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LOCAL DYNAMICS, CHAOS CONTROL AND FLIP BIFURCATION ANALYSIS OF A DISCRETE EPIDEMIC MODEL WITH VITAL DYNAMICS AND VACCINATION

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Abstract. We study the local dynamics and existence of bifurcation sets at equilibrium states, bifurcation analysis and chaos of the epidemic model with vital dynamics and vaccination in $\mathbb{R}^2_+ = \{(I,S) : I,\ S \geq 0\}$. More specifically, it is proved that discrete epidemic model has disease-free and endemic equilibrium states under model's parameters restriction(s), and we have studied local dynamical properties at equilibrium states by the theory of linear stability. Furthermore, first we have pointed out the bifurcations sets at equilibrium states, and then proved that at disease-free equilibrium state discrete epidemic model does not undergo flip bifurcation but it undergoes only flip bifurcation at endemic equilibrium state by center manifold theorem and bifurcation theory. Additional, hybrid control strategy is utilized to control chaos in the epidemic model due to the occurrence of flip bifurcations. Finally, numerical simulations are given to verify theoretical results.

Keywords: numerical simulation; discrete epidemic model; bifurcation; chaos.

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1. Introduction

Recently, epidemiology of contagious diseases and control has been greatly increased due to the use of mathematical modeling. The infections diseases transmission is different from

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non-infections diseases, so for their effective control emergency planning, control programme evolution and policy making is required which based on study design, analysis and interpretation of data. Mathematical modeling play a key role in this regard. The rapid spread of disease at a highest rate in a given population is considered as epidemic. The damages which can be caused by epidemic are health loss, life loss, financial and economic loss. The extend of contagious diseases is so crucial that it changes the population's demographics. In order to control and eradicate the disease, anticipation and mediation measures are therefore essential. Mathematical models are useful for examining how a disease behaves when it infects a population and for determining the conditions under which it will be eradicated. The writing around epidemic models that have been established and investigated for different sorts of illnesses is exceptionally wealthy. In recent years, discrete mathematical models have acquired more concern since epidemic information are gathered in discrete intervals, and moreover numerical schemes utilize discretization for tackling differential equations. In addition, discrete mathematical models show more complex dynamics, for instance, Balamuralitharan & Radha [1] have investigated the Hopf and transcritical bifurcations of the following epidemic model:

$$\begin{cases} \dot{I} = \pi - \beta IS - \mu I, \\ \dot{S} = \beta IS - (\mu + \gamma + \mu_t + r)S, \\ \dot{V} = (\gamma + r)S - \mu V, \end{cases} \tag{1}$$

where π, β, γ, μ, μ_t and r are respectively denote individuals S recruitment rate, infection I rate, removal rate of individuals I, removal rate, natural death rate and treatment rate. Pérez, Avila-Vales & García-Almeida [2] have explored bifurcation of following epidemic model:

$$\begin{cases} \dot{I} = rI\left(1 - \frac{I}{K}\right) - \frac{\beta IS}{1+\mu S}, \\ \dot{S} = \frac{\beta IS}{1+\mu S} - \Theta S - \frac{\lambda S}{1+\varepsilon S}, \end{cases} \tag{2}$$

where S, I denotes susceptible and infected individuals; K and r, respectively denote carrying capacity and growth rate of class S. Li & Li [3] have investigated Bogdanov-Takens bifurcation of the model:

$$
\begin{cases}
\dot{I} = A - dI - \frac{\beta I S^3}{b+aS+S^2}, \\
\dot{S} = \frac{\beta I S^3}{b+aS+S^2} - (d+\gamma+\varepsilon)S, \\
\dot{V} = \gamma S - dV,
\end{cases} \tag{3}
$$

where susceptible is S, I is infected and V is vaccinated individuals; A, d, γ and ε are birth rate of S, death rate, removal rate and per capita infection related to the death rate of the individuals. Parsamanesh & Erfanian [4] have investigated global dynamics of the epidemic model:

$$
\begin{cases}
\dot{I} = I(\frac{\beta(V-I-S)}{V} - (\mu+\gamma+\alpha)), \\
\dot{S} = qA + p(V-I) - (\mu+p+\varepsilon)S, \\
\dot{V} = A - \mu V - \alpha I,
\end{cases} \tag{4}
$$

with all positive parameters and initial conditions. Cao et al. [5] have explored bifurcations of the model:

$$
\begin{cases}
I_{t+1} = I_t + \bigwedge - \beta I_t S_t - dI_t, \\
S_{t+1} = S_t + \beta I_t S_t - (d+\gamma)S_t - m,
\end{cases} \tag{5}
$$

where $\bigwedge$, β, d and γ are the recruitment rate, transmission rate, natural death rate and spontaneous recovery rate of individuals I. For further study in this direction, we suggest the interested reader to the works of eminent researchers [6–12].

Inspired by the aforementioned research, hereafter we will give mathematical formulation of a desired discrete epidemic model by considering population are divided into three categories such as infected individuals I, susceptible individuals S and vaccinated individuals V. Now if Δt is considered to appropriate time increment then changes in model occurs at $t = 0, \Delta t, \cdots$ where at $t = n\Delta t$, N_t denotes total number of individuals. Furthermore, at $t = n\Delta t$ the numbers of individuals in other groups are designated as I_t, S_t and V_t. So, whole conceivable changes in the model and transmissions between its sub-populations along with their rates of transmissions are represented in Figure 1. Here the models's parameters μ_3, μ_1, μ_4 and μ_6 are respectively denote natural death rate, contact rate, cure rate and rate of immunity loss while μ_7 and μ_5 are rates of vaccination in individuals S_t and newcomers. So, based on Figure 1 and preassumptions the model's equations takes the form:

$$(6)\qquad \begin{cases} I_{t+1} = \frac{\mu_1 S_t I_t}{N_t} + (1-(\mu_3+\mu_4))I_t, \\ S_{t+1} = (1-\mu_5)\mu_3 N_t - \frac{\mu_1 S_t I_t}{N_t} + (1-(\mu_3+\mu_7))S_t + (\mu_4)I_t + \mu_6 V_t, \\ V_{t+1} = \mu_5\mu_3 N_t + \mu_7 S_t + (1-(\mu_3+\mu_6))V_t. \end{cases}$$

It is noted here that individuals S becomes infected at standard incidence rate $\frac{\mu_1 S_t I_t}{Nt}$. Hereafter, in order to summing model's equations, which are depicted in (6), one can observe that $N_{t+1} = N_t$, and so size of population N will continue constant. So, model (6) takes the following required form [13]:

$$(7)\qquad \begin{cases} I_{t+1} &= \frac{\mu_1 S_t I_t}{\mu_2} + (1-(\mu_3+\mu_4))I_t, \\ S_{t+1} &= ((1-\mu_5)\mu_3+\mu_6)\mu_2 - \frac{\mu_1 S_t I_t}{\mu_2} + (1-(\mu_3+\mu_7+\mu_6))S_t + (\mu_4-\mu_6)I_t, \end{cases}$$

by letting $V_t = N - S_t - I_t$ where $N = \mu_2$. So, in this study our aim is to explore dynamical characteristics of the epidemic model, which is depicted in (7). Our investigations for the model (7) include:

- Existence of equilibrium states of discrete epidemic model (7).
- Local dynamical behavior and bifurcation sets at equilibrium states.
- Flip bifurcation analysis at equilibrium states.
- Examine of chaos by hybrid control strategy.
- Numerical verification of theoretical results.

The organization of rest of the paper are as follows: local dynamics and bifurcation sets at equilibrium states are studied in Section 2 whereas Section 3 is about the study of flip bifurcation at equilibrium states. The chaos control by Hybrid control strategy is examined in Section 4. In order to confirm theoretical results, some simulations are presented in Section 5 whereas conclusion is given in Section 6.

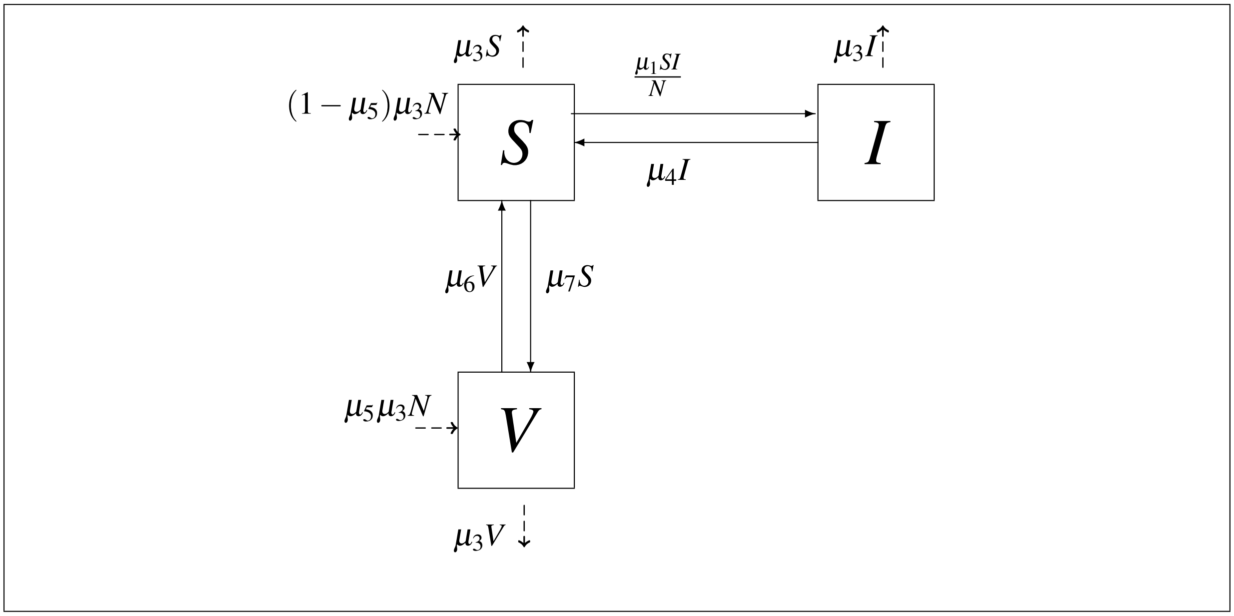


FIGURE 1. Flow chart of a epidemic model (6) along with the rates of transmissions

2. ANALYSIS OF STABILITY AT EQUILIBRIUM STATES AND BIFURCATION SETS

In this section, first we examine the equilibrium states and then we will study local behavior at equilibrium states for the discrete epidemic model (DEM) (7). Furthermore, at obtained non-hyperbolic condition(s), we will also identified the bifurcation sets in order to explore drastically change in the behavior of solution of DEM (7) by the variation of single involved model's parameters. Now in the following, we will find equilibrium states, in allowed parametric region $\mathbb{R}_+^2 = \{(I,S) : I,S \geq 0\}$. So, if equilibrium state (ES) of DEM (7) is (I,S) then

$$\begin{cases} I &= \frac{\mu_1 SI}{\mu_2} + (1-(\mu_3+\mu_4))I, \\ S &= ((1-\mu_5)\mu_3+\mu_6)\mu_2 - \frac{\mu_1 SI}{\mu_2} + (1-(\mu_3+\mu_7+\mu_6))S + (\mu_4-\mu_6)I. \end{cases} \tag{8}$$

It is noted here that system (8) satisfied obviously if $(I,S) = \left(0, \frac{((1-\mu_5)\mu_3+\mu_6)\mu_2}{\mu_3+\mu_7+\mu_6}\right)$. Therefore, DEM (7) has disease-free equilibrium state (DFES) $\left(0, \frac{((1-\mu_5)\mu_3+\mu_6)\mu_2}{\mu_3+\mu_7+\mu_6}\right)$ if $\mu_6 > \mu_3(\mu_5-1)$. On the other hand, the solution of following system, which is simplified form of (8), give the endemic equilibrium state (EES):

$$\begin{cases} \frac{\mu_1 S}{\mu_2} - (\mu_3 + \mu_4) = 0, \\ ((1-\mu_5)\mu_3 + \mu_6)\mu_2 - \frac{\mu_1 S I}{\mu_2} - (\mu_3 + \mu_7 + \mu_6)S + (\mu_4 - \mu_6)I = 0. \end{cases} \tag{9}$$

From (9), one gets

$$S = \frac{(\mu_3 + \mu_4)\mu_2}{\mu_1}, \tag{10}$$

and

$$I = \frac{((1-\mu_5)\mu_3 + \mu_6)\mu_2 - (\mu_3 + \mu_7 + \mu_6)S}{(\mu_3 + \mu_6)}. \tag{11}$$

Utilizing (10) into (11), one gets

$$I = \frac{((1-\mu_5)\mu_3 + \mu_6)\mu_1\mu_2 - (\mu_3 + \mu_7 + \mu_6)(\mu_3 + \mu_4)\mu_2}{\mu_1(\mu_3 + \mu_6)}. \tag{12}$$

From (10) and (12) one can obtains that if $\mu_6 > \frac{(\mu_3+\mu_7)(\mu_3+\mu_4)-(1-\mu_5)\mu_3\mu_1}{\mu_1-(\mu_3+\mu_4)}$ then DEM (7) has EES $\left(\frac{((1-\mu_5)\mu_3+\mu_6)\mu_1\mu_2-(\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4)\mu_2}{\mu_1(\mu_3+\mu_6)}, \frac{(\mu_3+\mu_4)\mu_2}{\mu_1}\right)$. Alternatively, the derived parametric condition $\mu_6 > \frac{(\mu_3+\mu_7)(\mu_3+\mu_4)-(1-\mu_5)\mu_3\mu_1}{\mu_1-(\mu_3+\mu_4)}$ corresponds to $\mathfrak{R}_0 := \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1}{(\mu_3+\mu_6+\mu_7)(\mu_3+\mu_4)} > 1$, where $\mathfrak{R}_0$ denotes the basic reproduction number. So, one can concludes that if $\mathfrak{R}_0 := \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1}{(\mu_3+\mu_6+\mu_7)(\mu_3+\mu_4)} > 1$ then DEM (7) has EES $\left(\frac{((1-\mu_5)\mu_3+\mu_6)\mu_1\mu_2-(\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4)\mu_2}{\mu_1(\mu_3+\mu_6)}, \frac{(\mu_3+\mu_4)\mu_2}{\mu_1}\right)$.

Now variation matrix $V|_{\mathrm{ES}}$ of the linearized system of DEM (7) under $(f_1, f_2) \mapsto (I_{t+1}, S_{t+1})$ is

$$V|_{\mathrm{ES}} := \begin{pmatrix} \frac{\mu_1 S}{\mu_2} + 1 - (\mu_3 + \mu_4) & \frac{\mu_1 I}{\mu_2} \\ (\mu_4 - \mu_6) - \frac{\mu_1 S}{\mu_2} & 1 - \frac{\mu_1 I}{\mu_2} - (\mu_3 + \mu_7 + \mu_6) \end{pmatrix}, \tag{13}$$

where

$$\begin{cases} f_1 := \frac{\mu_1 S_t I_t}{\mu_2} + (1 - (\mu_3 + \mu_4)) I_t, \\ f_2 := ((1-\mu_5)\mu_3 + \mu_6)\mu_2 - \frac{\mu_1 S_t I_t}{\mu_2} + (1 - (\mu_3 + \mu_7 + \mu_6)) S_t + (\mu_4 - \mu_6) I_t. \end{cases} \tag{14}$$

Now at equilibrium sates local dynamics of DEM (7) is explored by stability theory [14–16]. For DFES, (13) gives

$$V|_{\mathrm{DFES}} := \begin{pmatrix} 1 + \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1}{\mu_3+\mu_7+\mu_6} - (\mu_3 + \mu_4) & 0 \\ (\mu_4 - \mu_6) - \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1}{\mu_3+\mu_7+\mu_6} & 1 - (\mu_3 + \mu_7 + \mu_6) \end{pmatrix}, \tag{15}$$

with

$$\lambda_1 = 1 + \frac{((1-\mu_5)\,\mu_3 + \mu_6)\,\mu_1}{\mu_3+\mu_7+\mu_6} - (\mu_3+\mu_4)\,,\ \lambda_2 = 1-(\mu_3+\mu_7+\mu_6)\,. \tag{16}$$

Theorem 2.1. DFES of DEM (7) is

(i) a sink if

$$\frac{((1-\mu_5)\,\mu_3+\mu_6)\,\mu_1}{\mu_3+\mu_4} - \mu_3 - \mu_6 < \mu_7 < \frac{\left\{\begin{array}{r}((1-\mu_5)\,\mu_3+\mu_6)\,\mu_1 - (\mu_3+\mu_4)\times \\ (\mu_3+\mu_6)+2\,(\mu_3+\mu_6)\end{array}\right\}}{-2+\mu_3+\mu_4}, \tag{17}$$

and

$$0 < \mu_7 < 2 - \mu_3 - \mu_6; \tag{18}$$

(ii) a source if

$$\mu_7 > \max\left\{\frac{\left\{\begin{array}{r}((1-\mu_5)\,\mu_3+\mu_6)\,\mu_1 - (\mu_3+\mu_4)\times \\ (\mu_3+\mu_6)+2\,(\mu_3+\mu_6)\end{array}\right\}}{-2+\mu_3+\mu_4}, 2-\mu_3-\mu_6\right\}; \tag{19}$$

(iii) a saddle if

$$2-\mu_3-\mu_6 < \mu_7 < \frac{((1-\mu_5)\,\mu_3+\mu_6)\,\mu_1 - (\mu_3+\mu_4)\,(\mu_3+\mu_6)+2\,(\mu_3+\mu_6)}{-2+\mu_3+\mu_4}, \tag{20}$$

or

$$\frac{((1-\mu_5)\,\mu_3+\mu_6)\,\mu_1 - (\mu_3+\mu_4)\,(\mu_3+\mu_6)+2\,(\mu_3+\mu_6)}{-2+\mu_3+\mu_4} < \mu_7 < 2-\mu_3-\mu_6; \tag{21}$$

(iv) non-hyperbolic if

$$\mu_7 = 2-\mu_3-\mu_6, \tag{22}$$

or

$$\mu_7 = \frac{((1-\mu_5)\,\mu_3+\mu_6)\,\mu_1 - (\mu_3+\mu_4)\,(\mu_3+\mu_6)+2\,(\mu_3+\mu_6)}{-2+\mu_3+\mu_4}. \tag{23}$$

Proof. By linear stability theory, DFES is a sink if $|\lambda_1| = \left|1+\frac{((1-\mu_5)\mu_3+\mu_6)\mu_1}{\mu_3+\mu_7+\mu_6} - (\mu_3+\mu_4)\right| < 1$ and $|\lambda_2| = |1-(\mu_3+\mu_7+\mu_6)| < 1$. This implies that DFES is a sink if $\frac{((1-\mu_5)\mu_3+\mu_6)\mu_1}{\mu_3+\mu_4} - \mu_3 - \mu_6 < \mu_7 < \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1-(\mu_3+\mu_4)(\mu_3+\mu_6)+2(\mu_3+\mu_6)}{-2+\mu_3+\mu_4}$. Furthermore, similar calculation shows that DFES of DEM (7) is an unstable, saddle and non-hyperbolic if corresponding parametric condition(s) hold(s). □

Hereafter, for DFES of DEM (7) we will give two theorem regarding flip bifurcation sets based on conditions (22) and (23), as follows.

Theorem 2.2. If (22) holds then flip bifurcation set at DFES of DEM (7) is

$$\mathscr{F}_1|_{\text{DFES}} := \{(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7) : 2-\mu_3-\mu_6\}. \tag{24}$$

Proof. Recall that DFES of DEM (7) is non-hyperbolic if (22) holds and from (16) one has $\lambda_1|_{(22)} = 1+\frac{((1-\mu_5)\mu_3+\mu_6)\mu_1}{2}-(\mu_3+\mu_4) \neq 1$ or -1 but $\lambda_2|_{(22)} = -1$. This implies that at DFES criterion of eigenvalues for the occurrence of flip bifurcation holds and thus one has the required conclusion. □

Theorem 2.3. If (23) holds then flip bifurcation set at DFES of DEM (7) is

$$\begin{aligned}\mathscr{F}_2|_{\text{DFES}} :=& \Bigg\{(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7) : \\ & \mu_7 = \frac{\left\{\begin{array}{r}((1-\mu_5)\mu_3+\mu_6)\mu_1-(\mu_3+\mu_4)\times \\ (\mu_3+\mu_6)+2(\mu_3+\mu_6)\end{array}\right\}}{-2+\mu_3+\mu_4}\Bigg\}.\end{aligned} \tag{25}$$

Proof. It is same as the proof of Theorem 2.2. □

Hereafter, we examined the local dynamics at EES of DEM (7). So, for EES, (13) gives

$$V|_{\text{EES}} := \begin{pmatrix} 1 & \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1-(\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4)}{\mu_3+\mu_6} \\ -\mu_3-\mu_6 & 1+\frac{(\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4)-((1-\mu_5)\mu_3+\mu_6)\mu_1}{\mu_3+\mu_6}-(\mu_3+\mu_7+\mu_6) \end{pmatrix}, \tag{26}$$

with

$$\lambda^2-\Lambda_1\lambda+\Lambda_2=0, \tag{27}$$

where

$$\begin{aligned}\Lambda_1 =& 2+\frac{(\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4)-((1-\mu_5)\mu_3+\mu_6)\mu_1}{\mu_3+\mu_6}-(\mu_3+\mu_7+\mu_6), \\ \Lambda_2 =& 1+\frac{(\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4)-((1-\mu_5)\mu_3+\mu_6)\mu_1}{\mu_3+\mu_6}-(\mu_3+\mu_7+\mu_6)- \\ & (\mu_3+\mu_6)\left(\frac{(\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4)-((1-\mu_5)\mu_3+\mu_6)\mu_1}{\mu_3+\mu_6}\right).\end{aligned} \tag{28}$$

From (27), one has

$$\lambda_{1,2} = \frac{\Lambda_1 \pm \sqrt{\Delta}}{2}, \tag{29}$$

where

$$\begin{aligned}
\Delta =&\Lambda_1^2 - 4\Lambda_2,\\
=&\left(2 + \frac{(\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4) - ((1-\mu_5)\mu_3+\mu_6)\mu_1}{\mu_3+\mu_6} - (\mu_3+\mu_7+\mu_6)\right)^2 - 4\,(1\\
&-(\mu_3+\mu_7+\mu_6) + \frac{(\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4) - ((1-\mu_5)\mu_3+\mu_6)\mu_1}{\mu_3+\mu_6} - (\mu_3+\mu_6)\\
&\left.\frac{(\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4) - ((1-\mu_5)\mu_3+\mu_6)\mu_1}{\mu_3+\mu_6}\right),\\
=&\left((\mu_3+\mu_6) - \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1 - (\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4)}{\mu_3+\mu_6}\right)^2 + \mu_7\,(\mu_7\\
&\left.+2(\mu_3+\mu_6) + 2\left(\frac{((1-\mu_5)\mu_3+\mu_6)\mu_1 - (\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4)}{\mu_3+\mu_6}\right)\right) > 0.
\end{aligned} \tag{30}$$

So, real characteristics roots of (29) are

$$\begin{aligned}
\lambda_{1,2} =&\frac{1}{2}\left(2 + \frac{(\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4) - ((1-\mu_5)\mu_3+\mu_6)\mu_1}{\mu_3+\mu_6} - (\mu_3+\mu_7+\mu_6)\right)\\
&\pm\frac{1}{2}\sqrt{\begin{aligned}&\left((\mu_3+\mu_6) - \tfrac{((1-\mu_5)\mu_3+\mu_6)\mu_1 - (\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4)}{\mu_3+\mu_6}\right)^2\\ &+\mu_7\left(\mu_7 + 2(\mu_3+\mu_6) + 2\left(\tfrac{((1-\mu_5)\mu_3+\mu_6)\mu_1 - (\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4)}{\mu_3+\mu_6}\right)\right)\end{aligned}}.
\end{aligned} \tag{31}$$

In next Theorem, we give local behavior at EES of DEM (7).

Theorem 2.4. EES of DEM (7) is a

(i) sink if

$$\begin{aligned}
0 < \mu_7 < \min&\left\{\frac{((1-\mu_5)\mu_3+\mu_6)\mu_1}{\mu_3+\mu_4} - \mu_3 - \mu_6,\right.\\
&\left.\frac{((1-\mu_5)\mu_3+\mu_6)\mu_1(-2+\mu_3+\mu_6) + 4(\mu_3+\mu_6)}{-2\mu_4+2\mu_6+(\mu_3+\mu_4)(\mu_3+\mu_6)} - \mu_3 - \mu_6\right\};
\end{aligned} \tag{32}$$

(ii) source if

$$\mu_7 > \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1(-2+\mu_3+\mu_6) + 4(\mu_3+\mu_6)}{-2\mu_4+2\mu_6+(\mu_3+\mu_4)(\mu_3+\mu_6)} - \mu_3 - \mu_6; \tag{33}$$

(iii) non-hyperbolic if

$$\mu_7 = \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1(-2+\mu_3+\mu_6)+4(\mu_3+\mu_6)}{-2\mu_4+2\mu_6+(\mu_3+\mu_4)(\mu_3+\mu_6)} - \mu_3 - \mu_6. \tag{34}$$

Proof. It is same as the proof of Theorem 2.1. □

Hereafter, for EES of DEM (7), one has the following theorem regarding flip bifurcation set based on condition (34), as follows.

Theorem 2.5. If (34) holds then flip bifurcation set at EES of DEM (7) is

$$\begin{aligned} \mathscr{F}_3|_{\text{EES}} := \{&(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7): \\ &\mu_7 = \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1(-2+\mu_3+\mu_6)+4(\mu_3+\mu_6)}{-2\mu_4+2\mu_6+(\mu_3+\mu_4)(\mu_3+\mu_6)} - \mu_3 - \mu_6 \Big\}. \end{aligned} \tag{35}$$

Proof. Recall that if (34) holds then EES of DEM (7) is a non-hyperbolic. Therefore, from (31) one has $\lambda_1|_{(34)} = \frac{2(\mu_6-\mu_4)-((1-\mu_5)\mu_3+\mu_6)\mu_1(\mu_3+\mu_6)+3(\mu_3+\mu_4)(\mu_3+\mu_6)}{-2\mu_4+2\mu_6+(\mu_3+\mu_4)(\mu_3+\mu_6)} \neq 1$ or -1 but $\lambda_2|_{(34)} = -1$, and finally, we can conclude that at EES criterion of eigenvalues for the occurrence of flip bifurcation holds when model's parameters $(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7)$ passes (35). □

3. BIFURCATION

By bifurcation theory [17–26], we will explore flip bifurcation analysis at DFES and EES of DEM (7) in this section.

3.1. Flip bifurcation analysis at DFES.

Theorem 3.1. If (24) holds then at DFES, DEM (7) does not undergo flip bifurcation.

Proof. I is noted that DEM (7) is invariant under $I = 0$, and so it takes the form:

$$S_{t+1} = ((1-\mu_5)\mu_3+\mu_6)\mu_2 + (1-(\mu_3+\mu_7+\mu_6))S_t. \tag{36}$$

From (36), we have

$$f(\mu_7, S) := ((1-\mu_5)\mu_3+\mu_6)\mu_2 + (1-(\mu_3+\mu_7+\mu_6))S_t. \tag{37}$$

Finally, if $S = S^* = \frac{((1-\mu_5)\mu_3+\mu_6)\mu_2}{\mu_3+\mu_7+\mu_6}$ and $\mu_7 = \mu_7^* = 2-\mu_3-\mu_6$ then from (37), we get

$$\left.\frac{\partial f}{\partial S}\right|_{\mu_7=\mu_7^*=2-\mu_3-\mu_6,\ S=S^*=\frac{((1-\mu_5)\mu_3+\mu_6)\mu_2}{\mu_3+\mu_7+\mu_6}} := -1, \tag{38}$$

$$\left.\frac{\partial^2 f}{\partial S^2}\right|_{\mu_7=\mu_7^*=2-\mu_3-\mu_6,\ S=S^*=\frac{((1-\mu_5)\mu_3+\mu_6)\mu_2}{\mu_3+\mu_7+\mu_6}} := 0, \tag{39}$$

and

$$\left.\frac{\partial f}{\partial \mu_7}\right|_{\mu_7^*=2-\mu_3-\mu_6,\ S^*=\frac{((1-\mu_5)\mu_3+\mu_6)\mu_2}{\mu_3+\mu_7+\mu_6}} := -\frac{((1-\mu_5)\mu_3+\mu_6)\mu_2}{2} \neq 0. \tag{40}$$

The obtained condition (39) implies that no flip bifurcation occurs at DFES if (24) holds. □

Theorem 3.2. If (25) holds then at DFES, DEM (7) does not undergo flip bifurcation.

Proof. It is same as the proof of Theorem 3.1. □

3.2. Flip bifurcation analysis at EES.

Theorem 3.3. DEM (7) undergoes flip bifurcation at EES if (35) holds.

Proof. Recall that, if μ_7 is a bifurcation parameter then DEM (7) becomes

$$\begin{cases} I_{t+1} &= \frac{\mu_1 SI}{\mu_2} + (1-(\mu_3+\mu_4))I, \\ S_{t+1} &= ((1-\mu_5)\mu_3+\mu_6)\mu_2 - \frac{\mu_1 SI}{\mu_2} + (1-(\mu_3+\mu_7^*+\varepsilon+\mu_6))S + (\mu_4-\mu_6)I, \end{cases} \tag{41}$$

where $\mu_7 = \mu_7^* + \varepsilon$ and $\varepsilon << 1$. It is further noted that (41) takes the form:

$$\begin{aligned} u_{t+1} =& \Omega_{11}^1 u_t + \Omega_{12}^1 v_t + \frac{\mu_1}{\mu_2} u_t v_t, \\ v_{t+1} =& \Omega_{11}^2 u_t + \Omega_{12}^2 v_t - \frac{\mu_1}{\mu_2} u_t v_t - v_t \varepsilon, \end{aligned} \tag{42}$$

where

$$\begin{cases} \Omega_{11}^1 = \frac{\mu_1}{\mu_2} S + 1 - (\mu_3+\mu_4), \\ \Omega_{12}^1 = \frac{\mu_1}{\mu_2} I, \\ \Omega_{11}^2 = \mu_4 - \mu_6 - \frac{\mu_1}{\mu_2} S, \\ \Omega_{12}^2 = 1 - \frac{\mu_1}{\mu_2} I - (\mu_3+\mu_7^*+\mu_6), \end{cases} \tag{43}$$

by

$$u_t = I_t - I,\ v_t = S_t - S. \tag{44}$$

Now system (42) becomes

$$\begin{pmatrix} I_{t+1} \\ S_{t+1} \end{pmatrix} := \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} I_t \\ S_t \end{pmatrix} + \begin{pmatrix} \widehat{P}(u_t, v_t, \varepsilon) \\ \widehat{Q}(u_t, v_t, \varepsilon) \end{pmatrix}, \tag{45}$$

where

$$\begin{cases} \widehat{P} = & \left\{ \begin{array}{l} \frac{(2\mu_4 - 2\mu_6 - (\mu_3+\mu_4)(\mu_3+\mu_6))(\mu_3+\mu_6)}{(\mu_3+\mu_6)(((1-\mu_5)\mu_3+\mu_6)\mu_1 - 4(\mu_3+\mu_4)) - 4(-\mu_4+\mu_6)} \times \\ \left(\frac{\mu_1}{\mu_2} u_t v_t\right) \\ \quad - \frac{2(-2\mu_4+2\mu_6+(\mu_3+\mu_4)(\mu_3+\mu_6))}{(\mu_3+\mu_6)(4(\mu_3+\mu_4) - ((1-\mu_5)\mu_3+\mu_6)\mu_1) + 4(-\mu_4+\mu_6)} \times \\ \left(\frac{\mu_1}{\mu_2} u_t v_t + \varepsilon v_t\right) \end{array} \right\}, \\ \widehat{Q} = & \left\{ \begin{array}{l} \frac{(2\mu_6 - 2\mu_4 + (\mu_3+\mu_4)(\mu_3+\mu_6))(\mu_3+\mu_6)}{(\mu_3+\mu_6)(((1-\mu_5)\mu_3+\mu_6)\mu_1 - 4(\mu_3+\mu_4)) - 4(-\mu_4+\mu_6)} \times \\ \left(\frac{\mu_1}{\mu_2} u_t v_t\right) \\ - \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1(\mu_3+\mu_6) - 2(\mu_3+\mu_4)(\mu_3+\mu_6)}{(\mu_3+\mu_6)(((1-\mu_5)\mu_3+\mu_6)\mu_1 - 4(\mu_3+\mu_4)) - 4(-\mu_4+\mu_6)} \times \\ \left(\frac{\mu_1}{\mu_2} u_t v_t + \varepsilon v_t\right) \end{array} \right\}, \\ u_t = & \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1 - 2(\mu_3+\mu_4)}{2\mu_4 - 2\mu_6 - (\mu_3+\mu_4)(\mu_3+\mu_6)} I_t - \frac{2}{\mu_3+\mu_6} S_t, v_t = I_t + S_t, \\ u_t v_t = & \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1 - 2(\mu_3+\mu_4)}{2\mu_4 - 2\mu_6 - (\mu_3+\mu_4)(\mu_3+\mu_6)} \left(I_t^2 + I_t S_t\right) - \frac{2}{\mu_3+\mu_6}\left(S_t I_t + S_t^2\right), v_t\varepsilon = I_t\varepsilon + S_t\varepsilon, \end{cases} \tag{46}$$

by

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} := \begin{pmatrix} \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1 - 2(\mu_3+\mu_4)}{(2\mu_4 - 2\mu_6 - (\mu_3+\mu_4)(\mu_3+\mu_6))} & \frac{-2}{\mu_3+\mu_6} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} I_t \\ S_t \end{pmatrix}. \tag{47}$$

Now in a small neighborhood of $\varepsilon = 0$, the center manifold $F^C O$ at O is

$$F^C O = \left\{ (I_t, S_t) : S_t = v_0\varepsilon + v_1 I_t^2 + v_2 \varepsilon I_t + v_3 \varepsilon^3 + O\left((|I_t| + |\varepsilon|)^3\right) \right\}, \tag{48}$$

with

$$\begin{cases} v_0 & = 0 = v_3, \\ v_1 & = \left\{ \begin{array}{l} \frac{1}{1-\lambda_2}\Big(\frac{((1-\mu_5)\mu_3+\mu_6)\mu_1(\mu_3+\mu_6) - 2(\mu_3+\mu_4)(\mu_3+\mu_6)}{((1-\mu_5)\mu_3+\mu_6)\mu_1(\mu_3+\mu_6) - 4(\mu_3+\mu_4)(\mu_3+\mu_6) - 4(-\mu_4+\mu_6)} \times \\ \left(-\frac{\mu_1}{\mu_2} + \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1 - 2(\mu_3+\mu_4)}{-2\mu_4+2\mu_6+(\mu_3+\mu_4)(\mu_3+\mu_6)} \frac{\mu_1}{\mu_2} \right)\Big) \end{array} \right\}, \\ v_2 & = \left\{ \frac{1}{\lambda_2 - 1} \left(\frac{((1-\mu_5)\mu_3+\mu_6)\mu_1(\mu_3+\mu_6) - 2(\mu_3+\mu_4)(\mu_3+\mu_6)}{((1-\mu_5)\mu_3+\mu_6)\mu_1(\mu_3+\mu_6) - 4(\mu_3+\mu_4)(\mu_3+\mu_6) - 4(-\mu_4+\mu_6)} \right) \right\}. \end{cases} \tag{49}$$

So, we write (45) restrict to $F^C O$ as

$$f_1(I_t) = -I_t + m_1 I_t^2 + m_2 I_t \varepsilon + m_3 I_t^2 \varepsilon + m_4 I_t \varepsilon^2 + m_5 I_t^3 + O\left((|I_t| + |\varepsilon|)^4\right), \tag{50}$$

where

$$\begin{cases} m_1 = & \left\{ \dfrac{((1-\mu_5)\mu_3+\mu_6)\mu_1(\mu_3+\mu_6) - 2(\mu_3+\mu_4)(\mu_3+\mu_6)}{((1-\mu_5)\mu_3+\mu_6)\mu_1(\mu_3+\mu_6) - 4(\mu_3+\mu_4)(\mu_3+\mu_6) - 4(-\mu_4+\mu_6)} \times \left(\dfrac{\mu_1}{\mu_2}\right) + \dfrac{2((1-\mu_5)\mu_3+\mu_6)\mu_1 - 4(\mu_3+\mu_4)}{4(\mu_3+\mu_4)(\mu_3+\mu_6) - ((1-\mu_5)\mu_3+\mu_6)\mu_1(\mu_3+\mu_6) + 4(-\mu_4+\mu_6)} \left(\dfrac{\mu_1}{\mu_2}\right) \right\}, \\ m_2 = & \left\{ \dfrac{2(2\mu_4 - 2\mu_6) - 2(\mu_3+\mu_4)(\mu_3+\mu_6)}{4(\mu_3+\mu_4)(\mu_3+\mu_6) - ((1-\mu_5)\mu_3+\mu_6)\mu_1(\mu_3+\mu_6) + 4(-\mu_4+\mu_6)} \right\}, \\ m_3 = & \left\{ \dfrac{((1-\mu_5)\mu_3+\mu_6)\mu_1(\mu_3+\mu_6) + 4(-\mu_4+\mu_6)}{((1-\mu_5)\mu_3+\mu_6)\mu_1(\mu_3+\mu_6) - 4(\mu_3+\mu_4)(\mu_3+\mu_6) - 4(-\mu_4+\mu_6)} \times v_2\left(\dfrac{\mu_1}{\mu_2}\right) + \dfrac{2((1-\mu_5)\mu_3+\mu_6)\mu_1(\mu_3+\mu_6) + 8(-\mu_4+\mu_6)}{(\mu_3+\mu_6)(4(\mu_3+\mu_4)(\mu_3+\mu_6) - ((1-\mu_5)\mu_3+\mu_6)\mu_1(\mu_3+\mu_6) + 4(-\mu_4+\mu_6))} \times \left(v_2 \dfrac{\mu_1}{\mu_2}\right) \right\}, \\ m_4 = & 0, \\ m_5 = & \left\{ \dfrac{((1-\mu_5)\mu_3+\mu_6)\mu_1(\mu_3+\mu_6) + 4(-\mu_4+\mu_6)}{((1-\mu_5)\mu_3+\mu_6)\mu_1(\mu_3+\mu_6) - 4(\mu_3+\mu_4)(\mu_3+\mu_6) - 4(-\mu_4+\mu_6)} \times v_1\left(\dfrac{\mu_1}{\mu_2}\right) + \dfrac{2((1-\mu_5)\mu_3+\mu_6)\mu_1(\mu_3+\mu_6) + 8(-\mu_4+\mu_6)}{4(\mu_3+\mu_4)(\mu_3+\mu_6)^2 - ((1-\mu_5)\mu_3+\mu_6)\mu_1(\mu_3+\mu_6)^2 + 4(-\mu_4+\mu_6)(\mu_3+\mu_6)} \times \left(v_1 \dfrac{\mu_1}{\mu_2}\right) \right\}. \end{cases} \tag{51}$$

Finally, for the occurrence of flip bifurcation at EES, it is require that following discriminatory quantities are non-zero [17, 18]:

$$\ell_1 := \left(\frac{\partial^2 f_1}{\partial I_t \partial \varepsilon} + \frac{1}{2}\frac{\partial f_1}{\partial \varepsilon}\frac{\partial^2 f_1}{\partial I_t^2}\right)|_{(0,0)} = m_2 \neq 0, \tag{52}$$

and

(53)
$$\ell_2 := \left(\frac{1}{6}\frac{\partial^3 f_1}{\partial I_t^3} + \left(\frac{1}{2}\frac{\partial^2 f_1}{\partial I_t^2}\right)^2\right)|_{(0,0)} = m_1^2 + m_5.$$

From (53), if $\ell_2 \neq 0$ then at EES DEM (7) undergoes flip bifurcation and additionally, period-2 points bifurcate from EES are stable (unstable) if $\ell_2 > 0$ ($\ell_2 < 0$). □

4. Control of Chaos

In this section, hybrid control feedback method is utilized in order to control the chaos due to the emergence of flip bifurcation in DEM (7) by existing theory [27]. If DEM (7) undergoes flip bifurcation at EES then controlled DEM can be written as

(54)
$$\begin{cases} I_{t+1} = & \mu\left(\frac{\mu_1 S_t I_t}{\mu_2} + (1-(\mu_3+\mu_4))\,I_t\right) + (1-\mu)\,I_t, \\ S_{t+1} = & \mu\left(((1-\mu_5)\,\mu_3+\mu_6)\,\mu_2 - \frac{\mu_1 S_t I_t}{\mu_2} + \right. \\ & \left. (1-(\mu_3+\mu_7+\mu_6))\,S_t + (\mu_4-\mu_6)\,I_t\right) + (1-\mu)\,S_t, \end{cases}$$

where $0 < \mu < 1$. The $V|_{\text{EES}}$ evaluated at EES is

(55)
$$V|_{\text{EES}} = \begin{pmatrix} 1 & \mu\left(\frac{((1-\mu_5)\mu_3+\mu_6)\mu_1-(\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4)}{\mu_3+\mu_6}\right) \\ -\mu\,(\mu_3+\mu_6) & \mu\left(\frac{(\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4)-((1-\mu_5)\mu_3+\mu_6)\mu_1}{\mu_3+\mu_6} - (\mu_3+\mu_7+\mu_6)\right)+1 \end{pmatrix}.$$

The characteristics equation of $V|_{\text{EES}}$ is

(56)
$$\lambda^2 - \Lambda_1\lambda + \Lambda_2 = 0,$$

where

(57)
$$\begin{cases} \Lambda_1 = & 2+\mu\left(\frac{(\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4)-((1-\mu_5)\mu_3+\mu_6)\mu_1}{\mu_3+\mu_6} - (\mu_3+\mu_7+\mu_6)\right), \\ \Lambda_2 = & \mu\left(\frac{(\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4)-((1-\mu_5)\mu_3+\mu_6)\mu_1}{\mu_3+\mu_6} - (\mu_3+\mu_7+\mu_6)\right)+1+ \\ & \mu^2\left(((1-\mu_5)\,\mu_3+\mu_6)\,\mu_1 - (\mu_3+\mu_7+\mu_6)\,(\mu_3+\mu_4)\right). \end{cases}$$

So, based on linear stability theory, we have

Lemma 4.1. EES is a sink iff

$$\left|2+\mu\left(\frac{\begin{array}{c}(\mu_3+\mu_7+\mu_6)\,(\mu_3+\mu_4)\\ -((1-\mu_5)\,\mu_3+\mu_6)\,\mu_1\end{array}}{\mu_3+\mu_6} - (\mu_3+\mu_7+\mu_6)\right)\right|$$

$$< 2+\mu\left(\frac{(\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4)-((1-\mu_5)\mu_3+\mu_6)\mu_1}{\mu_3+\mu_6}-(\mu_3+\mu_7+\mu_6)\right)$$
$$+\mu^2\left(\begin{array}{c}((1-\mu_5)\mu_3+\mu_6)\mu_1-\\ (\mu_3+\mu_7+\mu_6)(\mu_3+\mu_4)\end{array}\right)<2.$$

5. Numerical Simulations

Example 5.1. *If* $\mu_1 = 3.5$, $\mu_2 = 1.5$, $\mu_3 = 0.001$, $\mu_4 = 0.8$, $\mu_5 = 0.005$, $\mu_6 = 0.055$, $\mu_7 \in [0.0025, 0.1]$ *with* $(I_0, S_0) = (0.83, 0.34)$ *then at* $\mu_7 = 0.05263275908836767$ *DEM (7) undergoes the flip bifurcation. The Maximum Lyapunov exponent with flip bifurcation diagrams are drawn in Figure 2. Further, at* $(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7) = (3.5, 1.5, 0.001, 0.8, 0.005, 0.055, 0.0$
$5263275908836767)$ *DEM (7) has* $\text{EES} = (0.8339361732414605, 0.34328571428571425)$ *and moreover, from (26) one gets:*

$$V|_{\text{EES}} = \begin{pmatrix} 0.9999999999999998 & 1.9458510708967411 \\ -0.05599999999999983 & -1.054483829985109 \end{pmatrix}, \tag{58}$$

with $\lambda_1 = -1$ *and* $\lambda_2 = 0.9455161700148913$*, and therefore, based on these simulations one can obtain that* $(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7) = (3.5, 1.5, 0.001, 0.8, 0.005, 0.055, 0.0526327590883$
$6767) \in \mathscr{F}_3|_{\text{EES}=(0.8339361732414605, 0.34328571428571425)}$. *Moreover, in this parametric domain, from (43), (49) and (51), one gets:*

$$\begin{cases} \Omega^1_{11} = 0.9999999999999998, \\ \Omega^1_{12} = 1.9458510708967411, \\ \Omega^2_{11} = -0.05599999999999983, \\ \Omega^2_{12} = -1.054483829985109, \end{cases} \tag{59}$$

$$\begin{cases} v_0 &= 0, \\ v_1 &= 0.0008845847547751073, \\ v_2 &= 0.01400240995804515, \\ v_3 &= 0, \end{cases} \tag{60}$$

and

$$\begin{cases} m_1 &= 2.2683904132033144,\\ m_2 &= -1.0280048199160903,\\ m_3 &= 1.1977210716228013,\\ m_4 &= 0,\\ m_5 &= 0.07566453229157898. \end{cases} \tag{61}$$

Using (61) in (52) and (53) one gets: $\ell_1 = -1.0280048199160903 \neq 0$ *and* $\ell_2 = 5.22125959900 4282 > 0$. *Since* $\ell_2 = 5.221259599004282 > 0$ *which imply that stable period-*2 *points bifurcate from* $\text{EES} = (0.8339361732414605, 0.34328571428571425)$. *So, our simulation in Example 5.1 agrees with theoretical results obtained in Theorems 2.4, 2.5 and 3.3.*

Example 5.2. *Finally, if* $\mu_1 = 3.5$, $\mu_2 = 1.5$, $\mu_3 = 0.001$, $\mu_4 = 0.8$, $\mu_5 = 0.005$, $\mu_6 = 0.055$, $\mu_7 = 5.221259599004282$ *with* $(I_0, S_0) = (0.83, 0.34)$ *then model (7) undergoes flip bifurcation. For this, by applying hybrid strategy to get stable orbit at* $\text{EES} = (0.8339361732414605, 0.34328571428571425)$. *For this, model (54) takes the form*

$$\begin{cases} I_{t+1} = & \mu\left(\frac{(3.5)S_tI_t}{1.5} + (1-(0.001+0.8))I_t\right) + (1-\mu)I_t,\\ S_{t+1} = & \mu\Big(((1-0.005)0.001+0.055)1.5 - \frac{3.5S_tI_t}{1.5} + \\ & (1-(0.001+0.05263275908836767\\ & +0.055))S_t + (0.8-0.055)I_t\Big) + (1-\mu)S_t, \end{cases} \tag{62}$$

where

$$V_{\text{EES}} = \begin{pmatrix} 1-0.000000000000000222045\mu & 1.94585\mu \\ -0.056\mu & 1-2.05448\mu \end{pmatrix}, \tag{63}$$

with

$$\begin{aligned} &\lambda^2 - (2-2.05448\mu)\lambda + 1 - 2.054483829985109\mu + \\ &0.1089676599702175\mu^2 = 0. \end{aligned} \tag{64}$$

Furthermore, roots of (64) satisfying $|\lambda_{1,2}| < 1$ *if* $0 < \mu < 1$. *So, for the allowed interval of control parameter* μ *the flip bifurcation is completely eliminated. If* $\mu = 0.9$ *then for controlled model (62), plots of* t *vs* I_t *and* S_t *are drawn in Figure 3.*

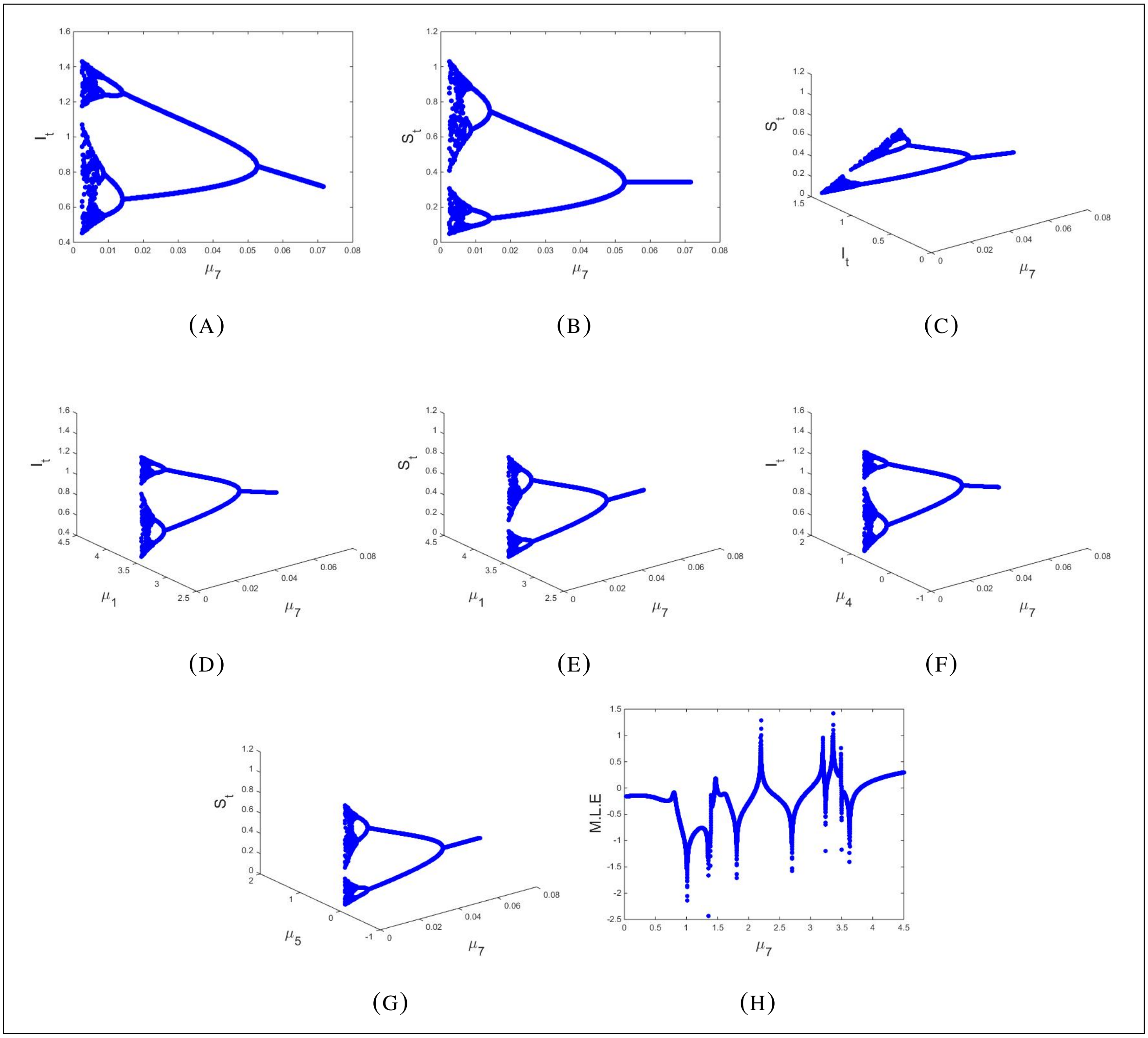
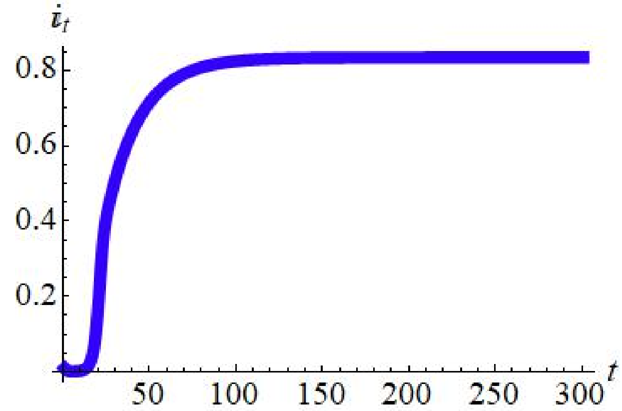
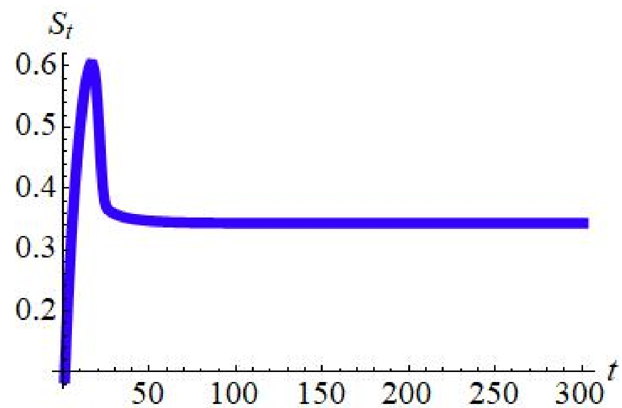


FIGURE 2. Flip B.Ds with MLE of DEM (7). 2a B.D for I_t. 2b B.D for S_t. 2c B.D for I_t and S_t. 2d B.D for μ_1 and I_t. 2e B.D for μ_1 and S_t.2f B.D for μ_4 and I_t.2g B.D for μ_5 and S_t. 2h MLEs



(A)



(B)

FIGURE 3. Plot of t vs I_t and S_t for controlled system (54)

6. CONCLUSION

The work is about local dynamics at equilibrium states, bifurcations and chaos in a discrete epidemic model (7) in the region: $\mathbb{R}^2_+ = \{(I,S) : I,\ S \geq 0\}$. We have studied local dynamics at DFES and EES of DEM (7), and explored that DFES of DEM (7) is a sink if $\frac{((1-\mu_5)\mu_3+\mu_6)\mu_1}{\mu_3+\mu_4} - \mu_3 - \mu_6 < \mu_7 < \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1-(\mu_3+\mu_4)(\mu_3+\mu_6)+2(\mu_3+\mu_6)}{-2+\mu_3+\mu_4}$ and $0 < \mu_7 < 2 - \mu_3 - \mu_6$, a source if $\mu_7 > \max\left\{\frac{((1-\mu_5)\mu_3+\mu_6)\mu_1-(\mu_3+\mu_4)(\mu_3+\mu_6)+2(\mu_3+\mu_6)}{-2+\mu_3+\mu_4}, 2-\mu_3-\mu_6\right\}$, a saddle if $2-\mu_3-\mu_6 < \mu_7 < \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1-(\mu_3+\mu_4)(\mu_3+\mu_6)+2(\mu_3+\mu_6)}{-2+\mu_3+\mu_4}$ or $\frac{\left\{\begin{array}{l}((1-\mu_5)\mu_3+\mu_6)\mu_1\\ \quad -(\mu_3+\mu_4)(\mu_3+\mu_6)\\ \qquad +2(\mu_3+\mu_6)\end{array}\right\}}{-2+\mu_3+\mu_4} < \mu_7 < 2-\mu_3-\mu_6$, non-hyperbolic if $\mu_7 = \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1-(\mu_3+\mu_4)(\mu_3+\mu_6)+2(\mu_3+\mu_6)}{-2+\mu_3+\mu_4}$ or $\mu_7 = 2-\mu_3-\mu_6$; EES is a sink if $\mu_7 < \min\left\{\frac{((1-\mu_5)\mu_3+\mu_6)\mu_1}{\mu_3+\mu_4} - \mu_3 - \mu_6, \frac{\left\{\begin{array}{c}((1-\mu_5)\mu_3+\mu_6)\mu_1\\(-2+\mu_3+\mu_6)+\\4(\mu_3+\mu_6)\end{array}\right\}}{\left\{\begin{array}{c}-2\mu_4+2\mu_6+(\mu_3+\mu_4)\\(\mu_3+\mu_6)\end{array}\right\}} - \right.$ $\left.\mu_3-\mu_6\right\}$, source if $\mu_7 > \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1(-2+\mu_3+\mu_6)+4(\mu_3+\mu_6)}{-2\mu_4+2\mu_6+(\mu_3+\mu_4)(\mu_3+\mu_6)} - \mu_3 - \mu_6$ and finally, non-hyperbolic if $\mu_7 = \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1(-2+\mu_3+\mu_6)+4(\mu_3+\mu_6)}{-2\mu_4+2\mu_6+(\mu_3+\mu_4)(\mu_3+\mu_6)} - \mu_3 - \mu_6$. Next in order to study bifurcation analysis, we first examined the bifurcation sets at equilibrium states (i) flip bifurcation set $\mathscr{F}_1|_{\text{DFES}} := \{(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7) : 2-\mu_3-\mu_6\}$ at DFES, (ii) at DFES of DEM (7), the flip bifurcation set is $\mathscr{F}_2|_{\text{DFES}} := \left\{(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7) : \mu_7 = \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1-(\mu_3+\mu_4)(\mu_3+\mu_6)+2(\mu_3+\mu_6)}{-2+\mu_3+\mu_4}\right\}$ (iii) at EES of two-dimensional discrete epidemic model (7), the flip bifurcation set is $\mathscr{F}_3|_{\text{EES}} := \left\{(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\ \mu_7) : \mu_7 = \frac{((1-\mu_5)\mu_3+\mu_6)\mu_1(-2+\mu_3+\mu_6)+4(\mu_3+\mu_6)}{-2\mu_4+2\mu_6+(\mu_3+\mu_4)(\mu_3+\mu_6)} - \mu_3 - \mu_6\right\}$, and then we have proved that at DFES flip bifurcation do not take place if $(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7) \in \mathscr{F}_1|_{\text{DFES}}$ and $\mathscr{F}_2|_{\text{DFES}}$ but at EES model undergoes flip bifurcation if $(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7) \in \mathscr{F}_3|_{\text{EES}}$. Furthermore, Hybrid control strategy is utilized to control chaos in the under study DEM (7) due to occurrence of flip bifurcation. Finally, numerical simulations are given to verify theoretical findings.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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