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ROLE OF REFUGE IN THE DYNAMIC OF TWO PREY – ONE PREDATOR FOOD WEB MODEL WITH FEAR AND NONLINEAR HARVESTING

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Abstract: This paper proposes and studies a food web model consisting of three species: two prey and one predator. The model takes into consideration the impact of fear, refuge for the first prey, which is dependent on the predator, and nonlinear harvesting of each prey. Further, the predator consumes the prey according to a Holling-type interaction. The model's boundedness is investigated. The conditions for all points of equilibrium that are biologically possible and local stability have been studied. The global stability of the model was examined using suitable Lyapunove functions. At last, the model is solved using numerical simulation for a variety of parameter values, and the results are shown graphically to confirm the analytical results.

Keywords: predator- prey system; Holling type; nonlinear harvesting; refuge; food web.

2020 AMS Subject Classification: 92D30, 92C42, 92D40.

1. INTRODUCTION

Biomathematics, which mixes mathematics and biology, uses mathematical methods to study complicated biological problems. This field includes a variety of topics, such as ecology, epidemiology, bioinformatics, pharmacokinetics, genetics, and more. Biomathematics employs mathematics to solve complicated biological problems and generate model predications of

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biological system dynamics. Mathematical models utilise differential equations, difference equations, and stochastic processes to understand and evaluate biological processes and systems. A large proportion of biomathematics scholars have researched the predator-prey model, which is important to scientific paradigms, human conversation, and environmental health. See [1-10]. Different biological processes, such as predation, fear, cannibalism, diseases, and migration, have the capability to affect the dynamical behavior of a model [11-13].

Studying the relationship between a predator and prey is crucial because the prey typically hides to prevent the predator from catching or attacking it. Ecologists refer to this type of action as refuge. The number of predators and prey in certain natural systems may influence prey refuge; for this reason, many research studies have used a predator–prey model in which the prey refuge depends on both species. For example, Haque et al. [14] deal with a prey–predator model with prey refuge in proportion to both species. Naji et al. [15] proposed a prey-predator model involving predator–dependent refuge in the prey population. In [16], Pratama discussed the dynamics of prey refuge in a diseased predator-prey model. Rahman et al. [17] investigated the effect of prey refuge with the Holling type IV functional response dependent prey-predator model.

Additionally, the predator's eating of prey is important. The amount of prey each predator consumes in a unit of time is called the functional response. Several scientists are studying functional responses in the prey-predator model, such as Khajanchi [18] analysed a stage-structure predator-prey model using Monod-Halane type response function. Lu et al. [19] investigated the periodic solution of a stage-structure predator-prey model with a Growley-Martin type functional response. Fordjour et al. [20] studied the dynamics of a predator-prey model with generalised Holling-type functional response and mutual interference. Chen and Young [21] discussed the impact of nonlinear harvesting and delay on predator-prey model using Beddington-De Angelis functional response.

Furthermore, it is important to consider species harvesting in a predator-prey model. Different harvesting techniques have been used. These include nonlinear, linear, and constant harvesting. Nonlinear harvesting is considered to be more appropriate than others from an economic and biological perspective [22-26].

Despite the previous studies, this work proposes and analyses a two prey-one predator model with fear, refuge dependent on the predator, and nonlinear harvesting. So, this work is structured as follows: Section 2 presents the formulation for the model. The Boundedness of the solution are studied in Section 3. Section 4 addresses the existence of all feasible points of equilibrium. In Section, local stability conditions are determined at every point of equilibrium. The Laypunove function utilized for verifying the global stability of the suggested model in Section 6. Section 7 confirms the completion of the numerical simulation of the theoretical results. The discussion and conclusion were in the last section.

2. MATHEMATICAL MODEL FORMULATION

This section searches for a scenario involving two prey, x and y, in which x is vulnerable and takes predator-dependent refuge, while both prey undergo nonlinear harvesting and predation by the same predator z. The model understudy is described as follows, with first prey, second prey, and predator density at time denoted by $x(t), y(t)$ and $z(t)$ respectively:

1- Both prey populations grow logistically without a predator or fear; however, a predator's fear of these populations can have multiple effects. The fear functions $\frac{1}{1+f_1 z}$ and $\frac{1}{1+f_2 z}$ influence the growth of the first and second prey, with f_1, f_2 representing their respective fear parameters.

2- When a predator is present, the prey feels fear of predation. As a result, a proportional amount of the first prey population needs refuge, depending on the predator. On the other hand, the functional response of the first prey and predator is Holling type 2, while the functional response of the second prey is Holling type 4 where, $\alpha_1, \alpha_2, \beta\ and\ \gamma > 0$, $0 < n_1 < 1$ and $0 \leq 1 - n_1 z \leq 1$.

3- Assume that the prey species experience nonlinear harvesting.

4- Supposed that there was interspecies competition.

Using the previous assumptions, the model is described as follows:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{rx}{1+f_1 z} - ax^2 - \frac{\alpha_1(1-n_1 z)xz}{\beta+(1-n_1 z)x} - \frac{q_1 E_1 x}{m_1 E_1 + m_2 x}\ , \\
\frac{dy}{dt} &= \frac{sy}{1+f_2 z} - by^2 - \frac{\alpha_2 yz}{1+\gamma y^2} - \frac{q_2 E_2 y}{m_3 E_2 + m_4 y}\ , \qquad (1) \\
\frac{dz}{dt} &= \frac{c_1 \alpha_1 (1-n_1 z)xz}{\beta+(1-n_1 z)x} + \frac{c_2 \alpha_2 yz}{1+\gamma y^2} - mz^2 - dz\ .
\end{aligned}$$

The initial condition $(X(0), Y(0), Z(0))$ should be in the first quadrant on a biological basis. Table 1 below shows the parameters set for model (1), which were assumed to have positive value:

Table 1- parameters Description.

Parameter	**Description**
r	The birth rate of first prey.
s	The birth rate of second prey.
a	The interspecies competition of the first prey.
b	The interspecies competition of the second prey.
m	The interspecies competition of the predator.
f_1	The fear rate of first prey of predator.
f_2	The fear rate of second prey of predator.
α_1	The attack rate of predator of the first prey.
α_2	The attack rate of predator of the second prey.
c_1	Coefficient of transformation from first prey towards predator.
c_2	Coefficient of transformation from second prey towards predator.
β	The half-saturation constant.
γ	Level of defense.
d	The natural death rate of the predator.
n_1	Coefficient of first prey refuge.
$q_i, i = 1,2$	Catch-ability coefficient of first, second prey.
$E_i, i = 1,2$	Effort of harvesting for the species.
$m_i, i = 1,2,3,4$	Suitable positive constants.

3. BOUNDEDNESS OF THE MODEL

The model domain is $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3, x(0) \geq 0, y(0) \geq 0, z(0) \geq 0\}$, and it is assumed that for $\forall t \geq 0$, the functions $x(t), y(t)$, and $z(t)$ with the derivatives of these functions are continuous, which implies that all of them are Lipschizain in $\mathbb{R}^3_+$, and there is a unique solution to model (1). The following theorem shows the bounds for this model (1) solution in $\mathbb{R}^3_+$.

Theorem (3.1):- The of model (1) solutions, which starting in $\mathbb{R}^3_+$ are uniformly bounded.

Proof: - Applying the first equation in model (1), the following is concluded:

$$\frac{dx}{dt} \leq rx\left[1 - \frac{x}{r/a}\right],$$

due to it solved the differential inequality above:

$x(t) \leq \frac{r}{a}$.

In the same way, the second equation leads to:

$y(t) \leq \frac{s}{b}$.

Next, let $M(t) = x(t) + y(t) + z(t)$,

therefore, by calculating it is derivative with respect to time, we obtain:

$$\frac{dM}{dt} \leq \frac{rx}{1 + f_1 z} - ax^2 + \frac{sy}{1 + f_2 z} - by^2 - \frac{q_1 E_1 x}{m_1 E_1 + m_2 x} - \frac{q_2 E_2 y}{m_3 E_2 + m_4 y} - dz,$$
$$\frac{dM}{dt} \leq \frac{r}{a} + \frac{s}{b} - L(x + y + z).$$

Then $\frac{dM}{dt} \leq J - LM$,

where $J = \frac{r}{a} + \frac{s}{b}$; and $L = \min\{q_1 E_1, q_2 E_2, d\}$,

so as $t \to \infty, M(t) \leq \frac{J}{L}$, hence the proof completed.

4. EXISTENCE OF POINTS OF EQUILIBRIUM

At maximum, the model (1) includes seven nonnegative points of equilibrium, $P_i, i = 0,1,2, \dots ,6$, whose forms and conditions for existing are listed below:

- A trivial point of equilibrium (TPE), $P_0 = (0,0,0)$, always presents.
- The first axial point of equilibrium (FAPE), $P_1 = (\bar{x}, 0,0)$, where $\bar{x}$ is a +ve root for 2nd-order equation that follows:

$$B_1^{[1]} \bar{x}^2 + B_2^{[1]} \bar{x} + B_3^{[1]} = 0, \tag{2.a}$$

Which

$$\left.\begin{array}{l} B_1^{[1]} = -a m_1 < 0 \\ B_2^{[1]} = r m_2 - a m_1 E_1 \\ B_3^{[1]} = E_1 (r m_1 - q_1) \end{array}\right\}. \tag{2.b}$$

As a result of the sign discarding rule, equation (2.a) has at least one positive root if the following condition are met:

$$r m_1 > q_1, \tag{2.c}$$

- The second axial point of equilibrium (SAPE), $P_2 = (0, \check{y}, 0)$, where $\check{y}$ is a +ve root for 2nd-order equation that follows:

$$B_1^{[2]}\check{y}^2 + B_2^{[2]}\check{y} + B_3^{[2]} = 0, \quad (3.a)$$

which

$$\left.\begin{array}{l} B_1^{[2]} = -bm_4 < 0 \\ B_2^{[2]} = sm_4 - bm_3 E_2 \\ B_3^{[2]} = E_2(sm_3 - q_2) \end{array}\right\}. \quad (3.b)$$

As a result of the sign discarding rule, equation (3.a) has at least one positive root if the following condition are met:

$$sm_3 > q_2 \ , \quad (3.c)$$

- The free predator point of equilibrium (PDFPE), $P_3 = (\bar{\bar{x}}, \bar{\bar{y}}, 0)$, where $\bar{\bar{x}}$ is a +ve root for 2nd-order equation that follows:

$$B_1^{[1]}\bar{\bar{x}}^2 + B_2^{[1]}\bar{\bar{x}} + B_3^{[1]} = 0, \quad (4.a)$$

$\check{y}$ is a +ve root for 2nd-order equation that follows:

$$B_1^{[2]}\bar{\bar{y}}^2 + B_2^{[2]}\bar{\bar{y}} + B_3^{[2]} = 0, \quad (4.b)$$

where $B_1^{[1]}$, $B_2^{[1]}$, $B_3^{[1]}$, $B_1^{[2]}$, $B_2^{[2]}$, and $B_3^{[2]}$ is it the same in (2.b) and (3.b) such that PDFPE exist if the conditions (2.c) and (3.c) are met.

- Second prey free point of equilibrium (SPFPE), $P_4 = (\tilde{x}, 0, \tilde{z})$, as

$$\tilde{x} = \frac{-\beta(d+m\tilde{z})}{(1-n_1\tilde{z})(d-c_1\alpha_1+m\tilde{z})} \ , \quad (5.a)$$

while $\tilde{z}$ is a +ve root for 9th-order equation that follows:

$$\begin{array}{l} B_1^{[3]}\tilde{z}^8 + B_2^{[3]}\tilde{z}^7 + B_3^{[3]}\tilde{z}^6 + B_4^{[3]}\tilde{z}^5 + B_5^{[3]}\tilde{z}^4 + B_6^{[3]}\tilde{z}^3 \\ +B_7^{[3]}\tilde{z}^2 + B_8^{[3]}\tilde{z} + B_9^{[3]} = 0 \end{array}, \quad (5.b)$$

where $B_i^{[3]}, i = 1, \dots, 9$ are calculated using MATLAB program and it will not give here due to their huge and complicated forms.

Then, if any one of the following conditions is satisfied, equation (5.b) has at least one positive root due to the sign discarding rule:

$$\left.\begin{array}{c} B_1^{[3]} > 0 \ and \ B_i^{[3]} < 0, \ \ i = 2, \dots, 9 \\ or \\ B_1^{[3]} < 0 \ and \ B_i^{[3]} > 0, \ \ i = 2, \dots, 9 \end{array}\right\}. \quad (5.c)$$

Assuming that there just one positive root of equation (5.b) represented by $\tilde{z}$, then SPFPE exist if provided that:

$$c_1\alpha_1 > d + m\tilde{z} \ . \quad (5.d)$$

- First prey free point of equilibrium (FPFPE), $P_5 = (0, \hat{y}, \hat{z})$, as

$$\hat{z} = \frac{c_2\alpha_2\hat{y} - d[\gamma\hat{y}^2 + 1]}{m(1+\gamma\hat{y}^2)}, \quad (6.a)$$

while $\hat{y}$ is a +ve root for 9nd-order equation that follows::

$$B_1^{[4]}\hat{y}^8 + B_2^{[4]}\hat{y}^7 + B_3^{[4]}\hat{y}^6 + B_4^{[4]}\hat{y}^5 + B_5^{[4]}\hat{y}^4 + B_6^{[4]}\hat{y}^3 + B_7^{[4]}\hat{y}^2 + B_8^{[4]}\hat{y} + B_9^{[4]} = 0, \quad (6.b)$$

where

$$B_1^{[4]} = -bg_1\gamma^3 mm_4,$$

$$B_2^{[4]} = \gamma^3 m^2 m_4 s - bm(g_1g_3\gamma^3 + g_2\gamma^2 m_4),$$

$$B_3^{[4]} = g_3\gamma^3 m^2 s - bm(g_2g_3\gamma^2 + 3g_1\gamma^2 m_4) - E_2 g_1\gamma^3 mq_2,$$

$$B_4^{[4]} = g_1g_5\gamma^2 m_4 - bm(3g_1g_3\gamma^2 + 2g_2\gamma m_4) + 3m_4\gamma^2 m^2 s - E_2g_2\gamma^2 mq_2,$$

$$B_5^{[4]} = g_5\gamma(g_2 + m_4 + g_1g_3\gamma) - bm(2g_2g_3\gamma + 3g_1\gamma m_4) - g_1g_4\gamma m_4 + 3g_3\gamma^2 m^2 s - 3E_2g_1\gamma^2 mq_2,$$

$$B_6^{[4]} = g_5\gamma(g_2g_3 + g_1m_4) - bm(g_2m_4 + 3g_1g_3\gamma) - g_4(g_2m_4 + g_1g_3\gamma) + g_1g_3\gamma m_4 + 3\gamma m^2 m_4 s - 2E_2g_2\gamma mq_2,$$

$$B_7^{[4]} = g_5(g_2g_3 + g_1m_4) - g_4(g_2g_3 + g_1m_4) - bm(g_2g_3 + g_1m_4) + g_1g_3g_5\gamma + 3\gamma m^2 g_3 s - 3E_2g_1\gamma mq_2,$$

$$B_8^{[4]} = g_5(g_2g_3 + g_1m_4) + m^2 m_4 s - g_1g_3g_4 - E_2g_2mq_2 - bg_1g_3m,$$

$$B_9^{[4]} = m^2 g_3 s + g_1g_3g_5 - E_2g_1mq_2 \ .$$

Such that,

$g_1 = m - f_2 d$, $g_2 = f_2c_2\alpha_2$, $g_3 = E_2m_3$, $g_4 = c_2{\alpha_2}^2$, $g_5 = \alpha_2 d$.

Then, if any one of the following conditions is satisfied, equation (6.b) has at least one positive root due to the sign discarding rule:

$$\left.\begin{array}{c} B_1^{[4]} > 0 \; and \; B_i^{[4]} < 0, \;\; i = 2, \dots, 9 \\ or \\ B_1^{[4]} < 0 \; and \; B_i^{[4]} > 0, \;\; i = 2, \dots, 9 \end{array}\right\}. \tag{6.c}$$

Assuming that there just one positive root of equation (6.b) represented by $\hat{y}$ then FPFPE exist if provided that:

$$c_2\alpha_2\hat{y} > [1 + \gamma\hat{y}^2]d. \tag{6.d}$$

- The positive point of equilibrium (PPE), $P_6 = (x^*, y^*, z^*)$, as

$$x^* = \frac{c_2\alpha_2\beta y^* - (1+\gamma y^{*2})(d\beta + m\beta z^*)}{(1-n_1 z^*)[(1+\gamma y^{*2})(d + mz^* - c_1\alpha_1) - c_2\alpha_2 y^*]} \;, \tag{7.a}$$

and (x^*, y^*) displays the point at which the isoclines obtained from model (1) positively intersect the following two isoclines when setting the first two equations equal to zero after substituting the value of x^* given in eq. (7.a):

$$\left.\begin{array}{l} f(y,z) = (u_1y^6 + u_2y^4 + u_3y^2 + u_4)z^8 + (u_5y^6 + u_6y^5 + u_7y^4 \\ \quad u_8y^3 + u_9y^2 + u_{10}y + u_{11})z^7 + (u_{12}y^6 + u_{13}y^5 + u_{14}y^4 + u_{15}y^3 \\ \quad +u_{16}y^2 + u_{17}y + u_{18})z^6 + (u_{19}y^6 + u_{20}y^5 + u_{21}y^4 \\ \quad +u_{22}y^3 + u_{23}y^2 + u_{24}y + u_{25})z^5 + (u_{26}y^6 + u_{27}y^5 \\ \quad +u_{28}y^4 + u_{29}y^3 + u_{30}y^2 + u_{31}y + u_{32})z^4 + (u_{33}y^6 \\ \quad +u_{34}y^5 + u_{35}y^4 + u_{36}y^3 + u_{37}y^2 + u_{38}y + u_{39})z^3 \\ \quad +(u_{40}y^6 + u_{41}y^5 + u_{42}y^4 + u_{43}y^3 + u_{44}y^2 + u_{45}y + u_{46})z^2 \\ \quad +(u_{47}y^6 + u_{48}y^5 + u_{49}y^4 + u_{50}y^3 + u_{51}y^2 + u_{52}y + u_{53})z \\ \quad +(u_{54}y^6 + u_{55}y^5 + u_{56}y^4 + u_{57}y^3 + u_{58}y^2 + u_{59}y + u_{60}) \\ g(y,z) = (k_1z + k_2)y^4 + (k_3z + k_4)y^3 + (k_5z + k_6)y^2 + (k_7z^2 \\ \quad +k_8z + k_9)y + (k_{10}z^2 + k_{11}z + k_{12}) \end{array}\right\} , \tag{7.b}$$

where u_i and k_j $\forall\, i = 0,1,\dots,60$, $j = 0,1,\dots,12$ were determined using MATLAB program, that will not give them here since they are huge and complicated forms.

When $z \to 0$, direct computation implies that:

$$\left.\begin{array}{c} u_{54}y^6 + u_{55}y^5 + u_{56}y^4 + u_{57}y^3 + u_{58}y^2 + u_{59}y + u_{60} = 0 \\ k_2y^4 + k_4y^3 + k_6y^2 + k_9y + k_{12} = 0 \end{array}\right\}. \tag{7.c}$$

Under the discarding rule of signs, all equation in (7.c) may has a unique positive root for the isocline given by $\bar{y}_1$ and $\bar{y}_2$ respectively, where the leading and free coefficients have sign opposite.

Then system (7.b) has unique intersection point denoted as (y^*, z^*) with the given conditions**:**

$$\left.\begin{array}{c} c_2\alpha_2\beta y^* > (1+\gamma y^{*2})(d\beta + m\beta z^*) \\ (1+\gamma y^{*2})(d+mz^*) > (1+\gamma y^{*2})c_1\alpha_1 + c_2\alpha_2 y^* \end{array}\right\}, \quad (7.d)$$

$$y_1^* < y_2^*, \quad (7.e)$$

$$\left.\begin{array}{c} \frac{\partial f}{\partial y} > 0 \; and \; \frac{\partial f}{\partial z} < 0 \\ or \\ \frac{\partial f}{\partial y} < 0 \; and \; \frac{\partial f}{\partial z} > 0 \end{array}\right\}, \quad (7.f)$$

$$\left.\begin{array}{c} \frac{\partial g}{\partial y} > 0 \; and \; \frac{\partial g}{\partial z} > 0 \\ or \\ \frac{\partial g}{\partial y} < 0 \; and \; \frac{\partial g}{\partial z} < 0 \end{array}\right\}. \quad (7.g)$$

Therefore, only the (PPE) exists in the $\mathbb{R}_+^3$.

5. LOCAL STABILITY ANALYSIS

The linearization method is used in this section to study the local stability of the model (1) .the Jacobian matrix (J.M) of the model (1) at (x, y, z) is:

$$J_i = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad (8)$$

where

$$a_{11} = \frac{r}{1+f_1 z} - 2ax - \frac{\beta\alpha_1 z(1-n_1 z)}{(\beta+(1-n_1 z)x)^2} - \frac{m_1 q_1 {E_1}^2}{(m_1 E_1 + m_2 x)^2},$$

$$a_{13} = -\left(\frac{rxf_1}{(1+f_1 z)^2} + \frac{\beta\alpha_1 x(1-2n_1 z) + \alpha_1 x^2(1-n_1 z)^2}{(\beta+(1-n_1 z)x)^2}\right),$$

$$a_{22} = \frac{s}{1+f_2 z} - 2by - \frac{\alpha_2 z - \gamma\alpha_2 y^2 z}{(1+\gamma y^2)^2} - \frac{m_3 q_2 {E_2}^2}{(m_3 E_2 + m_4 y)^2},$$

$$a_{23} = -\left(\frac{syf_2}{(1+f_2 z)^2} + \frac{\alpha_2 y}{1+\gamma y^2}\right),$$

$$a_{31} = \frac{\beta c_1 \alpha_1 z(1-n_1 z)}{(\beta+(1-n_1 z)x)^2},$$

$$a_{32} = \frac{c_2\alpha_2 z - \gamma\alpha_2 c_2 z y^2}{(1+\gamma y^2)^2},$$

$$a_{33} = \frac{c_1\beta\alpha_1 x(1-2n_1 z) + c_1\alpha_1 x^2(1-n_1 z)^2}{(\beta+(1-n_1 z)x)^2} + \frac{c_2\alpha_2 y}{1+\gamma y^2} - 2mz - d.$$

If each of the (J.M) eigenvalues has a negative sign, then a point of equilibrium is local asymptotically stable (LAS). Therefore, the next theorem provides the local stability conditions at each equilibrium point.

Theorem (5.1): The TPE is LAS if the below conditions are met:

$$\left.\begin{aligned} r &< \frac{q_1}{m_1} \\ s &< \frac{q_2}{m_3} \end{aligned}\right\}. \tag{9}$$

Proof: At TPE, the J.M is written as:

$$J_0 = \begin{bmatrix} r - \frac{q_1}{m_1} & 0 & 0 \\ 0 & s - \frac{q_2}{m_3} & 0 \\ 0 & 0 & -d \end{bmatrix}.$$

Hence, the J_0eigenvalues are:

$$\lambda_1 = r - \frac{q_1}{m_1}, \quad \lambda_2 = s - \frac{q_2}{m_3}, \quad \lambda_3 = -d.$$

When condition (9) holds, TPE is LAS.

Theorem (5.2): The FAPE is LAS if the following conditions satisfied:

$$\left.\begin{aligned} r &< 2a\bar{x} + \frac{m_1 q_1 {E_1}^2}{(m_1 E_1 + m_2 \bar{x})^2} \\ s &< \frac{q_2}{m_3} \\ \frac{c_1 \alpha_1 \bar{x}}{\beta + \bar{x}} &< d \end{aligned}\right\}. \tag{10}$$

Proof: At FAPE, the J.M can be written as:

$$J_1 = \begin{bmatrix} r - 2a\bar{x} - \frac{m_1 q_1 {E_1}^2}{(m_1 E_1 + m_2 \bar{x})^2} & 0 & -(rf_1\bar{x} + \frac{\alpha_1 \bar{x}}{(\beta + \bar{x})^2}) \\ 0 & s - \frac{q_2}{m_3} & 0 \\ 0 & 0 & \frac{c_1 \alpha_2 \bar{x}}{\beta + \bar{x}} - d \end{bmatrix}.$$

So J_1 is an upper triangular matrix with three eigenvalues:

$$\lambda_1 = r - 2a\bar{x} - \frac{m_1 q_1 {E_1}^2}{(m_1 E_1 + m_2 \bar{x})^2}, \quad \lambda_2 = s - \frac{q_2}{m_3}, \quad \lambda_3 = \frac{c_1 \alpha_2 \bar{x}}{\beta + \bar{x}} - d.$$

The real part of $\lambda_1, \lambda_2 and\, \lambda_3$ is negative only if condition (10) is met, and saddle point otherwise.

Theorem (5.3): The SAPE is LAS if below conditions holds:

$$\left.\begin{aligned} s &< 2b\breve{y} + \frac{m_3 q_2 {E_2}^2}{(m_3 E_2 + m_4 \breve{y})^2} \\ r &< \frac{q_1}{m_1} \\ \frac{c_2 \alpha_2 \breve{y}}{1 + \gamma \breve{y}^2} &< d \end{aligned}\right\}. \tag{11}$$

Proof: At SAPE, the J.M is written as follows**:**

$$J_2 = \begin{bmatrix} r - \frac{q_1}{m_1} & 0 & 0 \\ 0 & s - (2b\check{y} + \frac{m_3 q_2 {E_2}^2}{(m_3 E_2 + m_4 \check{y})^2}) & -(sf_2\check{y} + \frac{\alpha_2 \check{y}}{1+\gamma \check{y}^2}) \\ 0 & 0 & \frac{c_2 \alpha_2 \check{y}}{1+\gamma \check{y}^2} - d \end{bmatrix}.$$

J_2 is an upper triangular matrix with three eigenvalues:

$$\lambda_1 = r - \frac{q_1}{m_1}, \quad \lambda_2 = s - (2b\check{y} + \frac{m_3 q_2 {E_2}^2}{(m_3 E_2 + m_4 \check{y})^2}), \quad \lambda_3 = \frac{c_2 \alpha_2 \check{y}}{1+\gamma \check{y}^2} - d.$$

The real part of $\lambda_1, \lambda_2 and\ \lambda_3$ is negative only if condition (11) is satisfied, and saddle point otherwise.

Theorem (5.4): The PDFPE is LAS if the following conditions are met:

$$\left.\begin{aligned} r &< 2a\bar{\bar{x}} + \frac{m_1 q_1 {E_1}^2}{(m_1 E_1 + m_2 \bar{\bar{x}})^2} \\ s &< 2b\bar{\bar{y}} + \frac{m_3 q_2 {E_2}^2}{(m_3 E_2 + m_4 \bar{\bar{y}})^2} \\ \frac{c_1 \alpha_1 \bar{\bar{x}}}{\beta + \bar{\bar{x}}} + \frac{c_2 \alpha_2 \bar{\bar{y}}}{1+\gamma \bar{\bar{y}}^2} &< d \end{aligned}\right\}. \tag{12}$$

Proof: At PDFPE, the J.M is written as follow**:**

$$J_3 = \begin{bmatrix} r - (2a\bar{\bar{x}} + \frac{m_1 q_1 {E_1}^2}{(m_1 E_1 + m_2 \bar{\bar{x}})^2}) & 0 & -(rf_1\bar{\bar{x}} + \frac{\alpha_1 \bar{\bar{x}}}{(\beta + \bar{\bar{x}})^2}) \\ 0 & s - (2b\bar{\bar{y}} + \frac{m_3 q_2 {E_2}^2}{(m_3 E_2 + m_4 \bar{\bar{y}})^2}) & -(sf_2\bar{\bar{y}} + \frac{\alpha_2 \bar{\bar{y}}}{1+\gamma \bar{\bar{y}}^2}) \\ 0 & 0 & (\frac{c_1 \alpha_1 \bar{\bar{x}}}{\beta + \bar{\bar{x}}} + \frac{c_2 \alpha_2 \bar{\bar{y}}}{1+\gamma \bar{\bar{y}}^2}) - d \end{bmatrix}.$$

The eigenvalues of J_3 derived by its characteristic equation, can be written as follows:

$$\lambda_1 = r - (2a\bar{\bar{x}} + \frac{m_1 q_1 {E_1}^2}{(m_1 E_1 + m_2 \bar{\bar{x}})^2}), \quad \lambda_2 = s - (2b\bar{\bar{y}} + \frac{m_3 q_2 {E_2}^2}{(m_3 E_2 + m_4 \bar{\bar{y}})^2}),$$
$$\lambda_3 = \left(\frac{c_1 \alpha_1 \bar{\bar{x}}}{\beta + \bar{\bar{x}}} + \frac{c_2 \alpha_2 \bar{\bar{y}}}{1+\gamma \bar{\bar{y}}^2}\right) - d.$$

So, the real part of $\lambda_1, \lambda_2 and\ \lambda_3$ is negative only if condition (12) is satisfied, and saddle point otherwise.

Theorem (5.5): The SPFPE is LAS if below conditions holds:

$$\left.\begin{aligned} \frac{c_1 \beta \alpha_1 \tilde{x}(1-2n_1\tilde{z}) + c_1 \alpha_1 \tilde{x}^2 (1-n_1\tilde{z})^2}{(\beta + (1-n_1\tilde{z})\tilde{x})^2} &< 2m\tilde{z} + d \\ \frac{s}{1+f_2\tilde{z}} &< \alpha_2 \tilde{z} + \frac{q_2}{m_3} \\ \frac{r}{1+f_1\tilde{z}} &< 2a\tilde{x} + \frac{\beta \alpha_1 \tilde{z}(1-n_1\tilde{z})}{(\beta + (1-n_1\tilde{z})\tilde{x})^2} + \frac{m_1 q_1 {E_1}^2}{(m_1 E_1 + m_2 \tilde{x})^2} \end{aligned}\right\}. \tag{13}$$

Proof: At SPFPE, the J.M is written as follow**:**

$$J_4 = \begin{bmatrix} \tilde{a}_{11} & 0 & \tilde{a}_{13} \\ 0 & \tilde{a}_{22} & 0 \\ \tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} \end{bmatrix},$$

Where

$$\tilde{a}_{11} = \frac{r}{1+f_1\tilde{z}} - (2a\tilde{x} + \frac{\beta\alpha_1\tilde{z}(1-n_1\tilde{z})}{(\beta+(1-n_1\tilde{z})\tilde{x})^2} + \frac{m_1q_1{E_1}^2}{(m_1E_1+m_2\tilde{x})^2}),$$

$$\tilde{a}_{13} = -(\frac{r\tilde{x}f_1}{(1+f_1\tilde{z})^2} + \frac{\beta\alpha_1\tilde{x}(1-2n_1\tilde{z})+\alpha_1\tilde{x}^2(1-n_1\tilde{z})^2}{(\beta+(1-n_1\tilde{z})\tilde{x})^2}) < 0,$$

$$\tilde{a}_{22} = \frac{s}{1+f_2\tilde{z}} - (\alpha_2\tilde{z} + \frac{q_2}{m_3}),$$

$$\tilde{a}_{31} = \frac{\beta c_1\alpha_1\tilde{z}(1-n_1\tilde{z})}{(\beta+(1-n_1\tilde{z})\tilde{x})^2} > 0 \ ,$$

$$\tilde{a}_{32} = c_2\alpha_2\tilde{z},$$

$$\tilde{a}_{33} = \frac{c_1\beta\alpha_1\tilde{x}(1-2n_1\tilde{z})+c_1\alpha_1\tilde{x}^2(1-n_1\tilde{z})^2}{(\beta+(1-n_1\tilde{z})\tilde{x})^2} - (2m\tilde{z} + d).$$

Then the characteristic equation is:

$$(\tilde{a}_{22} - \lambda)(\lambda^2 + tr_1\lambda + det_1) = 0,$$

where $tr_1 = (\tilde{a}_{11} + \tilde{a}_{33}) < 0$, $det_1 = \tilde{a}_{11}\tilde{a}_{33} - (\tilde{a}_{13}\tilde{a}_{31}) > 0$.

Hence, by trace-determinate stability criterion, SPFPE is LAS if condition (13) is satisfied, which is saddle point otherwise.

Theorem (5.6): The FPFPE is LAS if the following conditions are met:

$$\left.\begin{array}{l} \frac{c_2\alpha_2\hat{y}}{1+\gamma\hat{y}^2} < 2m\hat{z} + d \\ \frac{r}{1+f_1\hat{z}} < \frac{\alpha_1\hat{z}(1-n_1\hat{z})}{\beta} + \frac{q_1}{m_1} \\ \gamma\hat{y}^2 < 1 \\ \frac{s}{1+f_2\hat{z}} < 2b\hat{y} + \left(\frac{\alpha_2\hat{z}-\gamma\alpha_2\hat{y}^2\hat{z}}{(1+\gamma\hat{y}^2)^2}\right) + \frac{m_3q_2{E_2}^2}{(m_3E_2+m_4\hat{y})^2} \end{array}\right\}. \qquad (14)$$

Proof: At FPFPE, the J.M can be written as:

$$J_5 = \begin{bmatrix} \hat{a}_{11} & 0 & 0 \\ 0 & \hat{a}_{22} & \hat{a}_{23} \\ \hat{a}_{31} & \hat{a}_{32} & \hat{a}_{33} \end{bmatrix},$$

where

$$\hat{a}_{11} = \frac{r}{1+f_1\hat{z}} - (\frac{\alpha_1\hat{z}(1-n_1\hat{z})}{\beta} + \frac{q_1}{m_1}),$$

$$\hat{a}_{22} = \frac{s}{1+f_2\hat{z}} - (2b\hat{y} + \left(\frac{\alpha_2\hat{z}-\gamma\alpha_2\hat{y}^2\hat{z}}{(1+\gamma\hat{y}^2)^2}\right) + \frac{m_3q_2{E_2}^2}{(m_3E_2+m_4\hat{y})^2}),$$

$\hat{a}_{23} = -(\frac{s\hat{y}f_2}{(1+f_2\hat{z})^2} + \frac{\alpha_2\hat{y}}{1+\gamma\hat{y}^2}) < 0,$

$\hat{a}_{31} = \frac{c_1\alpha_1\hat{z}(1-n_1\hat{z})}{\beta} > 0,$

$\hat{a}_{32} = \frac{c_2\alpha_2\hat{z}-\gamma c_2\alpha_2\hat{y}^2\hat{z}}{(1+\gamma\hat{y}^2)^2},$

$\hat{a}_{33} = \frac{c_2\alpha_2\hat{y}}{1+\gamma\hat{y}^2} - (2m\hat{z}+d).$

Then, the characteristic equation is:

$$(\hat{a}_{11}-\lambda)(\lambda^2 + tr_2\lambda + det_2\lambda) = 0,$$

where $tr_2 = (\hat{a}_{22}+\hat{a}_{33}) < 0$, $det_2 = \hat{a}_{22}\hat{a}_{33} - (\hat{a}_{23}\hat{a}_{32}) > 0$.

So, by trace-determinate stability criterion, FPFPE is LAS if condition (14) holds, which is saddle point otherwise.

Theorem (5.7): The PPE is LAS if the following conditions are satisfied:

$$\left.\begin{matrix} a_{ii}^* < 0, \ i = 1,2,3 \\ \gamma {y^*}^2 < 1 \end{matrix}\right\}, \qquad (15)$$

where a_{ii}^* given in proof.

Proof: At PPE, the J.M is written as:

$$J_6 = \begin{bmatrix} a_{11}^* & 0 & a_{13}^* \\ 0 & a_{22}^* & a_{23}^* \\ a_{31}^* & a_{32}^* & a_{33}^* \end{bmatrix},$$

where

$a_{11}^* = \frac{r}{1+f_1z^*} - 2ax^* - \frac{\beta\alpha_1z^*(1-n_1z^*)}{(\beta+(1-n_1z^*)x^*)^2} - \frac{m_1q_1{E_1}^2}{(m_1E_1+m_2x^*)^2},,$

$a_{13}^* = -(\frac{rx^*f_1}{(1+f_1z^*)^2} + \frac{\beta\alpha_1x^*(1-2n_1z^*)+\alpha_1{x^*}^2(1-n_1z^*)^2}{(\beta+(1-n_1z^*)x^*)^2}) < 0,$

$a_{22}^* = \frac{s}{1+f_2z^*} - 2by^* - \frac{\alpha_2z^*-\gamma\alpha_2{y^*}^2z^*}{(1+\gamma{y^*}^2)^2} - \frac{m_3q_2{E_2}^2}{(m_3E_2+m_4y^*)^2},$

$a_{23}^* = -(\frac{sy^*f_2}{(1+f_2z^*)^2} + \frac{\alpha_2y^*}{1+\gamma{y^*}^2}) < 0,$

$a_{31}^* = \frac{\beta c_1\alpha_1z^*(1-n_1z^*)}{(\beta+(1-n_1z^*)x^*)^2},$

$a_{32}^* = \frac{c_2\alpha_2z^*-\gamma c_2\alpha_2{y^*}^2z^*}{\left(1+\gamma{y^*}^2\right)^2},$

$a_{33}^* = \frac{c_1\beta\alpha_1x^*(1-2n_1z^*)+c_1\alpha_1{x^*}^2(1-n_1z^*)^2}{(\beta+(1-n_1z^*)x^*)^2} + \frac{c_2\alpha_2y^*}{1+\gamma{y^*}^2} - (2mz^*+d).$

Therefore, the characteristic equation is:

$$\lambda^3 + H_1^*\lambda^2 + H_2^*\lambda + H_3^* = 0,$$

where

$$H_1^* = -(a_{11}^* + a_{22}^* + a_{33}^*),$$
$$H_2^* = a_{11}^*a_{22}^* + (a_{22}^*a_{33}^* - a_{23}^*a_{32}^*) + (a_{11}^*a_{33}^* - a_{13}^*a_{31}^*),$$
$$\Delta = H_1^*H_2^* - H_3^* = -{a_{11}^*}^2(a_{22}^* + a_{33}^*) - {a_{22}^*}^2(a_{11}^* + a_{33}^*) - {a_{33}^*}^2(a_{22}^* + a_{11}^*) -2a_{11}^*a_{22}^*a_{33}^* + a_{23}^*a_{32}^*(a_{22}^* + a_{33}^*) + a_{13}^*a_{31}^*(a_{11}^* + a_{33}^*) \; .$$

So, by Routh-Hurwitz criterion, PPE is LAS if condition (15) holds, which is saddle point otherwise.

6. Global Stability Analysis

The global asymptotically stability (GAS) of all the points in the equilibrium for model (1) is studied in the following theorems by using the Lyapunov function to calculate the attractive basin.

Theorem (6.1): Since TPE is LAS it becomes a GAS when:

$$\left.\begin{array}{l} r < q_1E_1 \\ s < q_2E_2 \end{array}\right\} , \tag{16}$$

holds.

Proof: Assume that this positive value as a definite function:

$$W_0(t) = x(t) + y(t) + z(t).$$

Where $W_0(t): \mathbb{R}_+^3 \to \mathbb{R}$ is continuously differentiable with $W_0(0,0,0) = 0$, and $W_0(x,y,z) > 0, \; \forall \; (x,y,z) \neq (0, 0, 0)$.

Also,

$$\frac{dW_0}{dt} = \frac{rx}{1+f_1z} - ax^2 - \frac{\alpha_1(1-n_1z)xz}{\beta+(1-n_1z)x} - \frac{q_1E_1x}{(m_1E_1+m_2x)} + \frac{sy}{1+f_2z} - by^2 - \frac{\alpha_2yz}{1+\gamma y^2} - \frac{q_2E_2y}{(m_3E_2+m_4y)} + \frac{c_1\alpha_1(1-n_1z)xz}{\beta+(1-n_1z)x} + \frac{c_2\alpha_2yz}{1+\gamma y^2} - mz^2 - dz.$$

By doing more calculations, we get the following result:

$$\frac{dW_0}{dt} \leq -(q_1E_1 - r)x - (q_2E_2 - s)y - dz.$$

As a result, W_0 is a Lyapunov function when we obtain $\frac{dW_0}{dt} < 0$ from condition (16), hence TPE is GAS.

Theorem (6.2): Assume FAPE is LAS, therefore it's GAS in the sub region R_+^3 when:

$$\left.\begin{array}{r} s < \frac{q_2 E_2 y}{(m_3 E_2 + m_4 y)} \\ r\, f_1 \bar{x} + \alpha_1 \bar{x} < d \\ \frac{q_1 E_1 m_2}{(m_1 E_1 + m_2 x)(m_1 E_1 + m_2 \bar{x})} < a \end{array}\right\}, \quad (17)$$

are satisfied.

Proof: Suppose this positive value as a definite function:

$$W_1(t) = \left(x - \bar{x} - \bar{x} ln \frac{x}{\bar{x}}\right) + y + z.$$

Where $W_1(t): \mathbb{R}_+^3 \to \mathbb{R}$ is continuously differentiable with $W_1(\bar{x}, 0,0) = 0$, and $W_1(x, y, z) > 0$, $\forall$ $(x, y, z) \neq (\bar{x}, 0, 0)$.

Also,

$$\frac{dW_1}{dt} = \left(\frac{x - \bar{x}}{x}\right)\frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt}.$$

By using model (1) and more calculations, we obtain:

$$\frac{dW_1}{dt} \leq -\left(a - \frac{q_1 E_1 m_2}{(m_1 E_1 + m_2 x)(m_1 E_1 + m_2 \bar{x})}\right)(x - \bar{x})^2 - \left(\frac{q_2 E_2 y}{(m_3 E_2 + m_4 y)} - s\right) y$$
$$-\left(d - (r\, f_1 \bar{x} + \alpha_1 \bar{x})\right) z.$$

As a result, W_1 regarded as Lyapunov function when we get $\frac{dW_1}{dt} < 0$ from condition (17), hence FAPE is GAS.

Theorem (6.3): Assume SAPE is LAS, so it's GAS in the sub region R_+^3 under these conditions:

$$\left.\begin{array}{r} r < \frac{q_1 E_1}{(m_1 E_1 + m_2 x)} \\ s\, f_2 \check{y} + \alpha_2 \check{y} < d \\ \frac{q_2 E_2 m_4}{(m_3 E_2 + m_4 y)(m_3 E_2 + m_4 \check{y})} < b \end{array}\right\}. \quad (18)$$

Proof: Let this positive value as a definite function:

$$W_2(t) = x + \left(y - \check{y} - \check{y} ln \frac{y}{\check{y}}\right) + z.$$

Where $W_2(t): \mathbb{R}_+^3 \to \mathbb{R}$ is continuously differentiable with $W_2(0, \check{y}, 0) = 0$, and $W_2(x, y, z) > 0$, $\forall (x, y, z) \neq (0, \check{y}, 0)$.

Also,

$$\frac{dW_2}{dt} = \frac{dx}{dt} + \left(\frac{y - \check{y}}{y}\right)\frac{dy}{dt} + \frac{dz}{dt}.$$

By using model (1) and more calculations, we get:

$$\frac{dW_2}{dt} \leq -(\frac{q_1E_1}{(m_1E_1+m_2x)}-r)x-(b-\frac{q_2E_2m_4}{(m_3E_2+m_4y)(m_3E_2+m_4\check{y})})(y-\check{y})^2$$
$$-\left(d-(s\,f_2\check{y}+\alpha_2\check{y}\,)\right)z.$$

Therefore, W_2 regarded as Lyapunov function when obtained $\frac{dW_2}{dt}<0$ from condition (18), hence SAPE is GAS.

Theorem (6.4): Assuming that PDFPE is LAS, the following conditions define the basin of attraction for this point:

$$\left.\begin{array}{r}\frac{q_1E_1m_2}{(m_1E_1+m_2x)(m_1E_1+m_2\bar{\bar{x}})}<a\\ \frac{q_2E_2m_4}{(m_3E_2+m_4y)(m_3E_2+m_4\bar{\bar{y}})}<b\\ \left(r\,f_1+\alpha_1(1-n_1z)\right)\bar{\bar{x}}+(s\,f_2+\alpha_2)\bar{\bar{y}}<d\end{array}\right\}. \quad (19)$$

Proof: Suppose this positive value as a definite function:

$$W_3(t)=\left(x-\bar{\bar{x}}-\bar{\bar{x}}ln\frac{x}{\bar{\bar{x}}}\right)+\left(y-\bar{\bar{y}}-\bar{\bar{y}}ln\frac{y}{\bar{\bar{y}}}\right)+z.$$

Such that $W_3(t):\mathbb{R}_+^3\to\mathbb{R}$ is continuously differentiable with $W_3(\bar{\bar{x}},\bar{\bar{y}},0)=0$, and $W_3(x,y,z)>0,\ \forall\ (x,y,z)\neq(\bar{\bar{x}},\bar{\bar{y}},0)$.

So,

$$\frac{dW_3}{dt}=\left(\frac{x-\bar{\bar{x}}}{x}\right)\frac{dx}{dt}+\left(\frac{y-\bar{\bar{y}}}{y}\right)\frac{dy}{dt}+\frac{dz}{dt}.$$

Then, by using model (1) and more calculations, we obtained:

$$\frac{dW_3}{dt}\leq-(a-\frac{q_1E_1m_2}{(m_1E_1+m_2x)(m_1E_1+m_2\bar{\bar{x}})})(x-\bar{\bar{x}})^2-\left(b-\frac{q_2E_2m_4}{(m_3E_2+m_4y)(m_3E_2+m_4\bar{\bar{y}})}\right)(y-\bar{\bar{y}})^2$$
$$-\left(d-(\left(r\,f_1+\alpha_1(1-n_1z)\right)\bar{\bar{x}}+(s\,f_2+\alpha_2)\bar{\bar{y}}\right)z.$$

Consequently, in the region that satisfied condition (19), W_3 behaved as a Lyapunov function with respect to PDFPE and $\frac{dW_3}{dt}<0$, so PDFPE is GAS.

Theorem (6.5): Assuming that SPFPE is LAS, the following conditions define the basin of attraction for this point:

$$\left.\begin{array}{r}\frac{\alpha_1(1-n_1z)(1-n_1\tilde{z})\tilde{z}}{(\beta+(1-n_1z)x)(\beta+(1-n_1\tilde{z})\tilde{x})}+\frac{q_1E_1m_2}{(m_1E_1+m_2x)(m_1E_1+m_2\tilde{x})}<a\\ s<\frac{q_2E_2}{(m_3E_2+m_4y)}+c_2\alpha_2\tilde{z}\\ k_{12}^2\ \leq 4mk_{11}\end{array}\right\}. \quad (20)$$

Where

$k_{11} = a - \frac{\alpha_1(1-n_1 z)(1-n_1\tilde{z})\tilde{z}}{(\beta+(1-n_1 z)x)(\beta+(1-n_1\tilde{z})\tilde{x})} - \frac{q_1 E_1 m_2}{(m_1 E_1+m_2 x)(m_1 E_1+m_2\tilde{x})}$,

$k_{12} = \frac{r f_1}{(1+f_1 z)(1+f_1\tilde{z})} - \frac{\alpha_1(1-n_1 z)(1-n_1\tilde{z})\left((1-c_1)\beta+\tilde{x}\right)}{(\beta+(1-n_1 z)x)(\beta+(1-n_1\tilde{z})\tilde{x})}$.

Proof: Suppose this positive value as a definite function:

$$W_4(t) = \left(x - \tilde{x} - \tilde{x}\ln\frac{x}{\tilde{x}}\right) + y + \left(z - \tilde{z} - \tilde{z}\ln\frac{z}{\tilde{z}}\right).$$

Such that $W_4(t): \mathbb{R}_+^3 \to \mathbb{R}$ is continuously differentiable with $W_4(\tilde{x}, 0, \tilde{z}) = 0$, and $W_4(x, y, z) > 0, \ \forall \ (x, y, z) \neq (\tilde{x}, 0, \tilde{z})$.

Also,

$$\frac{dW_4}{dt} = \left(\frac{x-\tilde{x}}{x}\right)\frac{dx}{dt} + \frac{dy}{dt} + \left(\frac{z-\tilde{z}}{z}\right)\frac{dz}{dt}.$$

By using model (1) and more calculations, $\frac{dW_4}{dt}$ can be written as:

$$\frac{dW_4}{dt} \leq -[k_{11}(x-\tilde{x})^2 - k_{12}(x-\tilde{x})(z-\tilde{z}) + m(z-\tilde{z})^2] - \left(\frac{q_2 E_2}{(m_3 E_2+m_4 y)} + c_2\alpha_2\tilde{z} - s\right)y,$$

and by condition (20), we obtained:

$$\frac{dW_4}{dt} \leq -\left[\sqrt{k_{11}}\,(x-\tilde{x}) + \sqrt{m}(z-\tilde{z})\right]^2 - \left(\frac{q_2 E_2}{(m_3 E_2+m_4 y)} + c_2\alpha_2\tilde{z} - s\right)y.$$

As a result, in the region that satisfied condition (20), W_4 behaved as a Lyapunov function with respect to SPFPE and $\frac{dW_4}{dt} < 0$, hence SPFPE is GAS.

Theorem (6.6): Assuming that FPFPE is LAS, the following conditions define the basin of attraction for this point:

$$\left.\begin{aligned} \frac{\gamma\alpha_2\hat{z}(y+\hat{y})}{(1+\gamma y^2)(1+\gamma\hat{y}^2)} + \frac{q_2 E_2 m_4}{(m_3 E_2+m_4 y)(m_3 E_2+m_4\hat{y})} < b \\ r < \frac{q_1 E_1}{(m_1 E_1+m_2 x)} + \frac{c_1\alpha_1(1-n_1 z)\hat{z}}{\beta+(1-n_1 z)x} \\ G_{12}^2 \leq 4mG_{11} \end{aligned}\right\}. \qquad (21)$$

Where

$G_{11} = b - \frac{\gamma\alpha_2\hat{z}(y+\hat{y})}{(1+\gamma y^2)(1+\gamma\hat{y}^2)} - \frac{q_2 E_2 m_4}{(m_3 E_2+m_4 y)(m_3 E_2+m_4\hat{y})}$,

$G_{12} = \frac{s f_2}{(1+f_2 z)(1+f_2\hat{z})} + \frac{\alpha_2}{(1+\gamma y^2)} - \frac{c_2\alpha_2(1-\gamma y\hat{y})}{(1+\gamma y^2)(1+\gamma\hat{y}^2)}$.

Proof: Let the positive value as a definite function:

$$W_5(t) = x + \left(y - \hat{y} - \hat{y}ln\frac{y}{\hat{y}}\right) + \left(z - \hat{z} - \hat{z}ln\frac{z}{\hat{z}}\right).$$

Where $W_5(t): \mathbb{R}_+^3 \rightarrow \mathbb{R}$ is continuously differentiable with $W_5(0, \hat{y}, \hat{z}) = 0$, and $W_5(x, y, z) > 0, \ \forall \ (x, y, z) \neq (0, \hat{y}, \hat{z})$.

Also,

$$\frac{dW_5}{dt} = \frac{dx}{dt} + \left(\frac{y - \hat{y}}{y}\right)\frac{dy}{dt} + \left(\frac{z - \hat{z}}{z}\right)\frac{dz}{dt}.$$

Then by using model (1) and more calculations, $\frac{dW_5}{dt}$ can be written as:

$$\frac{dW_5}{dt} \leq -[G_{11}\,(y - \hat{y})^2 - G_{12}(y - \hat{y})(z - \hat{z}) + m(z - \hat{z})^2] - \left(\frac{q_1E_1}{(m_1E_1 + m_2x)} + \frac{c_1\alpha_1(1 - n_1z)\hat{z}}{\beta + (1 - n_1z)x} - r\right)x$$

and by condition (21), we obtained:

$$\frac{dW_5}{dt} \leq -\left[\sqrt{G_{11}}\,(y - \hat{y}) + \sqrt{m}(z - \hat{z})\right]^2 - \left(\frac{q_1E_1}{(m_1E_1+m_2x)} + \frac{c_1\alpha_1(1-n_1z)\hat{z}}{\beta+(1-n_1z)x} - r\right)x.$$

Consequently, in the region that satisfied condition (21), W_5 behaved as a Lyapunov function with respect to FPFPE and $\frac{dW_5}{dt} < 0$, hence FPFPE is GAS.

Theorem (6.7): Assuming that PPE is LAS, the following conditions define the basin of attraction for this point:

$$\left.\begin{array}{r} v_{11} > 0 \\ v_{22} > 0 \\ 2v_{11}v_{33} \geq v_{13}^2 \\ 2v_{22}v_{33} \geq v_{23}^2 \end{array}\right\}, \qquad (22)$$

Where

$$v_{11} = a - \frac{\alpha_1(1-n_1z)(1-n_1z^*)z^*}{(\beta+(1-n_1z)x)(\beta+(1-n_1z^*)x^*)} - \frac{q_1E_1m_2}{(m_1E_1+m_2x)(m_1E_1+m_2x^*)},$$

$$v_{22} = b - \frac{\gamma\alpha_2z^*(y+y^*)}{(1+\gamma y^2)(1+\gamma y^{*2})} - \frac{q_2E_2m_4}{(m_3E_2+m_4y)(m_3E_2+m_4y^*)},$$

$$v_{33} = m, \quad v_{13} = \frac{rf_1}{(1+f_1z)(1+f_1z^*)} - \frac{\alpha_1(1-n_1z)(1-n_1z^*)((1-c_1)\beta+x^*)}{(\beta+(1-n_1z)x)(\beta+(1-n_1z^*)x^*)},$$

$$v_{23} = \frac{sf_2}{(1+f_2z)(1+f_2z^*)} + \frac{\alpha_2}{(1+\gamma y^2)} - \frac{c_2\alpha_2(1-\gamma yy^*)}{(1+\gamma y^2)(1+\gamma y^{*2})}.$$

Proof: Suppose this positive value as a definite function:

$$W_6(t) = \left(x - x^* - x^* ln\frac{x}{x^*}\right) + \left(y - y^* - y^* ln\frac{y}{y^*}\right) + \left(z - z^* - z^* ln\frac{z}{z^*}\right).$$

Where $W_6(t): \mathbb{R}^3_+ \to \mathbb{R}$ is continuously differentiable with $W_6(x^*, y^*, z^*) = 0$, and $W_6(x, y, z) > 0,\ \forall\ (x, y, z) \neq (x^*, y^*, z^*)$.

Also,

$$\frac{dW_6}{dt} = \left(\frac{x - x^*}{x}\right)\frac{dx}{dt} + \left(\frac{y - y^*}{y}\right)\frac{dy}{dt} + \left(\frac{z - z^*}{z}\right)\frac{dz}{dt}.$$

So, using model (1) and more calculations, $\frac{dW_6}{dt}$ can be written as:

$$\frac{dW_6}{dt} \le -\left[v_{11}(x - x^*)^2 + v_{13}(x - x^*)(z - z^*) + \frac{v_{33}}{2}(z - z^*)^2\right] - \left[v_{22}(y - y^*)^2 + v_{23}(y - y^*)(z - z^*) + \frac{v_{33}}{2}(z - z^*)^2\right].$$

and by condition (22), we getting:

$$\frac{dW_6}{dt} \le -\left[\sqrt{v_{11}}\ (y - y^*) + \sqrt{\frac{v_{33}}{2}}(z - z^*)\right]^2 - \left[\sqrt{v_{22}}\ (y - y^*) + \sqrt{\frac{v_{33}}{2}}(z - z^*)\right]^2.$$

Therefore, in the region that satisfied condition (22), W_6 behaved as a Lyapunov function with respect to PPE and $\frac{dW_6}{dt} < 0$, hence PPE is GAS.

7. NUMERICAL SIMULATION

In this section, we used numerical simulations to verify our findings and improve our knowledge of how changing parameter values impact the dynamics of the system. We started this numerical simulation with a variety of initial conditions and then proceeded to use hypothetical parameters. MATLAB R2009b was utilized to show the trajectories were generated.

$$\begin{array}{lllll} r = 0.9, & f_1 = 0.05, & a = 0.04, & \alpha_1 = 0.4, & \beta = 0.6, \\ n_1 = 0.1, & q_1 = 0.2, & E_1 = 0.3, & m_1 = 0.4, & m_2 = 0.5, \\ s = 0.9, & f_2 = 0.001, & b = 0.05, & \alpha_2 = 0.2, & \gamma = 0.01, \\ q_2 = 0.2, & E_2 = 0.1, & m_3 = 0.2, & m_4 = 0.3, & \\ c_1 = 0.25, & c_2 = 0.15, & d = 0.01, & m = 0.06. & \end{array} \quad (23)$$

The solution of model (1) under set (23) converges asymptotically to $P_6 = (16.396,\ 9.933, 3.905)$ from various initial conditions as shown in Fig. 1.

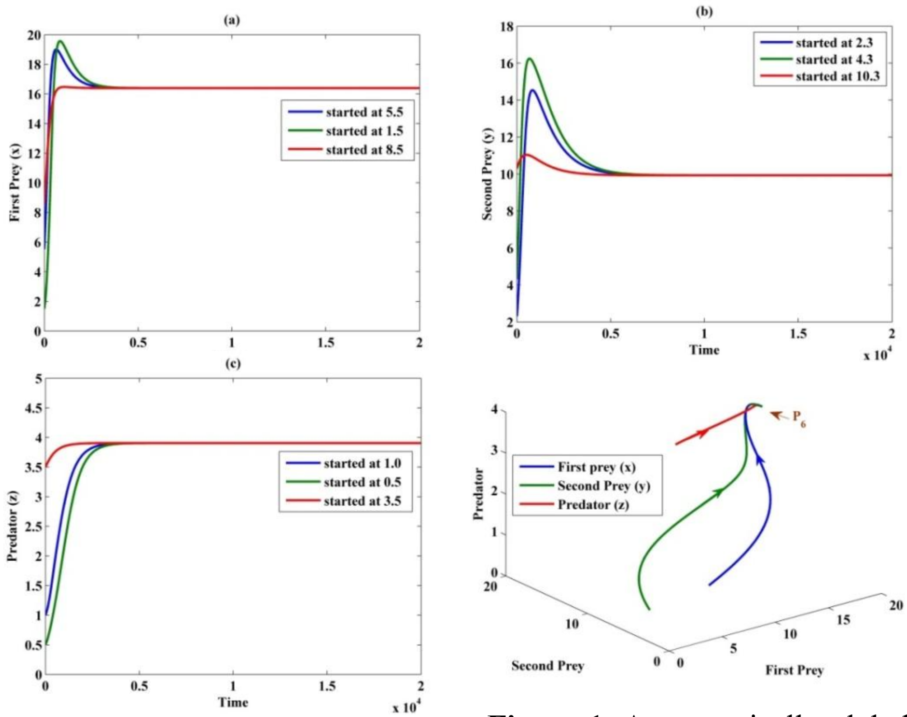
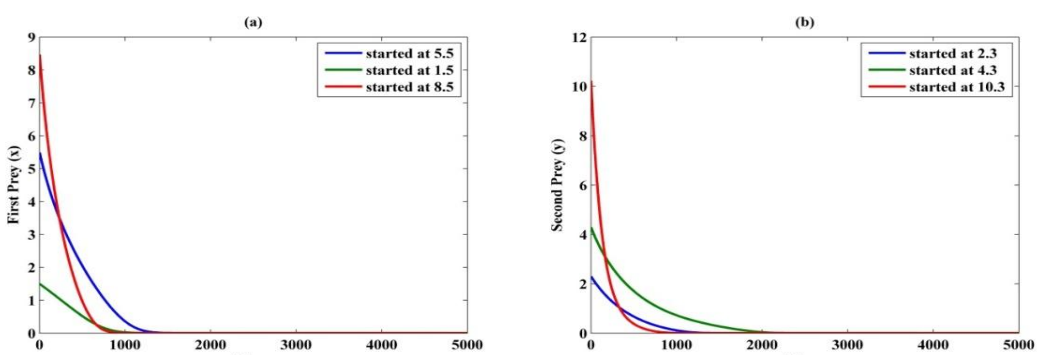


Figure 1. Asymptotically globally stable to PPE with data (23) and numerous initial conditions. (a) First prey trajectories. (b) Second prey trajectories. (c) Predator trajectories. (d) 3D-phase sketch of the model (1).

Obviously, Fig. 1 confirms the theoretical results, showing that the PPE is GAS.

However, the trajectory of model (1) approaches asymptotically to TPE $P_0 = (0\,,0\,,0\,)$, as illustrated in Fig. 2, as in set (23) with $r = 0.1, s = 0.1$.



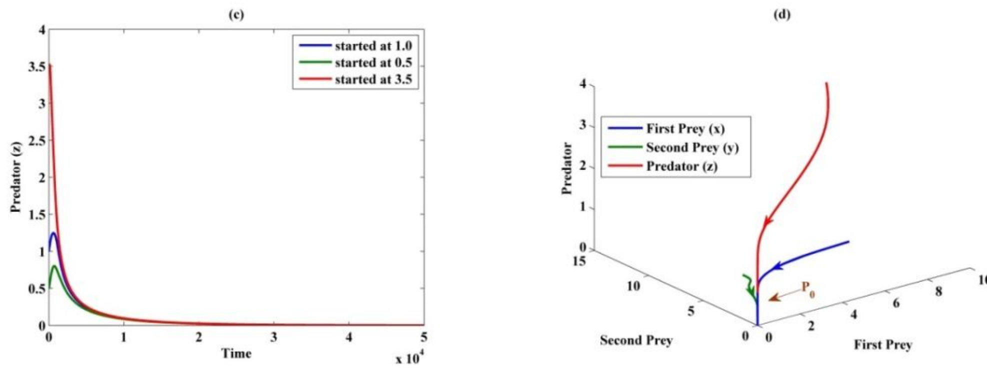


Figure 2. Asymptotically globally stable to TPE with data (23) and numerous initial conditions. (a) First prey trajectories. (b) Second prey trajectories. (c) Predator trajectories. (d) 3D-phase sketch of the model (1).

Additionally, numerical simulations have shown that when the parameters $s = 0.1$ and $d = 0.3$ are changed for the data (23), the model (1) trajectory approaches the global stable FAPE $P_1 = (22.367, 0, 0)$, which can be seen in Fig. 3.

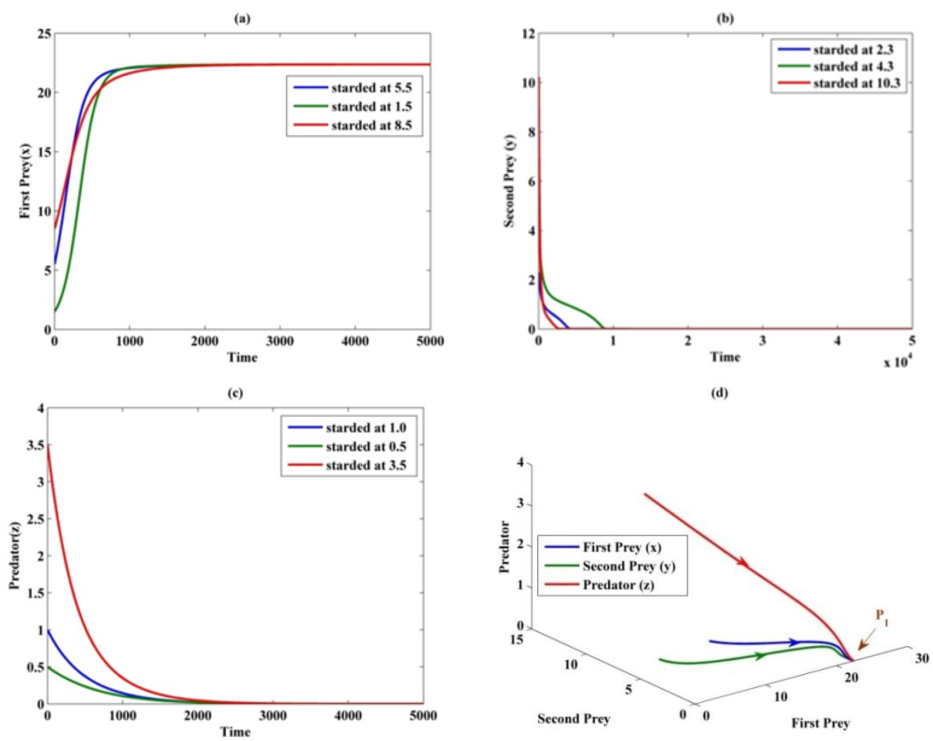


Figure 3. Asymptotically globally stable to FAPE with data (23) and numerous initial conditions. (a) First prey trajectories. (b) Second prey trajectories. (c) Predator trajectories. (d) 3D-phase sketch of the model (1).

Furthermore, Fig. 4 illustrates the changing values $r = 0.1$ and $d = 0.2$ with data (23) indicating that every trajectory for model (1) approaches asymptotically to global stable SAPE $P_2 = (0, 17.925, 0)$.

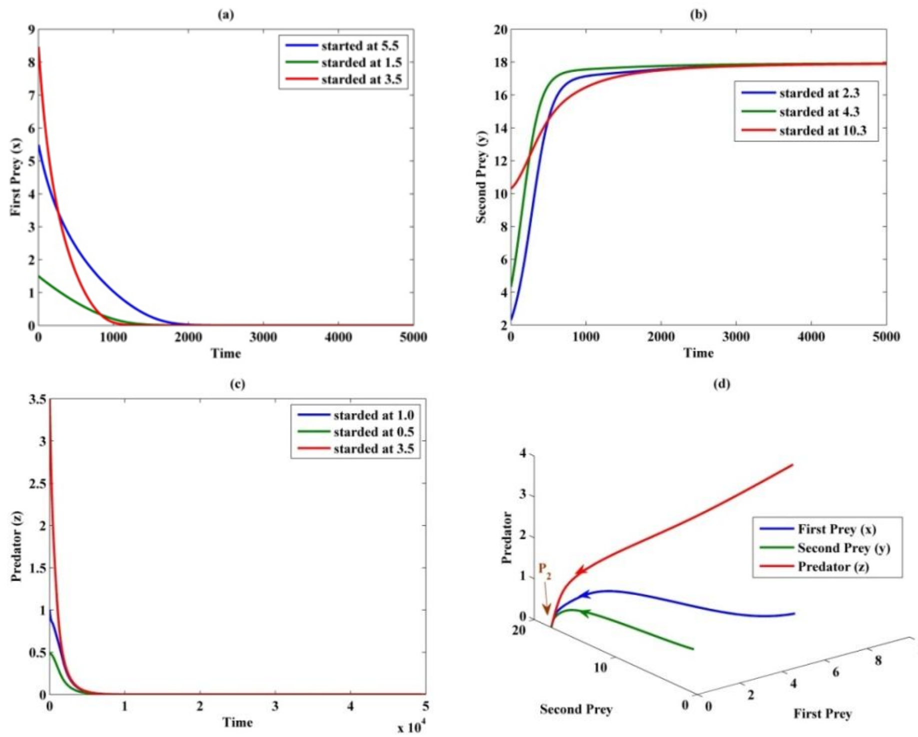
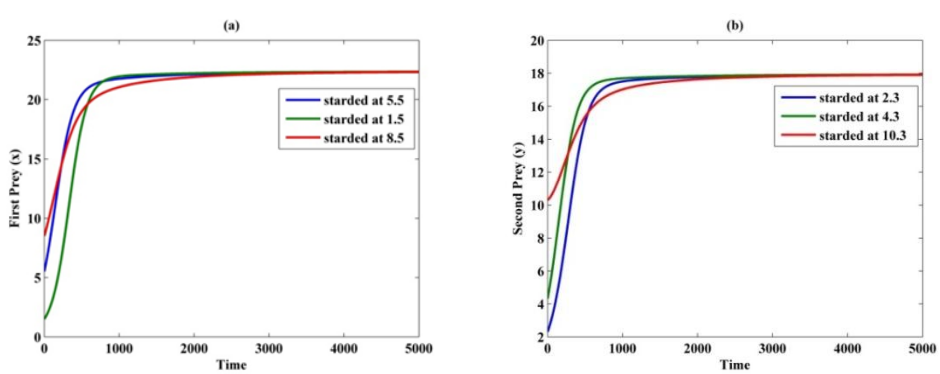


Figure 4. Asymptotically globally stable to SAPE with data (23) and numerous initial conditions. (a) First prey trajectories. (b) Second prey trajectories. (c) Predator trajectories. (d) 3D-phase sketch of the model (1).

The trajectory for model (1) approaches asymptotically to global stable PDFPE $P_3 = (22.367, 17.925, 0)$, as described in Fig. 5, while data (23) with $d = 0.3$ is used.



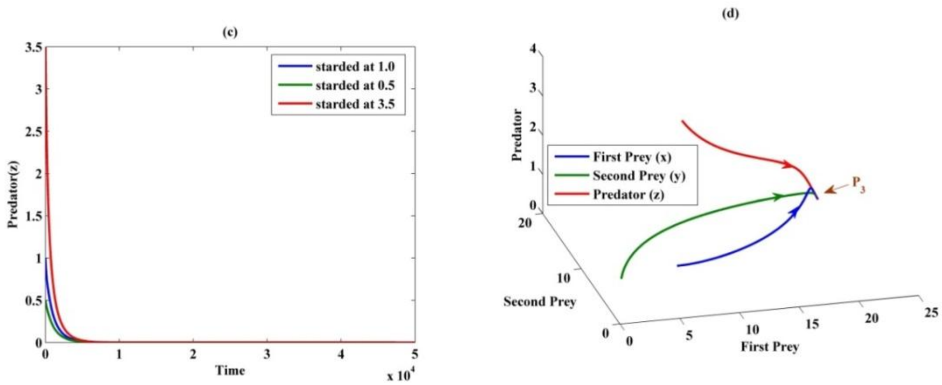


Figure 5. Asymptotically globally stable to PDFPE with data (23) and numerous initial conditions. (a) First prey trajectories. (b) Second prey trajectories. (c) Predator trajectories. (d) 3D-phase sketch of the model (1).

Instead, if data (23) with $s = 0.1$, then each for model (1) trajectory approaches asymptotically to global stable SPFPE $P_4 = (\,20.145, 0, 1.443\,)$, as dispelled in Fig. 6.

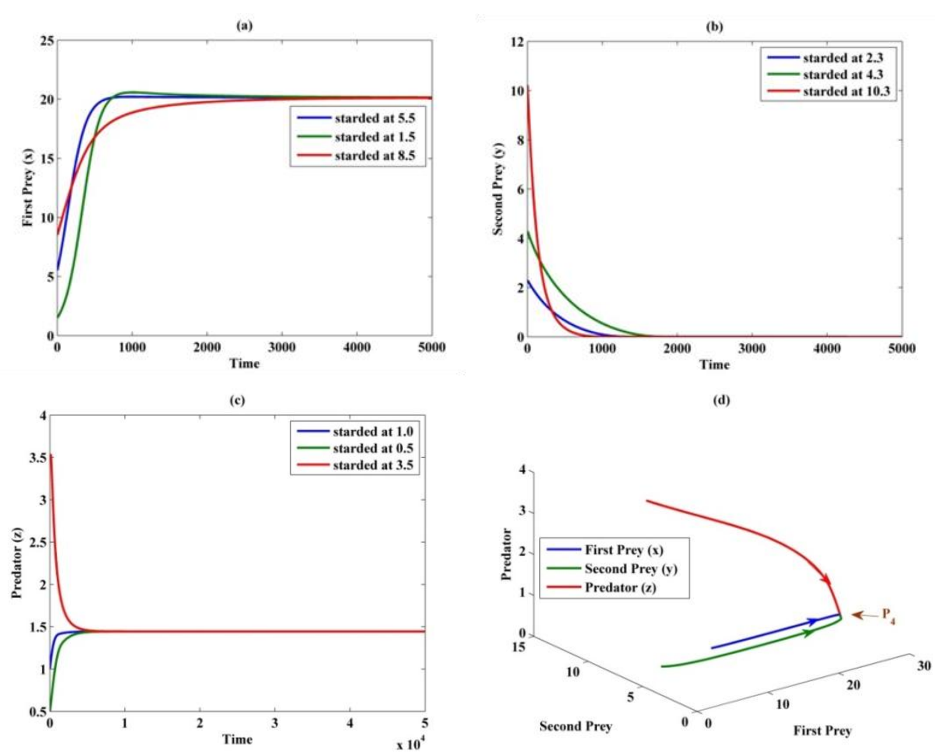


Figure 6. Asymptotically globally stable to SPFPE with data (23) and numerous initial conditions. (a) First prey trajectories. (b) Second prey trajectories. (c) Predator trajectories. (d) 3D-phase sketch of the model (1).

Moreover, when the data (23) is considered with $r = 0.1$ the trajectory approaches asymptotically to global stable FPFPE $P_5 = (\,0{,}15.348, 2.120\,)$, as seen in Fig. 7.

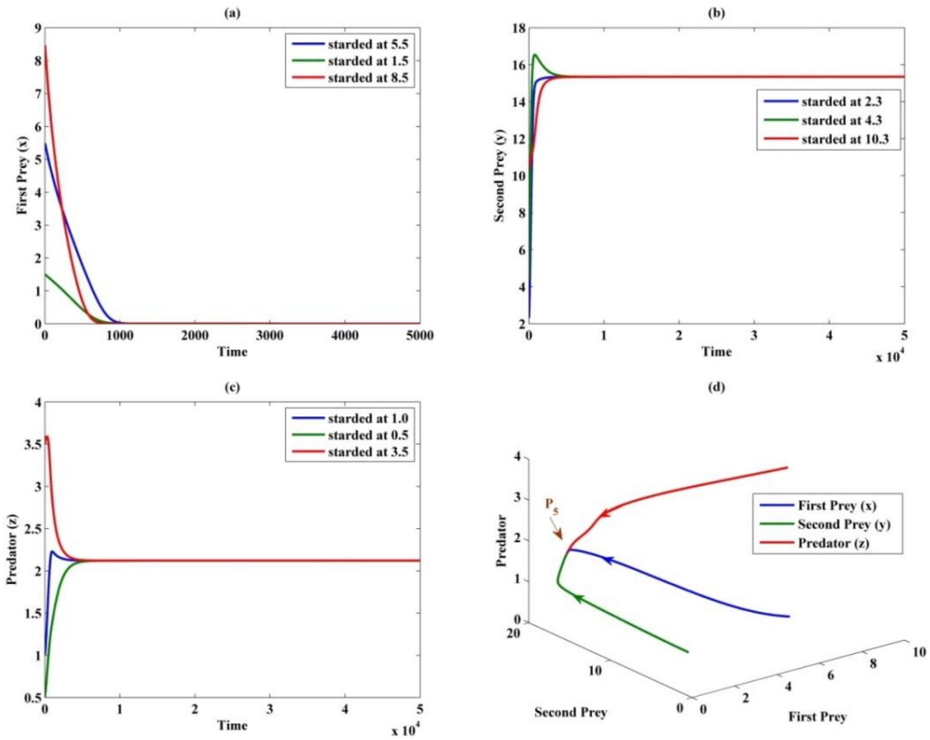


Figure 7. Asymptotically globally stable to FPFPE with data (23) and numerous initial conditions. (a) First prey trajectories. (b) Second prey trajectories. (c) Predator trajectories. (d) 3D-phase sketch of the model (1).

Currently, we have observed the following results to study how changing just one parameter at a time affects the dynamic behaviour of model (1).

For the parameters value given in data (23) and the range of α_1 beginning at $\alpha_1 \geq 0.9$, as seen in Fig. 8, model (1) goes asymptotically at $P_5 = (0, \hat{y}, \hat{z})$, which is FPFPE.

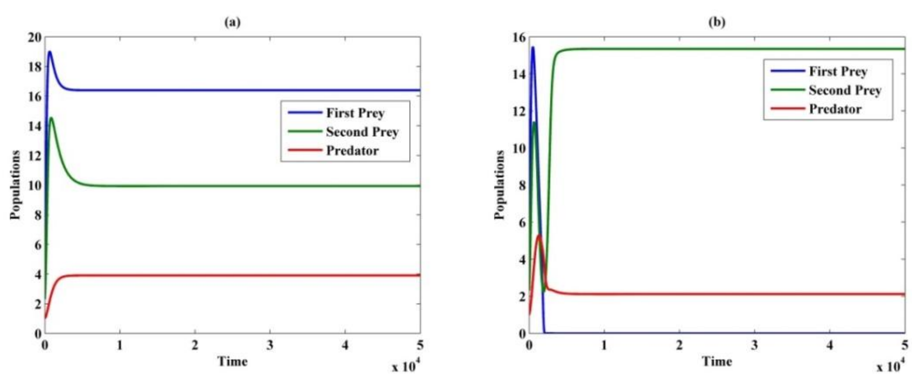


Figure 8. The solution to model (1) in data (23) over time, with different α_1 values. (a) Globally asymptotically stable PPE for $\alpha_1 = 0.4$. (b) Globally asymptotically stable FPFPE for $\alpha_1 = 0.9$.

Further, the trajectory of the model (1) converges asymptotically to the SPFPE $P_4 = (\tilde{x}, 0, \tilde{z})$ for the parameter value given in data (23) with variable α_2 at range $\alpha_2 \geq 0.4$, as indicated in Fig 9.

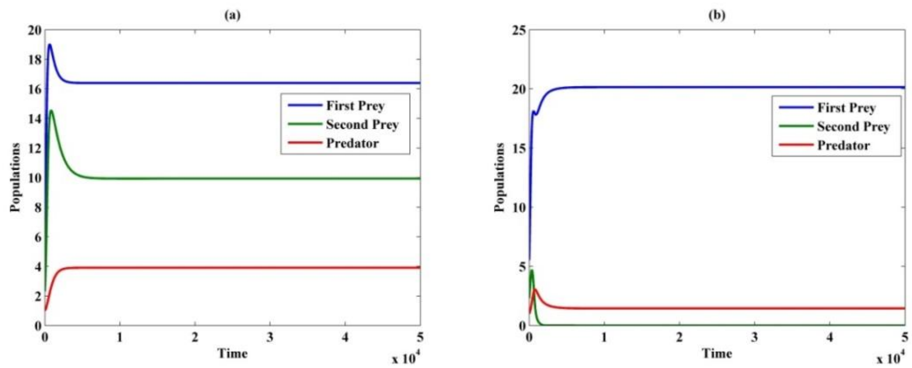


Figure 9. The solution to model (1) in data (23) over time, with different α_2 values. (a) Globally asymptotically stable PPE for $\alpha_2 = 0.2$. (b) Globally asymptotically stable SPFPE for $\alpha_2 = 0.4$.

Even so, model (1) approaches asymptotically to FPFPE $P_5 = (0, \hat{y}, \hat{z})$, as shown in Fig. 10, if the first prey changed the $m_2 \leq 0.01$ and all the other parameters remained the same from the data (23).

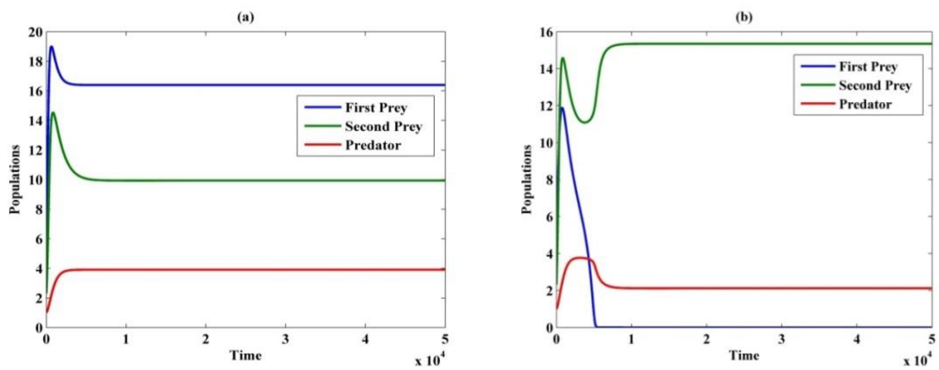


Figure 10. The solution to model (1) in data (23) over time, with different m_2 values. (a) Globally asymptotically stable PPE for $m_2 = 0.5$. (b) Globally asymptotically stable FPFPE for $m_2 = 0.01$.

The last part of the model (1) shows how the second prey changes in the range $m_4 \leq 0.02$, leading to the SPFPE $P_4 = (\tilde{x}, 0, \tilde{z})$, which can be seen in Fig. 11.

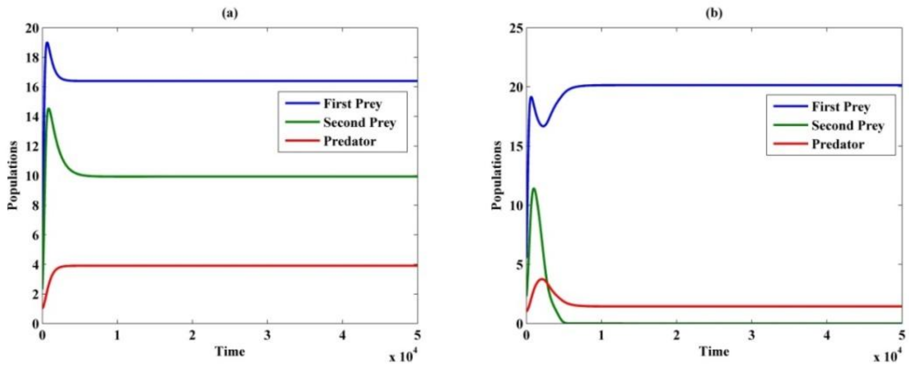


Figure 11. The solution to model (1) in data (23) over time, with different m_4 values. (a) Globally asymptotically stable PPE for $m_4 = 0.3$. (b) Globally asymptotically stable SPFPE for $m_4 = 0.02$.

8. CONCLUSION AND DISCUSSION

In this work, we developed and analyzed an ecological model that characterized the prey and predator, with fear and refuge dependent on predator and nonlinear harvesting in the prey. Three nonlinear autonomous ordinary differential equations built into the model describe the dynamics of three distinct species, namely the first prey (x), second prey (y), and predator (z). The Boundedness of model (1) was discussed. For every possible point of equilibrium, the existence condition was defined. Local and global stability investigations were done for these points. Lastly, numerical simulation is employed to estimate the control group of parameters that impact the model dynamics and validate the analytical results that were achieved.

Therefore, by numerically solving model (1) for various sets of initial points and parameters, we have reached the following conclusions; this process started with a hypothetical set of data (23):

1. Model (1) does not have a periodic dynamic; rather, its solution approaches one of the equilibrium points in asymptotic form.
2. Increasing the $\alpha_1 > 0.9$ destabilizes the PPE, and the model (1) approaches to FPFPE asymptotically.
3. Increasing the $\alpha_2 > 0.4$ destabilizes the PPE, and the model (1) approaches to SPFPE asymptotically.
4. Decreasing the $m_2 < 0.01$ destabilizes the PPE, and the model (1) approaches to FPFPE asymptotically.
5. Decreasing the $m_4 < 0.02$ destabilizes the PPE, and the model (1) approaches to SPFPE asymptotically.

6. Biased on the earlier explanation, it is clear that model (1) is highly responsive to variations in specific parameters. Consequently it is extremely controllable.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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