x_1})(x-x_1) + b\sigma_1(\tfrac{x}{y} - \tfrac{x_1}{y_1})(y-y_1) - \tfrac{a\beta S}{\alpha'+x}(x-x_1) - a\gamma I(x-x_1) \\ &\quad -\tfrac{bcS}{d+y}(y-y_1) + (\tfrac{\alpha_1 cy}{d+y} + \tfrac{\alpha_2\beta x}{\alpha'+x} - \mu - q_2E_2)S + w(\alpha_2\gamma x - q_3E_3 - \eta)I,\end{aligned}$$

as $\quad \mu+q_2E_2 \geq \frac{\alpha_1 c y_1}{d+y_1}+\frac{\alpha_2\beta x_1}{\alpha'+x_1}$ and $q_3E_3+\eta \geq \alpha_2\gamma x_1$ (see Proposition 5.2),

we get

$$
\begin{aligned}
D^\alpha V_1 \;<\; & -a\left(\frac{r_1+Ku}{K}\right)(x-x_1)^2-bv(y-y_1)^2-\frac{a\sigma_2 y}{x_1 x}(x-x_1)^2-\frac{b\sigma_1 x}{y_1 y}(y-y_1)^2\\
& -(an_1+bn_2-\frac{a\sigma_2}{x_1}-\frac{b\sigma_1}{y_1})(x-x_1)(y-y_1)-\frac{a\beta S}{\alpha'+x}(x-x_1)-a\gamma I(x-x_1)\\
& -\frac{bcS}{d+y}(y-y_1)+\frac{d\alpha_1 c(y-y_1)S}{(d+y)(d-y_1)}+\frac{\alpha'\alpha_2\beta(x-x_1)S}{(\alpha'+x)(\alpha'+x_1)}+w\alpha_2\gamma I(x-x_1).
\end{aligned}
$$

We replace a, b and w with there expressions. Supposing that

$an_1+bn_2-\frac{a\sigma_2}{x_1}-\frac{b\sigma_1}{y_1}\geq 0$, we find,

$$
\begin{aligned}
D^\alpha V_1 \;<\; & -\left(\frac{a(r_1+Ku)}{K}-\frac{an_1+bn_2}{2}\right)(x-x_1)^2-(bv-\frac{an_1+bn_2}{2})(y-y_1)^2\\
& -\frac{a\sigma_2 y}{x_1 x}(x-x_1)^2-\frac{b\sigma_1 x}{y_1 y}(y-y_1)^2-\frac{1}{2}(\frac{a\sigma_2}{x_1}+\frac{b\sigma_1}{y_1})((x-x_1)^2+(y-y_1)^2).
\end{aligned}
$$

Therefore, $D^\alpha V_1<0$ if $\frac{a\sigma_2}{x_1}+\frac{b\sigma_1}{y_1}\leq an_1+bn_2\leq 2min(bv,a\frac{r_1+ku}{k})$.

□

Proposition 5.7. *The equilibrium $P_2(x_2,y_2,0,I_2)$ is globally asymptotically stable if* $\frac{\alpha_2\alpha'\sigma_2}{(\alpha'+x_2)x_2}+\frac{\alpha_1 d\sigma_1}{(d+y_2)y_2}\leq\frac{\alpha_2\alpha'}{\alpha'+x_2}n_1+\frac{\alpha_1 d}{d+y_2}n_2\leq 2min(v,\frac{\alpha_2\alpha'}{\alpha'+x_2}\left(\frac{r_1+Ku}{K}\right))$.

Proof. We consider $V_2(x,y,S,I)$ the positive definite Lyapunov function, where:

$$
\begin{aligned}
V_2 \;=\; & \left(\frac{\alpha_2\alpha'}{\alpha'+x_2}\right)\left(x-x_2-x_2\ln\left(\frac{x}{x_2}\right)\right)+\left(\frac{\alpha_1 d}{d+y_2}\right)\left(y-y_2-y_2\ln\left(\frac{y}{y_2}\right)\right)+S\\
& +\left(\frac{\alpha'}{\alpha'+x_2}\right)\left(I-I_2-I_2\ln\left(\frac{I}{I_2}\right)\right).
\end{aligned}
$$

Using Lemma 2.4 we get:

$$
\begin{aligned}
D^\alpha V_2 \;<\; & \frac{\alpha_2\alpha'}{\alpha'+x_2}\frac{x-x_2}{x}D^\alpha x+\frac{\alpha_1 d}{d+y_2}\frac{y-y_2}{y}D^\alpha y+\frac{\alpha'}{\alpha'+x_2}\frac{I-I_2}{I}D^\alpha I+D^\alpha S,\\
D^\alpha V_2 \;<\; & \frac{\alpha_2\alpha'}{\alpha'+x_2}(x-x_2)(r_1\left(1-\frac{x}{K}\right)-\sigma_1+\sigma_2\frac{y}{x}-ux-\frac{\beta S}{\alpha'+x}-q_1E_1-n_1y-\gamma I)\\
& +\frac{\alpha_1 d}{d+y_2}(y-y_2)(r_2-\sigma_2+\sigma_1\frac{x}{y}-\frac{cs}{d+y}-vy-n_2x)\\
& +\frac{\alpha'}{\alpha'+x_2}(I-I_2)(\delta S+\alpha_2\gamma x-q_3E_3-\eta)\\
& +(\frac{\alpha_1 cy}{d+y}+\frac{\alpha_2\beta x}{\alpha'+x}-\delta I-\mu-q_2E_2)S,
\end{aligned}
$$

as $-\delta I\leq-\frac{\alpha'}{\alpha'+x_2}\delta I$, by using (3.3) we have $\mu+q_2E_2\geq\frac{\alpha_1 cy_2}{d+y_2}+\frac{\alpha_2\beta x_2}{\alpha'+x_2}-\frac{\alpha'}{\alpha'+x_2}\delta I_2$, so we get:

$$
\begin{aligned}
D^\alpha V_2 \;<\; & -\frac{\alpha_2\alpha'}{\alpha'+x_2}\left(\frac{r_1+Ku}{K}\right)(x-x_2)^2+\frac{\alpha_2\alpha'}{\alpha'+x_2}\sigma_2(\frac{y}{x}-\frac{y_2}{x_2})(x-x_2)\\
& -(\frac{\alpha_2\alpha'}{\alpha'+x_2}n_1+\frac{\alpha_1 d}{d+y_2}n2)(x-x_2)(y-y_2)-\frac{\alpha_2\alpha'}{\alpha'+x_2}\gamma(I-I_2)(x-x_2)\\
& -\frac{\alpha_2\alpha'\beta S}{(\alpha'+x_2)(\alpha'+x)}(x-x_2)+\frac{\alpha_1 d}{d+y_2}\sigma_1(\frac{x}{y}-\frac{x_2}{y_2})(y-y_2)-\frac{\alpha_1 d}{d+y_2}v(y-y_2)^2
\end{aligned}
$$

$$-\frac{\alpha_1 dcS}{(d+y_2)(d+y)}(y-y_2)+\frac{\alpha^{'}}{\alpha^{'}+x_2}\delta(I-I_2)S+\frac{\alpha^{'}}{\alpha^{'}+x_2}\alpha_2\gamma(x-x_2)(I-I_2)$$
$$+\alpha_1 cS(\frac{y}{d+y}-\frac{y_2}{d+y_2})+\alpha_2\beta S(\frac{x}{\alpha^{'}+x}-\frac{x_2}{\alpha^{'}+x_2})-\frac{\alpha^{'}}{\alpha^{'}+x_2}\delta S(I-I_2),$$

it's easy to show that ,

$\alpha_1 cS(\frac{y}{d+y}-\frac{y_2}{d+y_2})+\alpha_2\beta S(\frac{x}{\alpha^{'}+x}-\frac{x_2}{\alpha^{'}+x_2})=\frac{\alpha_1 dcS}{(d+y_2)(d+y)}(y-y_2)+\frac{\alpha_2\alpha^{'}\beta S}{(\alpha^{'}+x_2)(\alpha^{'}+x)}(x-x_2).$

So we get,

$$\begin{aligned} D^{\alpha}V_2 &< -\frac{\alpha_2\alpha^{'}(r_1+Ku)}{K(\alpha^{'}+x_2)}(x-x_2)^2-v\frac{\alpha_1 d}{d+y_2}(y-y_2)^2-\frac{\alpha_2\alpha^{'}\sigma_2 y}{(\alpha^{'}+x_2)x_2x}(x-x_2)^2-\frac{\alpha_1 d\sigma_1 x}{(d+y_2)y_2y}(y-y_2)^2 \\ &\quad -(\frac{\alpha_2\alpha^{'}}{\alpha^{'}+x_2}n_1+\frac{\alpha_1 d}{d+y_2}n_2-\frac{\alpha_2\alpha^{'}\sigma_2}{(\alpha^{'}+x_2)x_2}-\frac{\alpha_1 d\sigma_1}{(d+y_2)y_2})(x-x_2)(y-y_2), \end{aligned}$$

let $\frac{\alpha_2\alpha^{'}}{\alpha^{'}+x_2}n_1+\frac{\alpha_1 d}{d+y_2}n_2-\frac{\alpha_2\alpha^{'}\sigma_2}{(\alpha^{'}+x_2)x_2}-\frac{\alpha_1 d\sigma_1}{(d+y_2)y_2}\geq 0$ we find,

$$\begin{aligned} D^{\alpha}V_2 &< -\left(\frac{\alpha_2\alpha^{'}}{\alpha^{'}+x_2}\left(\frac{r_1+Ku}{K}\right)-\frac{1}{2}(\frac{\alpha_2\alpha^{'}}{\alpha^{'}+x_2}n_1+\frac{\alpha_1 d}{d+y_2}n_2)\right)(x-x_2)^2 \\ &\quad -(v-\frac{1}{2}(\frac{\alpha_2\alpha^{'}}{\alpha^{'}+x_2}n_1+\frac{\alpha_1 d}{d+y_2}n_2))(y-y_2)^2-\frac{\alpha_2\alpha^{'}\sigma_2 y}{(\alpha^{'}+x_2)x_2x}(x-x_2)^2 \\ &\quad -\frac{\alpha_1 d\sigma_1 x}{(d+y_2)y_1y}(y-y_2)^2-\frac{1}{2}(\frac{\alpha_2\alpha^{'}\sigma_2}{(\alpha^{'}+x_2)x_2}+\frac{\alpha_1 d\sigma_1}{(d+y_2)y_2})((x-x_2)^2+(y-y_2)^2). \end{aligned}$$

Therefore, $D^{\alpha}V_2<0$ if $\frac{\alpha_2\alpha^{'}\sigma_2}{(\alpha^{'}+x_2)x_2}+\frac{\alpha_1 d\sigma_1}{(d+y_2)y_2}\leq\frac{\alpha_2\alpha^{'}}{\alpha^{'}+x_2}n_1+\frac{\alpha_1 d}{d+y_2}n_2\leq 2min(v,\frac{\alpha_2\alpha^{'}}{\alpha^{'}+x_2}\left(\frac{r_1+Ku}{K}\right)).$

□

Proposition 5.8. *The equilibrium $P_3(x_3,y_3,S_3,0)$ is globally asymptotically stable if* $\frac{\alpha_2\alpha^{'}\sigma_2}{(\alpha^{'}+x_3)x_3}+\frac{\alpha_1 d\sigma_1}{(d+y_3)y_3}\leq\frac{\alpha_2\alpha^{'}}{\alpha^{'}+x_3}n_1+\frac{\alpha_1 d}{d+y_3}n_2\leq 2min(v,\frac{\alpha_2\alpha^{'}}{\alpha^{'}+x_3}\left(\frac{r_1+Ku}{K}\right)).$

Proof. Take the positive definite Lyapunov function:

$$V_3(x,y,S,I)=a\left(x-x_3-x_3\ln\left(\frac{x}{x_3}\right)\right)+b\left(y-y_3-y_3\ln\left(\frac{y}{y_3}\right)\right)+\left(S-S_3-S_3\ln\left(\frac{S}{S_3}\right)\right)+I,$$

with $a=\frac{\alpha_2\alpha^{'}}{\alpha^{'}+x_3}\leq\alpha_2$ and $b=\frac{\alpha_1 d}{d+y_3}\leq\alpha_1$. Using Lemma 2.4 we get:

$D^{\alpha}V_3<a\frac{x-x_3}{x}D^{\alpha}x+b\frac{y-y_3}{y}D^{\alpha}y+\frac{S-S_3}{S}D^{\alpha}S+D^{\alpha}I$, and from (3.1), we remplace $D^{\alpha}x$, $D^{\alpha}y$, $D^{\alpha}S$ and $D^{\alpha}I$ with there expressions:

$$\begin{aligned} D^{\alpha}V_3 &< a(x-x_3)(r_1\left(1-\frac{x}{K}\right)-\sigma_1+\sigma_2\frac{y}{x}-ux-\frac{\beta S}{\alpha'+x}-q_1E_1-n_1y-\gamma I) \\ &\quad +b(y-y_3)(r_2-\sigma_2+\sigma_1\frac{x}{y}-\frac{cs}{d+y}-vy-n_2x) \\ &\quad +(S-S_3)(\frac{\alpha_1 cy}{d+y}+\frac{\alpha_2\beta x}{\alpha'+x}-\delta I-\mu-q_2E_2)+(\delta S+\alpha_2\gamma x-q_3E_3-\eta)I, \end{aligned}$$

we have $\eta+q_3E_3\geq\delta S_3+\alpha_2\gamma x_3$ (see Proposition 5.4), so we get:

$$\begin{aligned}
D^{\alpha}V_3 &< -a\left(\tfrac{r_1+Ku}{K}\right)(x-x_3)^2 + a\sigma_2(\tfrac{y}{x}-\tfrac{y_3}{x_3})(x-x_3) - (an_1+bn_2)(x-x_3)(y-y_3)\\
&\quad -a\gamma I(x-x_3) - a\beta(\tfrac{S}{(\alpha'+x)} - \tfrac{S_3}{(\alpha'+x_3)})(x-x_3) + b\sigma_1(\tfrac{x}{y}-\tfrac{x_3}{y_3})(y-y_3)\\
&\quad -bv(y-y_3)^2 - bc(\tfrac{S}{d+y}-\tfrac{S_3}{d+y_3})(y-y_3) + \delta(S-S_3)I + \alpha_2\gamma(x-x_3)I\\
&\quad +\alpha_1 cd\tfrac{(y-y_3)(S-S_3)}{(d+y)(d+y_3)} + \alpha_2\beta\alpha'\tfrac{(x-x_3)(S-S_3)}{(\alpha'+x)(\alpha'+x_3)} - \delta I(S-S_3),
\end{aligned}$$

for $a=\frac{\alpha_2\alpha'}{\alpha'+x_3}$ and $b=\frac{\alpha_1 d}{d+y_3}$ we have

(5.6)

$$\begin{aligned}
\tfrac{-a\beta S(x-x_3)}{(\alpha'+x)} + \tfrac{a\beta S_3(x-x_3)}{(\alpha'+x_3)} + \tfrac{a\beta(x-x_3)(S-S_3)}{(\alpha'+x)} &= \tfrac{(-a\beta(\alpha'+x_3)+\alpha_2\beta\alpha')(x-x_3)(S-S_3)}{(\alpha'+x)(\alpha'+x_3)} + \tfrac{a\beta S_3(x-x_3)^2}{(\alpha'+x)(\alpha'+x_3)}\\
&= \tfrac{a\beta S_3(x-x_3)^2}{(\alpha'+x)(\alpha'+x_3)},
\end{aligned}$$

and

$$\begin{aligned}
\tfrac{-bcS(y-y_3)}{(d+y)} + \tfrac{bcS_3(y-y_3)}{d+y_3} + \tfrac{\alpha_1 cd(y-y_3)(S-S_3)}{(d+y)(d+y_3)} &= \tfrac{(-bc(d+y_3)+\alpha_1 cd)(y-y_3)(S-S_3)}{(d+y_3)(d+y)} + \tfrac{bcS_3(y-y_3)^2}{(d+y)(d+y_3)}\\
&= \tfrac{bcS_3(y-y_3)^2}{(d+y)(d+y_3)},
\end{aligned}$$

after simplification, for $an_1+bn_2-\frac{a\sigma_2}{x_3}-\frac{b\sigma_1}{y_3}\geq 0$ we find,

$$\begin{aligned}
D^{\alpha}V_3 &< -\left(a\left(\tfrac{r_1+Ku}{K}\right)-\tfrac{1}{2}(an_1+bn_2)\right)(x-x_3)^2 - (v-\tfrac{1}{2}(an_1+bn_2))(y-y_3)^2\\
&\quad -a\tfrac{\sigma_2 y}{x_3 x}(x-x_3)^2 - b\tfrac{\sigma_1 x}{y_3 y}(y-y_3)^2 - \tfrac{1}{2}(\tfrac{a\sigma_2}{x_3}+\tfrac{b\sigma_1}{y_3})((x-x_3)^2+(y-y_3)^2).
\end{aligned}$$

Therefore, $D^{\alpha}V_3<0$ if $\frac{b\sigma_1}{y_3}+\frac{a\sigma_2}{x_3}\leq an_1+bn_2\leq 2min(v,a\left(\frac{r_1+Ku}{K}\right))$. □

Proposition 5.9. *The equilibrium $P_4(x_4,y_4,S_4,I_4)$ is globally asymptotically stable if $\frac{\alpha_2\alpha'\sigma_2}{(\alpha'+x_4)x_4}+\frac{\alpha_1 d\sigma_1}{(d+y_4)y_4}\leq\frac{\alpha_2\alpha'}{\alpha'+x_4}n_1+\frac{\alpha_1 d}{d+y_4}n_2\leq 2min(v,\frac{\alpha_2\alpha'}{\alpha'+x_4}\left(\frac{r_1+Ku}{K}\right))$.*

Proof. Take the positive definite Lyapunov function:

$$\begin{aligned}
V_4 &= a\left(x-x_4-x_4\ln\left(\tfrac{x}{x_4}\right)\right) + b\left(y-y_4-y_4\ln\left(\tfrac{y}{y_4}\right)\right) + \left(S-S_4-S_4\ln\left(\tfrac{S}{S_4}\right)\right)\\
&\quad +\left(I-I_4-I_4\ln\left(\tfrac{I}{I_4}\right)\right),
\end{aligned}$$

with $a=\frac{\alpha_2\alpha'}{\alpha'+x_4}\leq\alpha_2$ and $b=\frac{\alpha_1 d}{d+y_4}\leq\alpha_1$. Using Lemma 2.4 we get:

$$D^{\alpha}V_4 < a\left(\frac{x-x_4}{x}\right)D^{\alpha}x + b\left(\frac{y-y_4}{y}\right)D^{\alpha}y + \left(\frac{S-S_4}{S}\right)D^{\alpha}S + \left(\frac{I-I_4}{I}\right)D^{\alpha}I,$$

from (3.1), we remplace $D^{\alpha}x$, $D^{\alpha}y$, $D^{\alpha}S$ and $D^{\alpha}I$ with there expressions:

$$\begin{aligned}
D^{\alpha}V_4 &< a(x-x_4)(r_1\left(1-\tfrac{x}{K}\right) - \sigma_1 + \sigma_2\tfrac{y}{x} - ux - \tfrac{\beta S}{\alpha'+x} - q_1E_1 - n_1 y - \gamma I)\\
&\quad +b(y-y_4)(r_2-\sigma_2+\sigma_1\tfrac{x}{y} - \tfrac{cs}{d+y} - vy - n_2 x) + (S-S_4)(\tfrac{\alpha_1 cy}{d+y} + \tfrac{\alpha_2\beta x}{\alpha'+x} - \delta I - \mu\\
&\quad -q_2E_2) + (I-I_4)(\delta S + \alpha_2\gamma x - q_3E_3 - \eta),
\end{aligned}$$

so we get:

$$\begin{aligned}
D^{\alpha}V_4 \;<\;& -a\left(\frac{r_1+Ku}{K}\right)(x-x_4)^2 + a\sigma_2\left(\frac{y}{x}-\frac{y_4}{x_4}\right)(x-x_4) - (an_1+bn_2)(x-x_4)(y-y_4)\\
& -a\gamma(I-I_4)(x-x_4) - a\beta\left(\frac{S}{(\alpha'+x)}-\frac{S_4}{(\alpha'+x_4)}\right)(x-x_4) + b\sigma_1\left(\frac{x}{y}-\frac{x_4}{y_4}\right)(y-y_4)\\
& -bv(y-y_4)^2 - bc\left(\frac{S}{d+y}-\frac{S_4}{d+y_4}\right)(y-y_4) + \delta(S-S_4)(I-I_4)\\
& +\alpha_2\gamma(x-x_4)(I-I_4) + \alpha_1 cd\frac{(y-y_4)(S-S_4)}{(d+y)(d+y_4)} + \alpha_2\beta\alpha'\frac{(x-x_4)(S-S_4)}{(\alpha'+x)(\alpha'+x_4)}\\
& -\delta(I-I_4)(S-S_4),
\end{aligned}$$

with (5.6) , after simplification, for $an_1+bn_2-\frac{a\sigma_2}{x_4}-\frac{b\sigma_1}{y_4}\geq 0$, we find,

$$\begin{aligned}
D^{\alpha}V_4 \;<\;& -\left(a\left(\frac{r_1+Ku}{K}\right)-\frac{1}{2}(an_1+bn_2)\right)(x-x_4)^2 - \left(v-\frac{1}{2}(an_1+bn_2)\right)(y-y_4)^2\\
& -a\frac{\sigma_2 y}{x_4 x}(x-x_4)^2 - b\frac{\sigma_1 x}{y_4 y}(y-y_4)^2 - \frac{1}{2}\left(\frac{a\sigma_2}{x_4}+\frac{b\sigma_1}{y_4}\right)\left((x-x_4)^2+(y-y_4)^2\right).
\end{aligned}$$

Therefore, $D^{\alpha}V_4<0$ if $\frac{a\sigma_2}{x_4}+\frac{b\sigma_1}{y_4}\leq an_1+bn_2\leq 2min\left(v,a\left(\frac{r_1+Ku}{K}\right)\right)$. □

6. NUMERICAL SIMULATIONS

In this part, we present a numerical simulations to confirm the theoretically obtained results. First the figures 2 – 5 present the model (3.1) simulations with respect to time, to verify the convergence of solutions to the equilibrium with different values of α. Then after assuming c with different values, the figure 6 present a numerical simulations to verify the impact of prey consumption in the protected area on the population.

After adding some values from [30] to the system (3.1) parameters and assuming the other to verify the stability conditions, in Table (5) we present the parametrs values.

Parameters	values in Fig. 2	values in Fig. 3	values in Fig. 4	values in Fig. 5
E_1,E_2,E_3	6, 5 and 4.8	3, 2 and 1.5	3, 3.2 and 4	3, 2.6 and 2.2
r_1,r_2	4 and 1	6 and 8	10 and 8	2.34 and 3
q_1,q_2,q_3	0.1, 0.2 and 0.4	0.1, 0.7 and 0.9	0.1, 0.2 and 0.48	0.1, 0.2 and 0.3
σ_1,σ_2	1 and 0.9	1 and 2	2 and 7.5	2 and 7.5
n_1,n_2	0.5 and 0.3	0.5 and 0.1	0.5 and 0.3	0.5 and 0.3
γ	5.5	5.5	5.33	8
δ	10	14	9.5	5
β	0.94	1	11	10
μ	1.5	1.8	1.8	2.41
η	60	60	60	2.68
d,$\alpha^{'}$	0.6 and 0.7	0.2 and 0.45	5.2 and 4.8	0.6999 and 0.70001
α_1,α_2	0.8 and 0.998	3.5 and 4	1.43 and 1.5	0.99 and 1
c	0.89	0.6	5	0.94
u	0.0001	0.4	0.1	0.01
v	0.333	0.9	0.4	0.4
K	3	5	6	0.7

TABLE 5. Parameters values

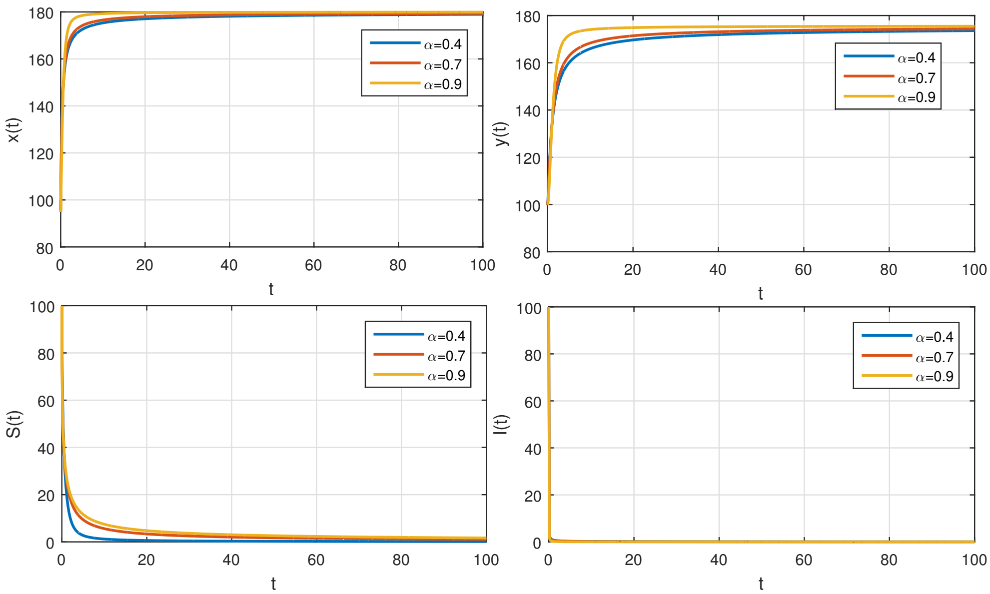

FIGURE 2. Convergence of solutions to the equilibrium P_1 with various values of α

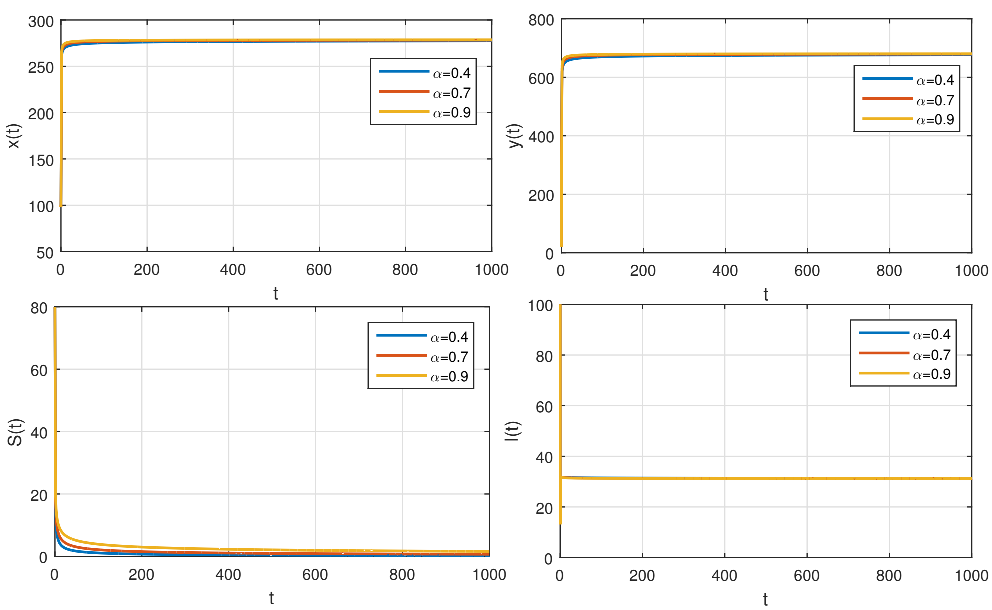

FIGURE 3. Convergence of solutions to the equilibrium P_2 with various values of α

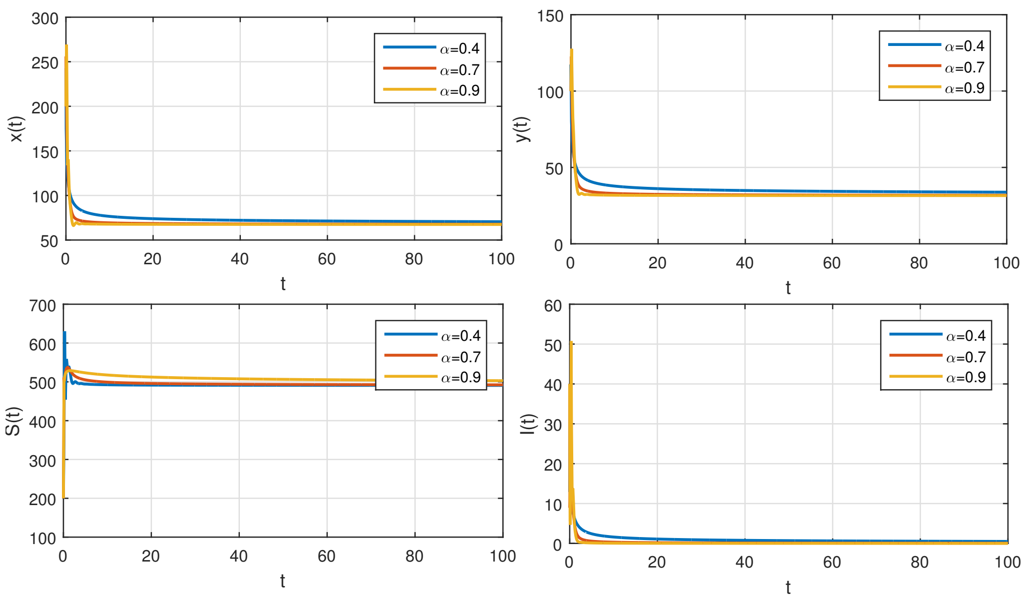

FIGURE 4. Convergence of solutions to the equilibrium P_3 with different values of α

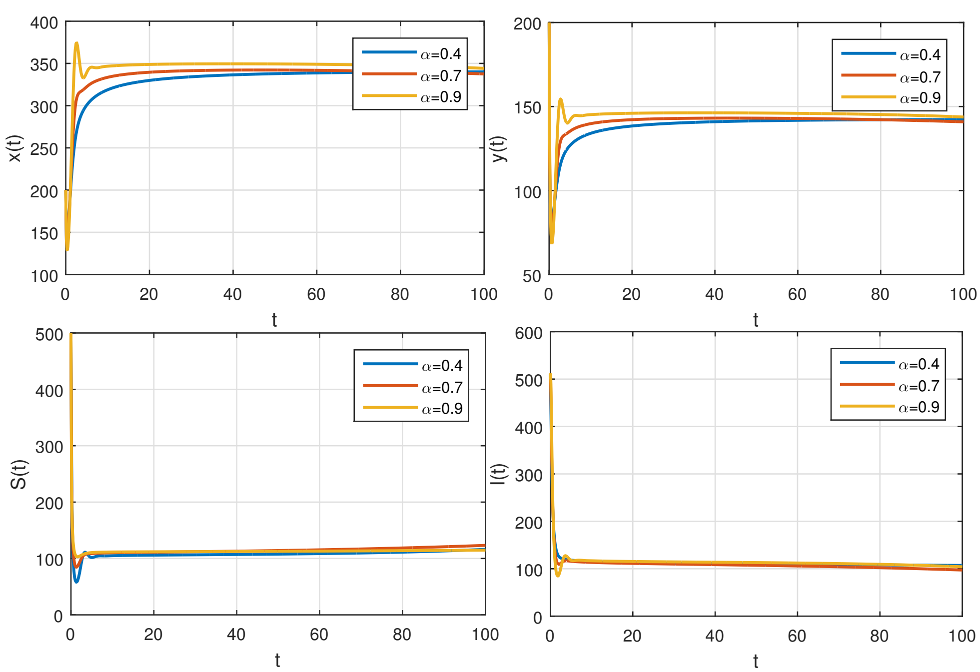

FIGURE 5. Convergence of solutions to the equilibrium P_4 with various values of α

By using the values of the Table 5 , the numerical simulation shown in the previous figures 2 – 5 , declares that population will increased and decreases over time to attain equilibrium for differences values of the fractional order derivative α, which proves the global stability of equilibrium points. The solutions of our model converge to the equilibrium more quickly by reducing the fractional order, which confirm that numerical solutions are continuously dependent on α.

For the figure 6, we use the values in Table 6 and we assume c with different values to verify the impact of prey consumption in the protected area on the population.

Parameters	values in Fig. 6
E_1,E_2,E_3	3, 2.6 and 2.2
r_1,r_2	2.34 and 3
q_1,q_2,q_3	0.1, 0.2 and 0.3
σ_1,σ_2	2 and 7.5
n_1,n_2	0.5 and 0.3
γ	8
δ	5
β	10
μ	2.41
η	2.68
d,α'	0.6999 and 0.70001
α_1,α_2	0.99 and 1

TABLE 6. Parameters values

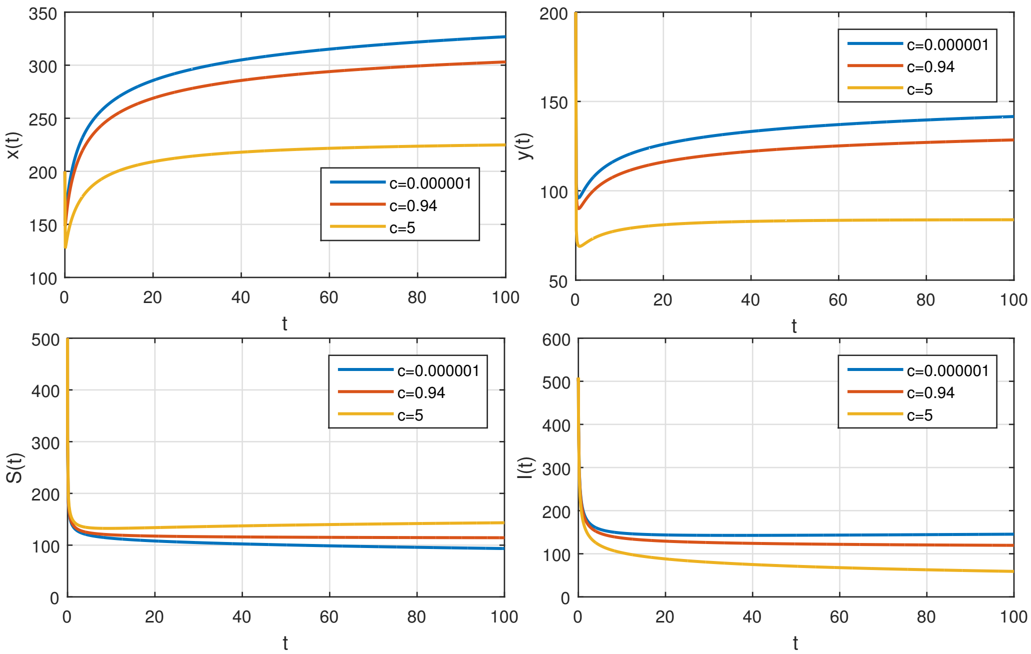

FIGURE 6. The impact of prey consumption in the reserved area with different values of c

The prey-predator competition has gradually formed a system of sustainable interactions, with predators having a direct and indirect influence on the populations of their prey by killing and eating a proportion of the individuals. Natural and unnatural regulatory systems, such as the creation of protected areas, have emerged in ecosystems to balance prey and predator populations. If there were no regulatory mechanisms, predatory animals could theoretically wipe out all their prey, and then disappear themselves due to lack of food. Using the values of the Table 5, the numerical simulation illustrated in figure 6 show that the penetration of susceptible predators into the reserved area, influences the quality of ecosystem functioning in that area. Indeed, they influence the structure and productivity of the prey. As the figure clearly shows, the predation coefficient has a significant impact on the population in both zones, the restricted zone and the free one. As long as the value of c increases, the number of prey x, prey y and predator I decreases, while the number of sensitive predators increases.

7. Conclusion

With a fractional order prey predators model, we've studied the positivity and the boundedness of the solutions provided of the model to check that the model is epidemiologically well posed. Then we sought the local and global stability of equilibriums by using the Lyapunov function. Therefore, from the numerical results it has been proved that the equilibriums are globally stable and the prey consumption in the protected zone have a significant impact on the population in both, the reserved and the free area, furthermore the penetration of susceptible predators in the restricted area influence the productivity and the quality of ecosystem functioning. The predator adapts its hunting strategies on the pack in the reserved area, which causes a delay. Indeed, the next item is based on a model where susceptible and infected predators will attack the reserved area with a time delay.

Conflict of Interests

The authors declare that there is no conflict of interests.

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