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STABILITY ANALYSIS OF CRYPTOSPORIDIOSIS DISEASE SIR MODEL UNDER STOCHASTIC PERTURBATIONS

LEONID SHAIKHET*

Department of Mathematics, Ariel University, Ariel 40700, Israel

Abstract. Using the known Cryptosporidiosis disease SIR model, a method of stability investigation for nonlinear systems of differential equations under stochastic perturbations is discussed. Stability conditions are obtained using the general method of Lyapunov functionals construction, the method of Linear Matrix Inequalities (LMIs) and MATLAB. The obtained results are illustrated via figures with numerical simulation of solutions of the system under consideration. The method can be applied for many other nonlinear systems of high order of nonlinearity in various applications.

Keywords: nonlinear systems; stochastic perturbations; Lyapunov functionals construction; linear matrix inequality; MATLAB; stability in probability.

2020 AMS Subject Classification: 37N25.

1. INTRODUCTION

Cryptosporidiosis of humans is an intestinal disease caused predominantly by Cryptosporidium infection. This disease is transmitted mainly via water and food and has major socioeconomic impact globally. So, it is not surprising that the problem of human cryptosporidiosis is given serious attention, this problem is very popular in research, see, for instance, [2, 3, 5, 6, 7, 8, 9, 11, 15, 16] and the references therein.

*Corresponding author

E-mail address: leonid.shaikhet@usa.net

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Here stability analysis is provided for the Cryptosporidiosis disease SIR model

$$
\begin{aligned}
\dot{S}(t) &= \Lambda + \gamma R(t) - (\mu + \rho E(t))S(t) - \frac{\lambda S(t)I(t)}{S(t)+I(t)+R(t)},\\
\dot{I}(t) &= \frac{\lambda S(t)I(t)}{S(t)+I(t)+R(t)} + \rho E(t)S(t) - (\alpha+\mu+\psi)I(t),\\
\dot{R}(t) &= \alpha I(t) - (\mu+\gamma)R(t),\\
\dot{E}(t) &= \theta I(t) - \nu E(t),
\end{aligned} \tag{1.1}
$$

described in [9]. In this model the total human population $N(t) = S(t) + I(t) + R(t)$ are sub-divided into sub-populations of susceptible individuals $S(t)$, individuals with Cryptosporidiosis $I(t)$, recovered human $R(t)$. The contaminated environment is denoted by $E(t)$. It is supposed that all parameters of the system (1.1) are positive.

It is supposed also that the system (1.1) is exposed to stochastic perturbations of the type of white noise. For stability investigation of the obtained system of nonlinear Ito's stochastic differential equations [4] is used the method, described in [12, 13]. This method can be used for many other nonlinear systems of stochastic differential equations in various applications.

2. Equilibria

Generally speaking, systems of nonlinear differential equations can have both the zero and nonzero solutions. The system (1.1) does not have the zero solution, but it may have several nonzero equilibria. It is natural, that for systems of the SIR type, only equilibria with all non-negative coordinates are of interest.

The equilibria of the system (1.1) are defined by the system of the four algebraic equations

$$
\begin{aligned}
\Lambda + \gamma R - (\mu+\rho E)S - \frac{\lambda SI}{S+I+R} &= 0,\\
\frac{\lambda SI}{S+I+R} + \rho ES - (\alpha+\mu+\psi)I &= 0,\\
\alpha I - (\mu+\gamma)R &= 0,\\
\theta I - \nu E &= 0.
\end{aligned} \tag{2.1}
$$

It is clear that assuming $E = 0$, we obtain the following solution of the system (2.1)

$$
Q_0^* = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right). \tag{2.2}
$$

Lemma 2.1. *(1) If $E \neq 0$ then the system* (2.1) *has two solutions:*

$$Q_1^* = (S_1^*, I_1^*, R_1^*, E_1^*), \quad Q_2^* = (S_2^*, I_2^*, R_2^*, E_2^*),$$

where

$$I_1^* = \alpha_1 - \sqrt{\alpha_1^2 - \alpha_2}, \quad I_2^* = \alpha_1 + \sqrt{\alpha_1^2 - \alpha_2}, \tag{2.3}$$

$$S_i^* = \frac{\Lambda}{\mu} - \beta_0 I_i^*, \quad R_i^* = \frac{\alpha}{\mu+\gamma} I_i^*, \quad E_i^* = \frac{\theta}{\nu} I_i^*, \quad i = 1, 2, \tag{2.4}$$

and

$$\alpha_1 = \frac{\beta_0\beta_1 + \beta_2\beta_3 - \beta_5}{2\beta_0\beta_2}, \quad \alpha_2 = \frac{\beta_1\beta_3 - \beta_4}{\beta_0\beta_2}, \tag{2.5}$$

$$\begin{gathered} \beta_0 = 1 + \frac{\alpha}{\mu+\gamma} + \frac{\psi}{\mu}, \quad \beta_1 = \lambda\mu + \frac{\rho\theta\Lambda}{\nu}, \quad \beta_2 = \frac{\rho\theta\psi}{\nu}, \quad \beta_3 = \frac{\Lambda}{\mu}, \\ \beta_4 = (\alpha + \mu + \psi)\Lambda, \quad \beta_5 = (\alpha + \mu + \psi)\psi. \end{gathered} \tag{2.6}$$

(2) The equilibrium Q_2^* has negative first coordinate: $S_2^* < 0$.

Proof. See Appendix. □

Remark 2.1. *Via* (2.3), (2.5) *for the existence and nonnegativity of I_1^* and I_2^* the following conditions must be satisfied:*

$$\beta_0\beta_1 + \beta_2\beta_3 \geq \beta_5, \quad \beta_1\beta_3 \geq \beta_4, \quad \alpha_1^2 \geq \alpha_2. \tag{2.7}$$

Note also that from the first and the last inequalities (2.7) *it follows*

$$\beta_0\beta_1 + \beta_2\beta_3 \geq \beta_5 + 2\sqrt{\beta_0\beta_2(\beta_1\beta_3 - \beta_4)}.$$

Remark 2.2. *Summing three first equations of the system* (2.1)*, we have $\Lambda - \mu(S + I + R) = \psi I \geq 0$. So, each equilibrium of the system* (1.1) *satisfies the condition*

$$S^* + I^* + R^* \leq \frac{\Lambda}{\mu}. \tag{2.8}$$

3. Stochastic Perturbations, Centralization and Linearization

Let us assume that the system (1.1) is exposed by stochastic perturbations of the type of white noise that are proportional to the deviation of the system (1.1) state $(S(t-h), I(t-h), R(t-h), E(t-h))$ with some delay $h \geq 0$ from its equilibrium (S^*, I^*, R^*, E^*). By that the system (1.1) is transformed to the system of Ito's stochastic differential equations with delay [4]

(3.1)
$$\begin{aligned}
dS(t) &= \Big(\Lambda + \gamma R(t) - (\mu + \rho E(t))S(t) - \frac{\lambda S(t)I(t)}{S(t)+I(t)+R(t)}\Big)dt + \sigma_1(S(t-h)-S^*)dw_1(t),\\
dI(t) &= \Big(\frac{\lambda S(t)I(t)}{S(t)+I(t)+R(t)} + \rho E(t)S(t) - (\alpha+\mu+\psi)I(t)\Big)dt + \sigma_2(I(t-h)-I^*)dw_2(t),\\
dR(t) &= (\alpha I(t) - (\mu+\gamma)R(t))dt + \sigma_3(R(t-h)-R^*)dw_3(t),\\
dE(t) &= (\theta I(t) - \nu E(t))dt + \sigma_4(E(t-h)-E^*)dw_4(t),
\end{aligned}$$

where $\sigma_1, ..., \sigma_4$ are constants and $w_1(t), ..., w_4(t)$ are mutually independent standard Wiener processes [4, 14].

Stochastic perturbations of the such type were first used in [1] and later in a lot of other works (see [14] and the references therein). By that the equilibrium (S^*, I^*, R^*, E^*) of the system (1.1) is also a solution of the system (3.1).

Using the new variables

$$y_1(t) = S(t) - S^*, \quad y_2(t) = I(t) - I^*, \quad y_3(t) = R(t) - R^*, \quad y_4(t) = E(t) - E^*, \tag{3.2}$$

let us transform the system (3.1) by the following way. Put

$$f(y(t)) = I^* y_1(t) + S^* y_2(t) + y_1(t)y_2(t), \qquad g(y(t)) = y_1(t) + y_2(t) + y_3(t). \tag{3.3}$$

Using (3.2), (3.3) and the simple equality

$$\frac{M+f}{N+g} = \frac{M}{N} - \frac{N^{-1}Mg - f}{N+g},$$

note that

$$\frac{S(t)I(t)}{S(t)+I(t)+R(t)} = \frac{S^*I^* + f(y(t))}{S^*+I^*+R^*+g(y(t))} = \frac{S^*I^*}{S^*+I^*+R^*} - Y(t), \tag{3.4}$$

where

$$Y(t)=\frac{(S^*+I^*+R^*)^{-1}S^*I^*g(y(t))-f(y(t))}{S^*+I^*+R^*+g(y(t))}. \tag{3.5}$$

Substituting (3.2) and (3.4) into the first equation of the system (3.1) and using (2.1), we obtain

$$\begin{aligned}
&\Lambda+\gamma R(t)-(\mu+\rho E(t))S(t)-\frac{\lambda S(t)I(t)}{S(t)+I(t)+R(t)}\\
&\quad=\Lambda+\gamma(y_3(t)+R^*)-(\mu+\rho E^*+\rho y_4(t))(y_1(t)+S^*)-\frac{\lambda S^*I^*}{S^*+I^*+R^*}+\lambda Y(t)\\
&\quad=\Lambda+\gamma y_3(t)+\gamma R^*-(\mu+\rho E^*)y_1(t)-(\mu+\rho E^*)S^*-\rho y_1(t)y_4(t)-\rho S^*y_4(t)\\
&\qquad-\frac{\lambda S^*I^*}{S^*+I^*+R^*}+\lambda Y(t)\\
&\quad=-(\mu+\rho E^*)y_1(t)+\gamma y_3(t)-\rho S^*y_4(t)-\rho y_1(t)y_4(t)+\lambda Y(t).
\end{aligned} \tag{3.6}$$

Similarly, for the second equation of the system (3.1) we have

$$\begin{aligned}
&\rho E(t)S(t)-(\alpha+\mu+\psi)I(t)+\frac{\lambda S(t)I(t)}{S(t)+I(t)+R(t)}\\
&\quad=\rho(y_4(t)+E^*)(y_1(t)+S^*)-(\alpha+\mu+\psi)(y_2(t)+I^*)+\frac{\lambda S^*I^*}{S^*+I^*+R^*}-\lambda Y(t)\\
&\quad=\rho y_1(t)y_4(t)+\rho E^*y_1(t)+\rho S^*y_4(t)+\rho E^*S^*-(\alpha+\mu+\psi)y_2(t)-\lambda Y(t)\\
&\qquad-(\alpha+\mu+\psi)I^*+\frac{\lambda S^*I^*}{S^*+I^*+R^*}\\
&\quad=\rho E^*y_1(t)-(\alpha+\mu+\psi)y_2(t)+\rho S^*y_4(t)+\rho y_1(t)y_4(t)-\lambda Y(t).
\end{aligned} \tag{3.7}$$

Using (3.6), (3.7) and similar transformation of two last equations of the system (3.1), as a result we obtain the system of nonlinear Ito's stochastic differential equations with the zero solution

$$\begin{aligned}
dy_1(t)&=(-(\mu+\rho E^*)y_1(t)+\gamma y_3(t)-\rho S^*y_4(t)-\rho y_1(t)y_4(t)+\lambda Y(t))dt\\
&\qquad+\sigma_1y_1(t-h)dw_1(t),\\
dy_2(t)&=(\rho E^*y_1(t)-(\alpha+\mu+\psi)y_2(t)+\rho S^*y_4(t)+\rho y_1(t)y_4(t)-\lambda Y(t))dt\\
&\qquad+\sigma_2y_2(t-h)dw_2(t),\\
dy_3(t)&=(\alpha y_2(t)-(\mu+\gamma)y_3(t))dt+\sigma_3y_3(t-h)dw_3(t),\\
dy_4(t)&=(\theta y_2(t)-\nu y_4(t))dt+\sigma_4y_4(t-h)dw_4(t),
\end{aligned} \tag{3.8}$$

and the initial condition

$$y(s) = (y_1(s), y_2(s), y_3(s), y_4(s))' = \phi(s), \quad s \in [-h, 0],$$

where $'$ is the sigh of transposition.

It is clear that stability of the equilibrium (S^*, I^*, R^*, E^*) of the system (3.1) is equivalent to stability of the zero solution of the system (3.8).

To get a linear part of the system (3.8) note that the function $Y(t)$ (3.5) has the form

$$\frac{N^{-1}S^*I^*g - f_0 - f_1}{N+g}, \qquad N = S^* + I^* + R^*,$$

where $g = g(y)$ is the linear function (3.3), f_0 and f_1 are respectively the linear and nonlinear parts of the function $f = f(y)$ (3.3). Using the linearization

$$\frac{1}{N+g} = \frac{1}{N} - \frac{g}{N^2} + o(y),$$

where $o(y)$ is a nonlinear part, and (3.3), we have

$$\begin{aligned}
Y(t) =& (N^{-1}S^*I^*g(y) - f_0(y) - f_1(y)) \left(\frac{1}{N} - \frac{g(y)}{N^2} + o(y)\right) \\
=& N^{-2}S^*I^*g(y) - N^{-1}f_0(y) + o(y) \\
=& N^{-2}S^*I^*(y_1(t) + y_2(t) + y_3(t)) - N^{-1}(I^*y_1(t) + S^*y_2(t)) + o(y) \\
=& -N^{-1}I^*(1 - N^{-1}S^*)y_1(t) - N^{-1}S^*(1 - N^{-1}I^*)y_2(t) + N^{-2}S^*I^*y_3(t) + o(y) \\
=& -N^{-2}I^*(I^* + R^*)y_1(t) - N^{-2}S^*(S^* + R^*)y_2(t) + N^{-2}S^*I^*y_3(t) + o(y).
\end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.8) and rejecting nonlinear terms we obtain the linear part of the system (3.8) in the form

$$\begin{aligned}
dz(t) &= Az(t)dt + \sum_{i=1}^{4} C_i z(t-h) dw_i(t), \\
z(s) &= \phi(s), \quad s \in [-h, 0],
\end{aligned} \tag{3.10}$$

where $z(t) = (z_1(t), z_2(t), z_3(t), z_4(t))'$, the matrix C_i has all zero elements besides of $c_{ii} = \sigma_i$, $i = 1, ..., 4$ and the elements of the matrix A are

$$
\begin{aligned}
& a_{11} = -\lambda N^{-2} I^*(I^* + R^*) - (\mu + \rho E^*),\\
& a_{12} = -\lambda N^{-2} S^*(S^* + R^*),\\
& a_{13} = \lambda N^{-2} S^* I^* + \gamma, \qquad a_{14} = -\rho S^*,\\
& a_{21} = \lambda N^{-2} I^*(I^* + R^*) + \rho E^*,\\
& a_{22} = \lambda N^{-2} S^*(S^* + R^*) - (\alpha + \mu + \psi),\\
& a_{23} = -\lambda N^{-2} S^* I^*, \qquad a_{24} = \rho S^*,\\
& a_{31} = a_{34} = 0, \quad a_{32} = \alpha, \quad a_{33} = -(\mu + \gamma),\\
& a_{41} = a_{43} = 0, \quad a_{42} = \theta, \quad a_{44} = -\nu.
\end{aligned} \tag{3.11}
$$

In particular, for the equilibrium $Q_0^* = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$ the matrix A takes the form:

$$
A = \begin{bmatrix} -\mu & -\lambda & \gamma & -\mu^{-1}\Lambda\rho \\ 0 & \lambda - (\alpha + \mu + \psi) & 0 & \mu^{-1}\Lambda\rho \\ 0 & \alpha & -(\mu + \gamma) & 0 \\ 0 & \theta & 0 & -\nu \end{bmatrix}. \tag{3.12}
$$

4. Stability

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a complete probability space, $\{\mathfrak{F}_t, t \geq 0\}$ be a nondecreasing family of sub-σ-algebras of $\mathfrak{F}$, i.e., $\mathfrak{F}_{t_1} \subset \mathfrak{F}_{t_2} \subset \mathfrak{F}$ for $t_1 < t_2$, $\mathbf{E}$ be the mathematical expectation with respect to the measure $\mathbf{P}$ [4, 14].

Definition 4.1. [14] *The zero solution of the equation* (3.8) *is called stable in probability if for any $\varepsilon > 0$ and $\varepsilon_1 \in (0,1)$ there exists a $\delta > 0$ such that the solution $y(t) = (y_1(t), y_2(t), y_3(t), y_4(t))'$ of the equation* (3.8) *satisfies the inequality $\mathbf{P}\{\sup_{t \geq 0} |y(t)| > \varepsilon\} < \varepsilon_1$ for any initial function $\phi(s)$ such that $\mathbf{P}\{\sup_{s \in [-h,0]} |\phi(s)| < \delta\} = 1$.*

Definition 4.2. [14] *The zero solution of the equation* (3.10) *is called:*

- mean square stable if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|z(t,\phi)|^2 < \varepsilon$, $t \geq 0$, provided that $\|\phi\|^2 = \sup_{s \in [-h,0]} \mathbf{E}|\phi(s)|^2 < \delta$;

- *asymptotically mean square stable if it is mean square stable and* $\lim_{t\to\infty} \mathbf{E}|z(t,\phi)|^2 = 0$ *for an each initial function* ϕ.

Remark 4.1. *It is known* [14] *that the investigation of stability in probability of the zero solution of a system of nonlinear stochastic differential equations with an order of nonlinearity higher than one can be reduced to the investigation of asymptotic mean square stability of the zero solution of the linear part of this system. So, to obtain conditions for stability in probability of the zero solution of the system of nonlinear stochastic differential equations* (3.8) *it is enough to get conditions for asymptotic mean square stability of the zero solution of the linear stochastic differential equation* (3.10).

Let $z(t)$ be a value of the solution of the equation (3.10) in the time moment t, $z_t = z(t+s)$, $s<0$, be a trajectory of the solution of the equation (3.10) until the time moment t, H_2 be a space of the initial functions $\phi(s)$ of the equation (3.10) such that $\|\phi\| < \infty$. Consider a functional $V(t,\varphi) : [0,\infty) \times H_2 \to \mathbf{R}_+$ that can be presented in the form $V(t,\varphi) = V(t,\varphi(0),\varphi(s))$, $s<0$, and for $\varphi = z_t$ put

$$(4.1) \qquad V_\varphi(t,z) = V(t,\varphi) = V(t,z_t) = V(t,z,z(t+s)), \quad z = \varphi(0) = z(t), \quad s<0.$$

Denote by D the set of the functionals, for which the function $V_\varphi(t,z)$ defined in (4.1) has a continuous derivative with respect to t and two continuous derivatives with respect to z. Let ∇V_φ and $\nabla^2 V_\varphi$ be respectively the first and the second derivatives of the function $V_\varphi(t,z)$. For the functionals from D the generator L of the equation (3.10) has the form [4, 14]

$$(4.2) \quad LV(t,z_t) = \frac{\partial V_\varphi(t,z(t))}{\partial t} + \nabla V'_\varphi(t,z(t))Az(t) + \frac{1}{2}\sum_{i=1}^{4} z'(t-h)C'_i \nabla^2 V_\varphi(t,z(t))C_i z(t-h).$$

Theorem 4.1. [14] *Let there exist a functional* $V(t,\varphi) \in D$, *positive constants* c_1, c_2, c_3, *such that the following conditions hold:*

$$\mathbf{E}V(t,z_t) \geq c_1 \mathbf{E}|z(t)|^2, \quad \mathbf{E}V(0,\phi) \leq c_2\|\phi\|^2, \quad \mathbf{E}LV(t,z_t) \leq -c_3 \mathbf{E}|z(t)|^2.$$

Then the zero solution of the equation (3.10) *is asymptotically mean square stable.*

Theorem 4.2. *Let there exists a positive definite matrix P, for which the following linear matrix inequality (LMI) holds*

$$PA + A'P + \sum_{i=1}^{4} C_i' P C_i < 0. \tag{4.3}$$

Then the equilibrium (S^*, I^*, R^*, E^*) *of the system* (3.1) *is stable in probability.*

Proof. Following the general method of Lyapunov functionals construction [14], let us consider a Lyapunov functional $V(t, z_t)$ in the form $V = V_1 + V_2$, where $V_1(z) = z'Pz$, $P > 0$. Via (4.2) for the equation (3.10) we have

$$LV_1(z(t)) = 2z'(t)PAz(t) + \sum_{i=1}^{4} z'(t-h)C_i'PC_iz(t-h).$$

Choosing the additional functional V_2 in the form $V_2(t, z_t) = \sum_{i=1}^{4} \int_{t-h}^{t} z'(s)C_i'PC_iz(s)ds$ with

$$LV_2(t, z_t) = \sum_{i=1}^{4} [z'(t)C_i'PC_iz(t) - z'(t-h)C_i'PC_iz(t-h)],$$

for the functional $V = V_1 + V_2$ we obtain

$$LV(t, z_t) = z'(t)\left(PA + A'P + \sum_{i=1}^{4} C_i'PC_i\right) z(t).$$

From this and the LMI (4.3) it follows that the constructed above functional $V(t, z_t)$ satisfies the conditions of Theorem 4.1, therefore, the zero solution of the equation (3.10) is asymptotically mean square stable. Via Remark 4.1 it means that the zero solution of the equation (3.8) is stable in probability that is equivalent to stability in probability of the equilibrium (S^*, I^*, R^*, E^*) of the system (3.1). The proof is completed. □

Remark 4.2. *Note that the stability condition* (4.3) *does not depend on the delay in the stochastic terms of the system* (3.1).

5. Numerical Examples

This section presents some examples that illustrate stability properties of the stochastically perturbed SIR model (3.1). Note that for numerical simulation of the solution of the system (3.1) the Euler-Maruyama scheme [10] and the described in [14] special algorithm for numerical simulation of the trajectories of the Wiener process are used. Note also that the examples have a purely mathematical sense without considering the medical essence of the model under consideration.

Example 5.1. *Consider the system* (3.1) *with the following values of the parameters:*

$$\Lambda = 10, \quad \mu = 5, \quad h = 0.1, \quad \gamma = \rho = \lambda = \alpha = \psi = \theta = \nu = 1. \tag{5.1}$$

From (2.2) *it follows that by the parameters* (5.1) *we have the equilibrium* $Q_0^* = (2,0,0,0)$. *Via MATLAB it was shown that for the matrix A, given in* (3.12)*,* (5.1)*, and the levels of noises*

$$\sigma_1 = 3, \quad \sigma_2 = 1.78, \quad \sigma_3 = 2.52, \quad \sigma_4 = 1$$

there exists the positive definite matrix

$$P = \begin{bmatrix} 21.84 & -2.24 & 2.08 & -7.18 \\ -2.24 & 14.05 & 0.80 & 25.95 \\ 2.08 & 0.80 & 15.82 & -1.69 \\ -7.18 & 25.95 & -1.69 & 156.63 \end{bmatrix},$$

for which the LMI (4.3) *holds. Via Theorem 4.2 it means that the equilibrium* $Q_0^* = (2,0,0,0)$ *of the system* (3.1) *is stable in probability.*

In Fig.1 100 trajectories of the solution $(S(t), I(t), R(t), E(t))$ *of the system* (3.1) *are shown with the initial conditions*

$$(S_0, I_0, R_0, I_0) = (S(s), I(s), R(s), E(s)) = (2.8, 2.65, 1.4, 1.75), \quad s \in [-0.1, 0].$$

One can see that all trajectories converge to the equilibrium $Q_0^* = (2,0,0,0)$.

Example 5.2. *Consider the system* (1.1) *with the following values of the parameters:*

$$\Lambda = 10, \quad \theta = 2, \quad h = 0.1, \quad \gamma = \rho = \lambda = \alpha = \psi = \mu = \nu = 1. \tag{5.2}$$

By that the conditions (2.7) *hold:*

$$\beta_0\beta_1 + \beta_2\beta_3 = 2.5 * 21 + 2 * 10 = 72.5 > \beta_5 = 3,$$

$$\beta_1\beta_3 = 21 * 10 = 210 > \beta_4 = 30,$$

$$\alpha_1^2 = 6.95 * 6.95 = 48.3 > \alpha_2 = 36,$$

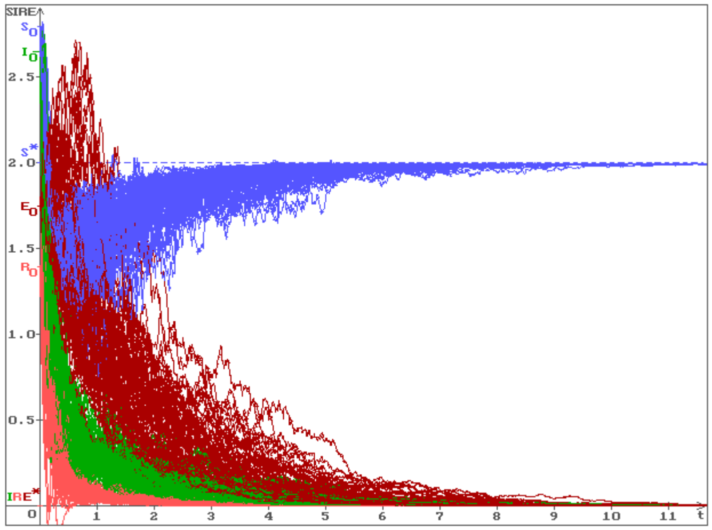

FIGURE 1. 100 trajectories of the solution $S(t)$ (blue), $I(t)$ (green), $R(t)$ (red), $E(t)$ (brown), of the system (3.1) converge to the stable equilibrium $Q_0^*(2,0,0,0)$

and the positive equilibrium $Q_1^* = (1.39, 3.44, 1.72, 6.88)$ *there exists, for which the condition* (2.8) *holds too:*

$$S^* + I^* + R^* = 1.39 + 3.44 + 1.72 = 6.55 < \frac{\Lambda}{\mu} = 10.$$

Via MATLAB it was shown that for the matrix A, given in (3.11)*,* (5.2)*, and the levels of noises*

$$\sigma_1 = 2.5, \quad \sigma_2 = 1.8, \quad \sigma_3 = 1.7, \quad \sigma_4 = 1.2$$

there exists the positive definite matrix

$$P = \begin{bmatrix} 298.77 & 208.99 & 91.29 & 102.50 \\ 208.99 & 284.85 & 86.83 & 146.81 \\ 91.29 & 86.83 & 177.02 & 30.19 \\ 102.50 & 146.81 & 30.19 & 224.34 \end{bmatrix},$$

for which the LMI (4.3) *holds. Via Theorem 4.2 it means that the equilibrium* $Q_1^* = (1.39, 3.44, 1.72, 6.88)$ *of the system* (3.1) *is stable in probability.*

In Fig.2 100 trajectories of the solution $(S(t), I(t), R(t), E(t))$ *of the system* (3.1) *are shown with the initial conditions*

$$(S_0, I_0, R_0, E_0) = (S(s), I(s), R(s), E(s)) = (6.5, 5.5, 0.5, 4.5), \quad s \in [-0.1, 0].$$

One can see that all trajectories converge to the equilibrium $Q_1^* = (1.39, 3.44, 1.72, 6.88)$.

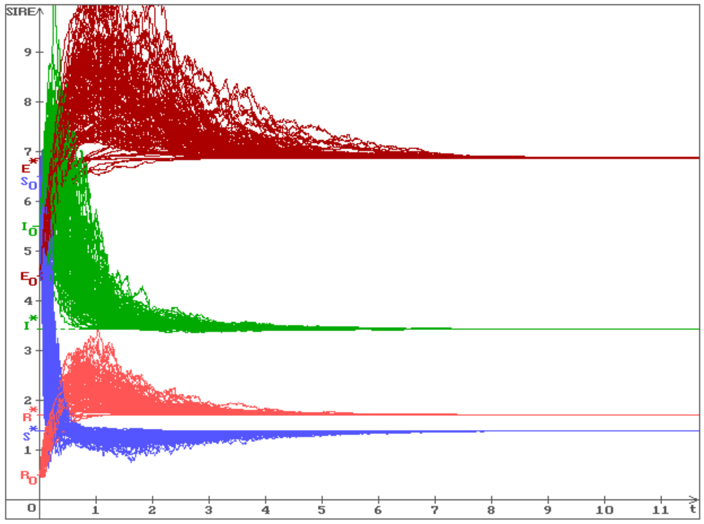

FIGURE 2. 100 trajectories of the solution $S(t)$ (blue), $I(t)$ (green), $R(t)$ (red), $E(t)$ (brown), of the system (3.1) converge to the stable equilibrium $Q_1^* = (1.39, 3.44, 1.72, 6.88)$

6. APPENDIX. PROOF OF LEMMA 2.1

(1) From two last equations (2.1) it follows

$$R = \frac{\alpha}{\mu+\gamma} I, \qquad E = \frac{\theta}{\nu} I. \tag{6.1}$$

Summing two first equations (2.1), we have

$$\Lambda + \gamma R - \mu S - (\alpha + \mu + \psi) I = 0. \tag{6.2}$$

Substituting (6.1) into (6.2) and using (2.6), we obtain

$$\begin{aligned} S =& \frac{\Lambda}{\mu} + \frac{\gamma}{\mu} R - \left(1 + \frac{\alpha}{\mu} + \frac{\psi}{\mu}\right) I \\ =& \frac{\Lambda}{\mu} + \frac{\gamma}{\mu} \frac{\alpha}{\mu+\gamma} I - \left(1 + \frac{\alpha}{\mu} + \frac{\psi}{\mu}\right) I \\ =& \frac{\Lambda}{\mu} - \beta_0 I. \end{aligned} \tag{6.3}$$

Via (6.3), (6.1), (2.6) we have

$$\begin{aligned} S+I+R &= \frac{\Lambda}{\mu} - \left(\beta_0 - 1 - \frac{\alpha}{\mu+\gamma}\right) I \\ &= \frac{\Lambda - \psi I}{\mu} \le \frac{\Lambda}{\mu}. \end{aligned} \tag{6.4}$$

Substituting (6.1), (6.3), (6.4) into the second equation (2.1), we obtain the equation for I:

$$\left(\frac{\lambda\mu}{\Lambda-\psi I} + \frac{\rho\theta}{\nu}\right)\left(\frac{\Lambda}{\mu} - \beta_0 I\right) = \alpha + \mu + \psi$$

or

$$\left(\lambda\mu + \frac{\rho\theta\Lambda}{\nu} - \frac{\rho\theta\psi}{\nu} I\right)\left(\frac{\Lambda}{\mu} - \beta_0 I\right) = (\alpha+\mu+\psi)(\Lambda - \psi I)$$

or

$$(\beta_1 - \beta_2 I)(\beta_3 - \beta_0 I) = \beta_4 - \beta_5 I, \tag{6.5}$$

where $\beta_0, ..., \beta_5$ are defined in (2.6).

Presenting the equation (6.5) in the form

$$\beta_0\beta_2 I^2 - (\beta_0\beta_1 + \beta_2\beta_3 - \beta_5) I + \beta_1\beta_3 - \beta_4 = 0$$

or

$$I^2 - 2\alpha_1 I + \alpha_2 = 0,$$

where α_1, α_2 are defined in (2.5), we obtain (2.3). By that (2.4) follows from (6.1), (6.3).

(2) Supposing that $S_2^* \ge 0$, from (2.4), (2.3) we have $\frac{\Lambda}{\mu\beta_0} \ge I_2^*$, i.e., $\frac{\Lambda}{\mu\beta_0} - \alpha_1 \ge \sqrt{\alpha_1^2 - \alpha_2}$ or

$$\left(\frac{\Lambda}{\mu\beta_0}\right)^2 - 2\alpha_1 \frac{\Lambda}{\mu\beta_0} \ge -\alpha_2.$$

From here and (2.5), (2.6) we obtain

$$\begin{aligned} \left(\frac{\Lambda}{\mu\beta_0}\right)^2 + \alpha_2 \ge 2\alpha_1 \frac{\Lambda}{\mu\beta_0} &= \left(\frac{\beta_1}{\beta_2} + \frac{\Lambda}{\mu\beta_0} - \frac{\beta_5}{\beta_0\beta_2}\right)\frac{\Lambda}{\mu\beta_0} \\ &= \left(\frac{\beta_1}{\beta_2} - \frac{\beta_5}{\beta_0\beta_2}\right)\frac{\Lambda}{\mu\beta_0} + \left(\frac{\Lambda}{\mu\beta_0}\right)^2. \end{aligned}$$

Therefore, via (2.5)

$$\alpha_2 = \frac{\beta_1\beta_3 - \beta_4}{\beta_0\beta_2} \ge \left(\frac{\beta_1}{\beta_2} - \frac{\beta_5}{\beta_0\beta_2}\right)\frac{\Lambda}{\mu\beta_0}.$$

Multiplying the obtained inequality by β_2 and substituting β_3, we obtain

$$\beta_1 \frac{\Lambda}{\mu\beta_0} - \frac{\beta_4}{\beta_0} \geq \left(\beta_1 - \frac{\beta_5}{\beta_0}\right) \frac{\Lambda}{\mu\beta_0} = \beta_1 \frac{\Lambda}{\mu\beta_0} - \frac{\beta_5}{\beta_0} \frac{\Lambda}{\mu\beta_0},$$

from where it follows that $\beta_4 \leq \dfrac{\beta_5\Lambda}{\beta_0\mu}$. Substituting β_4 and β_5 from (2.6) we obtain the wrong inequality

$$\beta_0 = 1 + \frac{\alpha}{\mu+\gamma} + \frac{\psi}{\mu} \leq \frac{\psi}{\mu}.$$

It means that the assumption $S_2^* \geq 0$ is wrong. So, $S_2^* < 0$. The proof is completed.

Conflict of Interests

The author declares that there is no conflict of interests.

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