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MARSHALL-OLKIN ALPHA POWER TRANSFORMED EXTENDED EXPONENTIAL DISTRIBUTION

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Abstract. This study presents the Marshall-Olkin Alpha Power Transformed Extended Exponential Distribution, a new statistical model that improves the flexibility of the standard exponential distribution using the Marshall-Olkin Alpha Power Transformed Extended-X family of distributions. MOAPTE_{Ex} distribution depends on the parameters θ, λ, and α. The lack of closed-form solutions and the requirement for numerical methods are highlighted as we examine the Maximum Likelihood Estimation (MLE) method for parameter estimation. The performance of many estimating strategies, such as maximum product spacing (MPS), least squares (LS), and MLE, across a range of sample sizes is assessed; this is done using a Monte Carlo simulation exercise. The results show that MLE is the most reliable method, particularly for larger samples, while MPS performs worse for smaller samples. Applications to actual datasets provide additional validation of the MOAPTE_{Ex} distribution, showing its efficacy in simulating fiber strength datasets where outer-performed the other competing models.

Keywords: quantile function; least square method; maximum likelihood estimation; maximum product spacing; Marshall-Olkin alpha power transformed extended-X family; exponential distribution.

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1. Introduction

Statistical modeling plays a crucial role in the analysis and interpretation of complex data in various disciplines, including finance, engineering, healthcare, and social sciences. The exponential distribution with the CDF and PDF defined as;

$$f(x;\lambda) = 1 - e^{-\lambda x}; \quad \lambda, x > 0, \tag{1}$$

$$f(x;\lambda) = \lambda e^{-\lambda x}; \quad \lambda, x > 0, \tag{2}$$

this is particularly notable for its application in modeling the time until an event occurs, such as the lifespan of products or the time until failure in reliability studies. Despite its widespread use, the exponential distribution is often criticized for its lack of flexibility, as it assumes a constant hazard rate and may not adequately capture the variability observed in real-world data. Researchers have developed a range of generalized distributions that incorporate additional parameters to better fit empirical data such as [1] proposed the generalization of exponential distribution using beta function, [2] introduced the logistic–exponential survival distribution, A new generalized Exponential family of distributions using Kumaraswamy method [3], and later Alpha Power Exponential distribution was proposed by [4], [5] proposed Exponentiated Generalized Gull Alpha Power Exponential (EGGAPE) distribution as an extension of Exponential distribution, recently a novel extended inverse-exponential distribution was introduced by [6] . [7] proposed Marshall-Olkin family of distributions with the aim of increasing flexibility and applicability by adding parameters that can modify the shape and behavior of the base distribution. This family has been successfully applied to various contexts, enhancing the modeling capabilities of traditional distributions including Exponential distributions. Many Researchers has utilized the Marshall-Olkin family of distributions to extend the other distribution such as Marshall–Olkin extended weibull distribution proposed by [8], the Marshall-Olkin Fréchet distribution [9], Marshall-Olkin Extended Gumbel Type-II Distribution [10] as a new family of distribution, [11] proposed Exponentiated Marshall-Olkin exponential distribution in application to COVID-19 second wave in Nepal, A discrete Kumaraswamy Marshall-Olkin exponential distribution [12], The Marshall–Olkin alpha power family of distributions with applications [13] with consideration on exponential distribution, and EGMO-exponential (EGMO-E) distribution [14].

Recently, a new generator family of distribution called Alpha Power Transformed Extended-X family of distribution with application to COVID-19 pandemic status in china proposed by [15] the authors considered the Weilbull distribution as the baseline distribution,researchers effectively to manage skewness and kurtosis, making it possible to model COVID-19 data set that deviate from the assumptions of standard distributions. The integration of the Marshall-Olkin method with the Alpha Power transformed Extended-X family of distribution leads to the development of the Marshall-Olkin Alpha Power Transformed Extended-X distribution with a CDF and PDF defined as;

$$F_{MOAPTEx}(x) = \begin{cases} \dfrac{\alpha^{\left(1-\frac{1-G(x)}{e^{G(x)}}\right)}-1}{(\alpha-1)\theta+(1-\theta)\left(\alpha^{\left(1-\frac{1-G(x)}{e^{G(x)}}\right)}-1\right)}; & \text{for } \theta>0, \quad \alpha>0, \quad \alpha\neq 1 \\ \frac{2G(x)}{\theta} & \text{for } \alpha=1 \end{cases} \tag{3}$$

$$f_{MOAPTEx}(x) = \begin{cases} \dfrac{\theta(\log\alpha)g(x)[2-G(x)]\alpha^{\left(1-\frac{1-G(x)}{e^{G(x)}}\right)}}{(\alpha-1)e^{G(x)}\left[\theta+(1-\theta)(\alpha-1)^{-1}\left(\alpha^{\left(1-\frac{1-G(x)}{e^{G(x)}}\right)}-1\right)\right]^2}; & \text{for } \theta>0, \quad \alpha>0, \quad \alpha\neq 1 \\ \frac{g(x)[2-G(x)]}{\theta e^{G(x)}} & \text{for } \alpha=1 \end{cases} \tag{4}$$

Where $G(x)$ and $g(x)$ is the CDF and PDF of a baseline distribution.

We use the one-parameter exponential distribution as the baseline distribution to introduce the three-parameter exponential distribution which adds flexibility to exponential distribution. This new family of distribution aims to provide a more adaptable framework for modeling a wide range of data behaviors, particularly those encountered in fiber strength analysis and other applications in materials science.

The motivation for this research is rooted in the necessity for advanced statistical tools that can accommodate the complexities of real-world data using the two data sets proves the superior performance of the MOAPTE$_{Ex}$ when compared to other distributions. By introducing the MOAPTE$_{Ex}$ distribution, this study aspires to enhance the toolkit available to statisticians and researchers, facilitating more accurate data analysis and interpretation in various applications.

2. Marshall-Olkin Alpha Power Transformed Extended-Exponential Distribution

In this section we apply the MOAPTE-X family of distribution to the exponential distribution, the Cumulative Density Function (CDF) of Marshall - Olkin Alpha Power Transformed Extended-Exponential Distribution obtained by substituting equation (1) into (3), is defined as;

(5)

$$F_{MOAPTE_{Ex}}(x)=\begin{cases}\dfrac{\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)}-1}{(\alpha-1)\left(\theta+(1-\theta)(\alpha-1)^{-1}\left(\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)}-1\right)\right)}; & \text{for } \theta>0,\quad \alpha>0,\quad \alpha\neq 1\\ \\ \frac{2(1-e^{-\lambda x})}{\theta} & \text{for}\quad \alpha=1\end{cases}$$

And its PDF reduces to;

(6)

$$f_{MOAPTE_{Ex}}(x)=\begin{cases}\dfrac{\theta\lambda(\log\alpha)e^{-\lambda x}[1+e^{-\lambda x}]\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)}}{(\alpha-1)e^{1-e^{-\lambda x}}\left[\theta+(1-\theta)(\alpha-1)^{-1}\left(\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)}-1\right)\right]^2}; & \text{for}\quad \theta>0,\quad \lambda>0,\\ & \alpha>0,\quad \alpha\neq 1\\ \frac{\lambda e^{-\lambda x}[1+e^{-\lambda x}]}{\theta e^{1-e^{-\lambda x}}} & \text{for}\quad \alpha=1\end{cases}$$

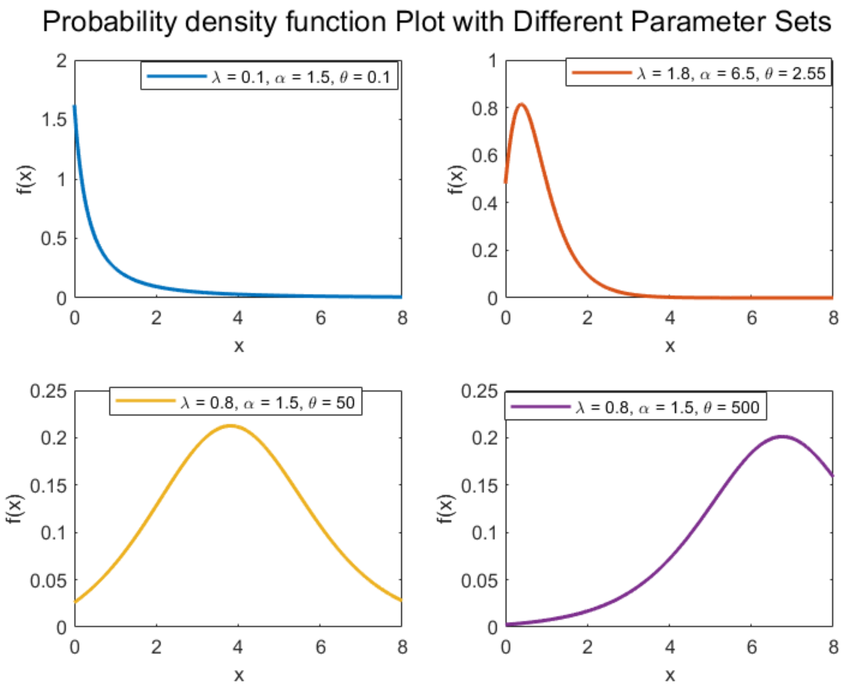
The survival function of MOAPTE-Ex is respectively given as;

$$S(x)=\frac{\theta\alpha-\theta\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)}}{(\alpha-1)\theta+(1-\theta)\left(\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)}-1\right)};\quad \theta>0,\quad \alpha>0,\quad \alpha\neq 1 \tag{7}$$

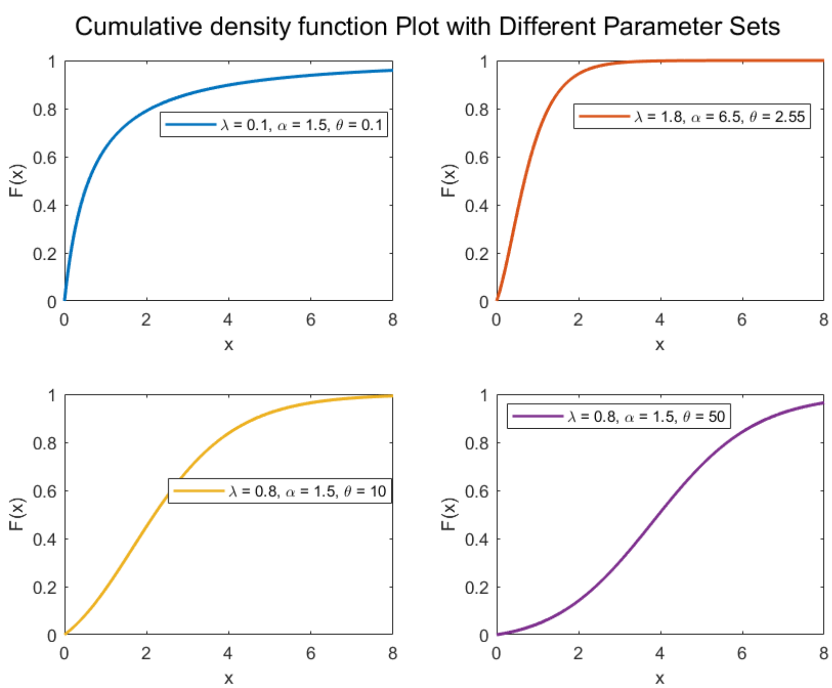
Where $\theta,\alpha,\beta,\lambda>0$ and $\alpha\neq 1$.

Fig. 1a shows the the plots of MOAPTE$_{Ex}$ distribution's shape depending on its parameter values. Fig. 1a indicate that MOAPTE$_{Ex}$ has monotone decreasing form (right skewed), symmetrical, and left skewed, this illustrates flexibility of MOAPTE$_{Ex}$ distribution to adapt its shape to various parameter choices.

The CDF of MOAPTE$_{Ex}$ is a monotone increase as shown in Fig. 1b and the survival function plots are monotone decreases see Fig. 3.



(A) Probability density function



(B) Cumulative density function

FIGURE 1. Probability density function (a) and Cumulative density function (b) plots of the MOAPTE$_{Ex}$ distribution with different parameter values

Hazard Function

The random variables $x_1, x_2, x_3, \ldots, x_n$ follows a MOAPTE$_{Ex}$ distribution denoted as $X \sim$ MOAPTE$_{Ex}(\theta, \alpha, \lambda)$, then the hazard function $h(x)$ for this distribution obtained by substituting Eq. (6) and (7) into Eq. (8), given;

$$h_{MOAPTE_{Ex}}(x) = \frac{f_{MOAPTE_{Ex}}(x)}{S_{MOAPTE_{Ex}}(x)}, \tag{8}$$

the hazard rate function can be expressed as

$$h(x,\alpha,\theta,\lambda) = \frac{\lambda(\log\alpha)e^{-\lambda x}[1+e^{-\lambda x}]\left(\alpha^{\left(\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)}-1\right)}{e^{1-e^{-\lambda x}}\left[\theta+(1-\theta)(\alpha-1)^{-1}\left(\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)}-1\right)\right]}, \quad \theta>0, \quad \lambda>0, \quad \alpha>0, \quad \alpha\neq 1 \tag{9}$$

The hazard rate function of the MOAPTE$_{Ex}$ distribution appears in Fig. (2) in several shapes that vary at different parameter values. The MOAPTE$_{Ex}$ distribution can accommodate the data set with decreasing hazard rate depending on the value of the parameters, also can exhibit increasing hazard rate shapes,bathtubs and up-down bathtub shape, This demonstrates how flexible this distribution is and can be applicable to the data with different behaviors.

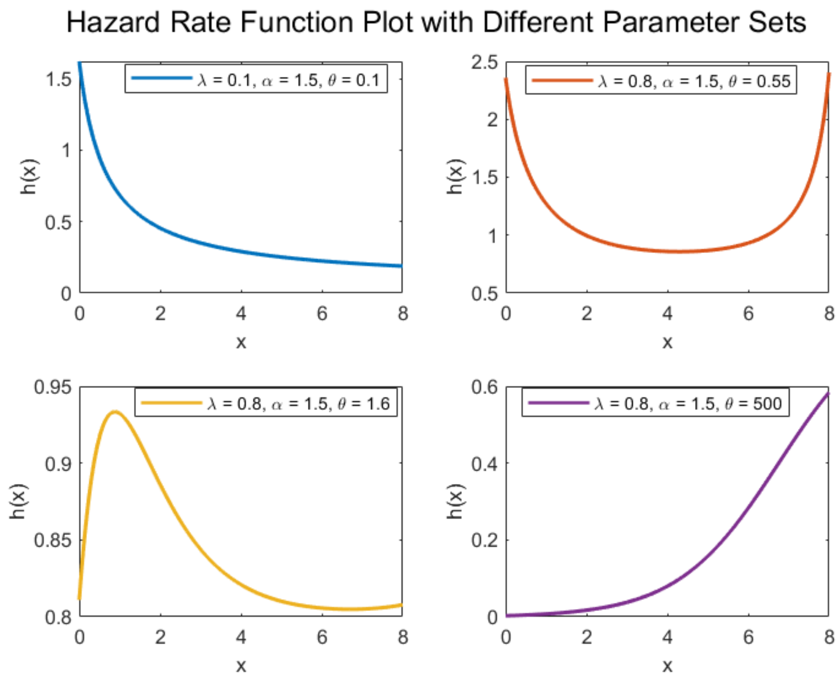


FIGURE 2. Hazard rate function plots of the MOAPTE-Ex distribution with different parameter values

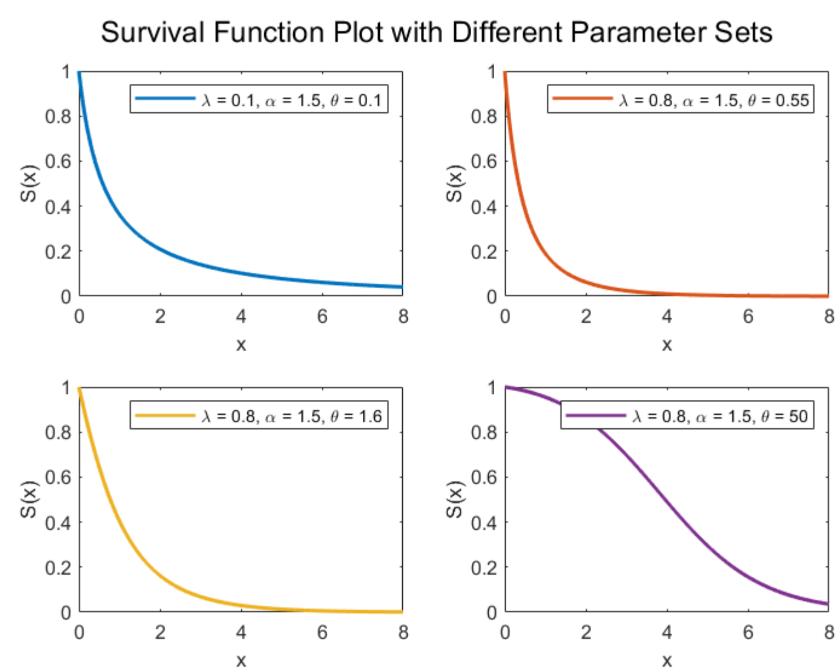


FIGURE 3. Survival Function plots of the MOAPTE-Ex distribution with different parameter values

3. Statistical Properties of Marshall-Olkin Alpha Power Transformed Extended-Exponential Distribution

3.1. Quantile function.

The quantile function of the MOAPTE$_{Ex}$ distribution random variable X is $Q_X(u) = F_X^{-1}(u), 0 < u < 1$, and for any $\theta, \alpha, \beta, \lambda > 0$ and $\alpha \neq 1$.

$$Q_X(u) = \frac{-1}{\lambda}\log\left(W\left[\left(1 - \frac{\log\left(\frac{1-u(1-\theta\alpha)}{1-u(1-\theta)}\right)}{\log(\alpha)}\right)e\right]\right), \quad 0 < u < 1 \tag{10}$$

The Median

The median of MOAPTE$_{Ex}$ distribution obtained by putting u = 0.5 into equation (10) ($Q_X(u)$) respectively.

$$Q_X(0.5) = \frac{-1}{\lambda}\log\left(W\left[\left(1 - \frac{\log\left(\frac{1-0.5(1-\theta\alpha)}{1-0.5(1-\theta)}\right)}{\log(\alpha)}\right)e\right]\right)$$

For any $\theta, \alpha, \lambda > 0$ and $\alpha \neq 1$.

Table (1) shows the quantile values of the MOAPTE$_{Ex}$ distribution at different parameter set, with increasing values of λ, θ and α, the quantile values rise for all quantile levels (0.1, 0.2,..., 0.9).

TABLE 1. Quantile Table for Different Parameter Combinations

	α = 0.2, λ = 1.5			**θ = 0.6, λ = 1.1**			**θ =0.6, α = 0.2**		
u	(θ = 0.1)	(θ = 1.5)	(θ = 9)	(α = 0.2)	(α = 5.5)	(α = 10.5)	(λ = 0.1)	(λ = 5.1)	(λ = 14.6)
0.1	0.00183	0.02639	0.13354	0.01478	0.07282	0.10323	0.16258	0.00319	0.00111
0.2	0.00410	0.05649	0.25330	0.03252	0.14943	0.20116	0.35783	0.00702	0.00244
0.3	0.00699	0.09157	0.37191	0.05429	0.23306	0.30181	0.59716	0.01171	0.00407
0.4	0.01084	0.13354	0.49655	0.08171	0.32787	0.41142	0.89878	0.01763	0.00618
0.5	0.01610	0.18559	0.63423	0.11750	0.43978	0.53721	1.29270	0.02535	0.00885
0.6	0.02385	0.25331	0.79447	0.16671	0.57873	0.69009	1.83381	0.03598	0.01256
0.7	0.03635	0.34791	0.99371	0.23959	0.76365	0.88996	2.63555	0.05168	0.01806
0.8	0.05990	0.49654	1.26774	0.36234	1.03857	1.18232	3.98570	0.07817	0.02730
0.9	0.12225	0.79448	1.72994	0.63335	1.54966	1.71599	6.96681	0.13660	0.04772

3.2. Rényi entropy. Renyi entropy analysis provides understanding about the uncertainty and complexity inherent in the underlying probability distribution. Consider a continuous variable named X with PDF, denoted by $f(x)$ of MOAPTE$_{Ex}$, the Rényi entropy of order δ is defined as:

(11)

$$R_{\delta}(X) = \frac{1}{1-\delta} \log \int_0^{\infty} f^{\delta}(x)dx \quad \text{where} \quad \delta > 0, \delta \neq 1$$

$$= \frac{\delta \log(\theta\lambda(\log\alpha))}{(1-\delta)} \log \int_0^{\infty} \left(\frac{e^{-\lambda x}[1+e^{-\lambda x}]\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)}}{(\alpha-1)e^{1-e^{-\lambda x}}\left[\theta+(1-\theta)(\alpha-1)^{-1}\left(\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)}-1\right)\right]^2} \right)^{\delta} dx.$$

where $\quad \delta > 0, \delta \neq 1$

Rényi entropy behave differently under various combinations of the parameters α, λ, and θ as shown in Table (2). The Rényi entropy values across parameter sets of varying θ and λ consistently decrease with increasing δ, indicating reduced uncertainty. A larger values of θ lead to more concentrated distributions, and higher values of α and λ result in more dispersed distributions.

TABLE 2. Rényi Entropy of Different Parameter Sets

	α = 1.8, λ = 3.1			**θ = 0.1, λ = 1.1**			**θ =0.01, α = 2.5**		
δ	θ = 1.1	θ = 5.5	θ =12.5	α = 0.4	α = 1.5	α = 5.5	λ = 0.9	λ = 1.5	λ = 9.5
2.5	5.3911	4.282	3.8443	0.5043	1.0255	6.6371	3.8414	6.8288	40.2986
4	4.6439	3.7587	3.4098	0.8436	1.1804	5.6388	3.4052	5.7925	32.5677
5.5	4.3742	3.5639	3.2448	0.953	1.2156	5.2852	3.2393	5.4262	29.9698
7	4.2303	3.4575	3.1535	1.0057	1.2258	5.0992	3.1474	5.234	28.6617
8.5	4.1392	3.3891	3.094	1.0361	1.2281	4.9829	3.0876	5.1139	27.8721
10	4.0757	3.3407	3.0516	1.0555	1.2272	4.9026	3.0449	5.031	27.3429
11.5	4.0286	3.3043	3.0196	1.0687	1.2251	4.8434	3.0127	4.97	26.9631

3.3. Order statistics. Given a sample of size n from the MOAPTE$_{Ex}$ distribution, let $X_1, X_2, \ldots, X_n$ be the random variables representing the sampled observations. The order statistics are denoted as $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ where $X_{(i)}$ is the i^{th} order statistics.

Given

$$f(x_{i:n}) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} (1-F(x))^{n-i} \tag{12}$$

Substituting equation (6) and (5) into (12) yields:

$$\begin{aligned} f(x_{i:n}) = {} & \frac{n!\theta\lambda(\log\alpha)e^{-\lambda x}[1+e^{-\lambda x}]}{(i-1)!(n-i)!(\alpha-1)^n e^{1-e^{-\lambda x}}} \alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)} \left(\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)} - 1\right)^{i-1} \\ & \times \left[\theta + (1-\theta)(\alpha-1)^{-1}\left(\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)} - 1\right)\right]^{-(n+1)} \left(\theta\alpha - \theta\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)}\right)^{n-i} \end{aligned} \tag{13}$$

When $i = 1$ the minimum order statistics for MOAPTE$_{Ex}$ distribution) is given as:

$$\begin{aligned} f(x_{1:n}) = {} & \frac{n!\theta\lambda(\log\alpha)e^{-\lambda x}[1+e^{-\lambda x}]}{(n-1)!(\alpha-1)^n e^{1-e^{-\lambda x}}} \alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)} \left(\theta\alpha - \theta\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)}\right)^{n-1} \\ & \times \left[\theta + (1-\theta)(\alpha-1)^{-1}\left(\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)} - 1\right)\right]^{-(n+1)} \end{aligned} \tag{14}$$

For $i = n$ the maximum order statistics for MOAPTE$_{Ex}$ distribution is given as:

$$\begin{aligned} f(x_{n:n}) = {} & \frac{n!\theta\lambda(\log\alpha)e^{-\lambda x}[1+e^{-\lambda x}]}{(n-1)!(\alpha-1)^n e^{1-e^{-\lambda x}}} \alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)} \left(\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)} - 1\right)^{n-1} \\ & \times \left[\theta + (1-\theta)(\alpha-1)^{-1}\left(\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)} - 1\right)\right]^{-(n+1)} \end{aligned} \tag{15}$$

3.4. Mode. By determining the value of x that maximizes the function, we can find the mode of MOAPTE$_{Ex}$ distribution from equation 6 is given as:

$$\begin{aligned} \frac{d}{dx} f_{MOAPTE-Ex}(x) = {} & \log(\alpha)\,\lambda\,\mathrm{e}^{\mathrm{e}^{-\lambda x}-1}\,\mathrm{e}^{-\lambda x} + \lambda\,\mathrm{e}^{\mathrm{e}^{-\lambda x}-1}\,\mathrm{e}^{-2\lambda x} - \lambda - \lambda\,\mathrm{e}^{-\lambda x} - \frac{\lambda\,\mathrm{e}^{-\lambda x}}{\mathrm{e}^{-\lambda x}+1} \\ & - \frac{2\,\alpha^{1-\mathrm{e}^{\mathrm{e}^{-\lambda x}-1}\,\mathrm{e}^{-\lambda x}}\log(\alpha)\,\sigma_1\,(\theta-1)}{\left(\theta(\alpha-1) - (\theta-1)\left(\alpha^{1-\mathrm{e}^{\mathrm{e}^{-\lambda x}-1}\,\mathrm{e}^{-\lambda x}} - 1\right)\right)} = 0 \end{aligned} \tag{16}$$

The value that satisfies the derivative equation derived from the distribution's probability density function for any given x is the mode of the MOAPTE$_{Ex}$ distribution. However, because of the complexity of the equation, analysis of this mode may prove to be challenging, using numerical optimization technique provide a practical and effective solution. Finding the value of x that yields the highest probability density and minimizing the negative of the PDF will yield a reliable estimate of the mode of the distribution.

3.5. The r^{th} Moments of MOAPTE$_{Ex}$ distribution. We need to calculate the $E(x^r)$ based on $f_{\text{MOAPTE}_{Ex}}(x)$. The r^{th} moments of MOAPTE$_{Ex}$ distribution is obtained by inserting equation 6 into the following equation;

$$E(x^r) = \int_0^\infty x^r f_{MOAPTE_{Ex}}(x)dx$$

The r^{th} moment of the MOAPTE$_{Ex}$ distribution is given by:

$$\begin{aligned}\mathbb{E}[x^r] = &\int_0^\infty x^r \theta\lambda(\log\alpha)e^{-\lambda x}\left[1+e^{-\lambda x}\right]\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)}(\alpha-1)^{-1}e^{-(1-e^{-\lambda x})} \\ &\times\left[\theta+(1-\theta)(\alpha-1)^{-1}\left(\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)}-1\right)\right]^{-2}dx\end{aligned} \tag{17}$$

where $\theta, \alpha, \lambda > 0$ and $\alpha \neq 1$.

Then expanding equation (17) using

$$(1-z)^{-2} = \sum_{k=0}^\infty (k+1)z^k, (1-z)^k = \sum_{j=0}^k \binom{k}{j}(-1)^j z^j, \alpha^z = \sum_{m=0}^\infty \frac{(\log\alpha)^m}{m!}z^m,$$

$$e^x = \sum_{n=0}^\infty \frac{x^n}{n!}, \text{and } \int_0^\infty x^r e^{-\beta x}dx = \frac{\Gamma(r+1)}{\beta^{r+1}}$$

Let A = $\left[\theta+(1-\theta)(\alpha-1)^{-1}\left(\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)}-1\right)\right]^{-2}$ from equation (17), Then, $A = \left[\theta+(1-\theta)(\alpha-1)^{-1}\left(\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)}-1\right)\right]^{-2}$ can be expressed in term of $(1-z)^{-2}$ as;

$$\begin{aligned}A &= \left(1-\left[(1-\theta)-(1-\theta)(\alpha-1)^{-1}\left(\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)}-1\right)\right]\right)^{-2} \\ &= \sum_{k=0}^\infty (k+1)(1-\theta)^k \sum_{j=0}^k \binom{k}{j}(-1)^{j+i}\left(\frac{1}{\alpha-1}\right)^j \sum_{i=0}^j \binom{j}{i}\left(\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)}\right)^{j-i}\end{aligned} \tag{18}$$

substituting equation 18 into 17 results:

(19)
$$\mathbb{E}[x^r] = \frac{\theta\lambda(\log\alpha)}{\alpha-1}\sum_{k=0}^{\infty}(k+1)(1-\theta)^k\sum_{j=0}^{k}\binom{k}{j}(-1)^{j+i}\left(\frac{1}{\alpha-1}\right)^j\sum_{i=0}^{j}\binom{j}{i}\int_0^\infty x^r e^{-\lambda x}\left[1+e^{-\lambda x}\right]$$
$$\times e^{-(1-e^{-\lambda x})}\left(\alpha^{\left(1-\frac{e^{-\lambda x}}{e^{1-e^{-\lambda x}}}\right)}\right)^{j-i+1} dx$$

For furthermore expansion in equation 19 yields:

(20)
$$\begin{aligned}\mathbb{E}[x^r] &= \frac{\theta\lambda(\log\alpha)}{\alpha-1}\sum_{k=0}^{\infty}(k+1)(1-\theta)^k\sum_{j=0}^{k}\binom{k}{j}(-1)^{j+i+v}\left(\frac{1}{\alpha-1}\right)^j\sum_{i=0}^{j}\binom{j}{i}\sum_{u=0}^{\infty}\frac{(\log\alpha)^u}{u!}\times \\ &(j-i+1)^u\sum_{v=0}^{u}\binom{u}{v}\sum_{n=0}^{\infty}\frac{(v-1)^n}{n!}e^{-(v+1)}\int_0^\infty x^r e^{-(v+1+n)\lambda x}\left[1+e^{-\lambda x}\right]dx \\ &= \frac{\theta\lambda(\log\alpha)}{\alpha-1}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}\sum_{u=0}^{\infty}\sum_{v=0}^{u}\sum_{n=0}^{\infty}\frac{(k+1)(1-\theta)^k\binom{k}{j}(-1)^{j+i+v}(\log\alpha)^u(v-1)^n}{u!n!(\alpha-1)^j} \\ &\quad (j-i+1)^u e^{-(v+1)}\int_0^\infty x^r e^{-(v+1+n)\lambda x}\left[1+e^{-\lambda x}\right]dx \\ &= \psi_{k,j,i,u,v,n}\int_0^\infty x^r e^{-(v+1+n)\lambda x}dx + \int_0^\infty x^r e^{-(v+2+n)\lambda x}dx\end{aligned}$$

Therefore, the moment of MOAPTE_{Ex} distribution given as:

$$\mathbb{E}[x^r] = \psi_{k,j,i,u,v,n}\left(\frac{\Gamma(r+1)}{[(v+1+n)\lambda]^{r+1}}+\frac{\Gamma(r+1)}{[(v+2+n)\lambda]^{r+1}}\right) \tag{21}$$

where

$$\psi_{k,j,i,u,v,n} = \frac{\theta\lambda(\log\alpha)}{\alpha-1}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}\sum_{u=0}^{\infty}\sum_{v=0}^{u}\sum_{n=0}^{\infty}\frac{(k+1)(1-\theta)^k\binom{k}{j}(-1)^{j+i+v}(\log\alpha)^u(v-1)^n}{u!n!(\alpha-1)^j}$$
$$(j-i+1)^u e^{-(v+1)}$$

For $k > j$ and $\theta,\alpha,\lambda > 0$ and $\alpha \neq 1$

3.5.1. *The mean.* The mean of the MOAPTE_{Ex} distribution is given from equation 21 by substituting $r = 1$ gives:

$$\mathbb{E}[x] = \psi_{k,j,i,u,v,n}\left(\frac{1}{[(v+1+n)\lambda]^2}+\frac{1}{[(v+2+n)\lambda]^2}\right)$$

Table 3 shows the statistical measurements including mean, standard deviation, skewness, and kurtosis for various parameter sets.

TABLE 3. The first five Moments,standard deviation,Skewness and kurtosis of the MOAPTE$_{Ex}$ distribution for some Parameter sets

Statistic	**α = 0.6, λ = 0.2**			**θ = 0.5, λ = 0.2**			**θ =2.5, α = 0.6**		
	θ = 0.41	θ = 5.5	θ =10.5	α = 0.06	α = 5.6	α = 14.6	λ = 0.3	λ = 2.28	λ = 10.2
M1	0.2238	0.0497	0.0277	0.2423	0.1313	0.0923	0.1342	0.2816	0.0874
M2	0.1302	0.0334	0.0188	0.1290	0.0857	0.0623	0.0877	0.1408	0.0155
M3	0.0904	0.0251	0.0142	0.0854	0.0634	0.0470	0.0650	0.0883	0.0043
M4	0.0688	0.0201	0.0114	0.0631	0.0503	0.0377	0.0515	0.0627	0.0016
M5	0.0554	0.0168	0.0096	0.0499	0.0416	0.0315	0.0427	0.0481	0.0007
Std. Deviation	0.2831	0.1757	0.1341	0.2652	0.2617	0.2318	0.2640	0.2481	0.0888
CV	1.2650	3.5369	4.8374	1.0945	1.9932	2.5107	1.9667	0.8812	1.0157
Skewness	1.1182	3.7558	5.2522	1.0731	1.9092	2.5119	1.8740	0.9154	2.1793
Kurtosis	3.0395	16.3793	30.6763	3.1387	5.3217	8.0736	5.1785	3.0074	10.1665

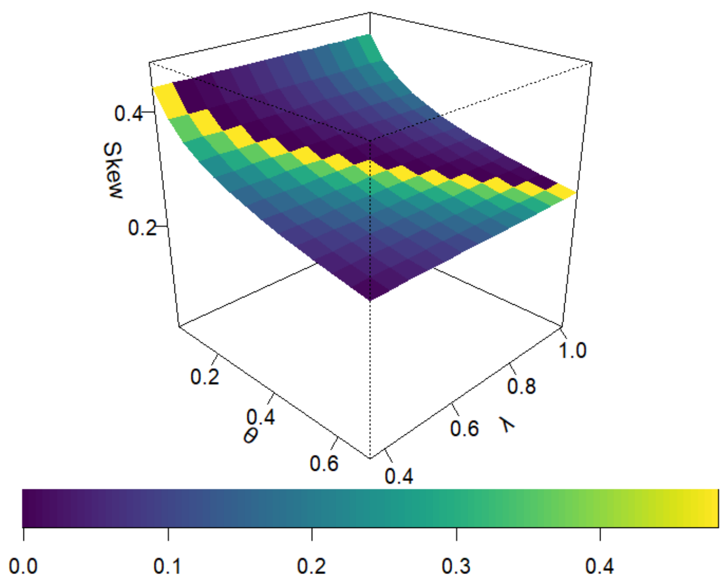
3.6. Skewness and Kurtosis. Quantile-based Bowley's measure of skewness as proposed by [16] which measures the asymmetry and Moors Kurtosis [17] for assessing tail behavior of the distribution, are useful properties of the Marshall - Olkin Alpha Power Transformed Extended-Exponential distributions that provide insights into the shape of the distribution and are obtained mathematically as:

$$S_k(B) = \frac{Q(1/4) + Q(3/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)}$$

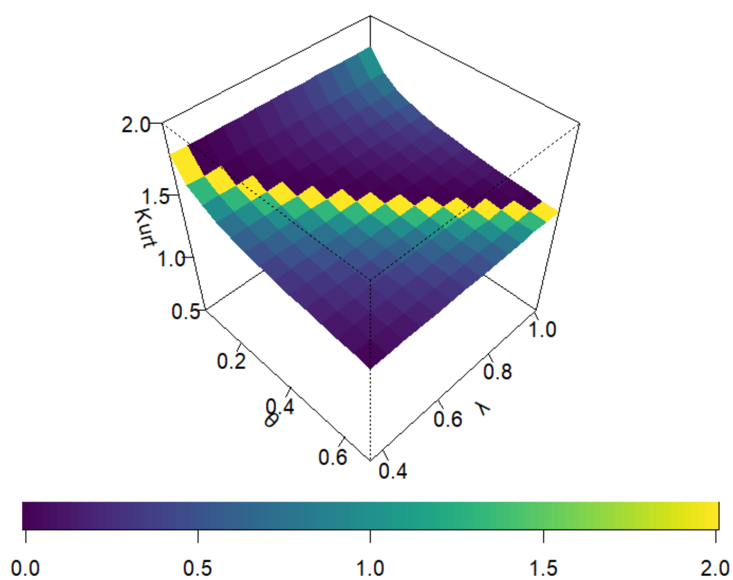
$$K_M = \frac{Q_{0.375} - Q_{0,625} + Q_{0.875} - Q_{0.125}}{Q_{0.75} - Q_{0.25}}$$

where Q is the quantile function of the MOAPTE$_{Ex}$ distribution.

Figure 4a for skewness and 4b for the kurtosis of the MOAPTE$_{Ex}$ distribution with fixed value of $\alpha = 7.09$ and different values of θ and λ.



(A)



(B)

FIGURE 4. Plot for the $MOAPTE_{Ex}$ Moors Kurtosis (b) and Bowley Skewness (a) with ($\alpha = 7.09$)

4. Parameters Estimation of MOAPTE$_{Ex}$

In this section, the Maximum Likelihood Estimation (MLE), Maximum Product Spacing (MPS), and Least Squares (LS) was used to estimate the parameters θ, α and λ of the MOAPTE$_{Ex}$ distribution.

4.1. Maximum Likelihood Estimation. Given the equation 6 respectively with a sample of n independent and identically distributed observations $\{x_1, x_2, \ldots, x_n\}$, the log-likelihood function of the MOAPTE$_{Ex}$ distribution is defined as

$$\mathscr{L}(\theta,\lambda,\alpha \mid x_1,x_2,\ldots,x_n) = \sum_{i=1}^{n} \log\left(f_{\text{MOAPTE}_{\text{Ex}}}(x_i)\right)$$

Substituting the expression for $f_{\text{MOAPTE}_{\text{Ex}}}(x)$:

$$\mathscr{L}(\theta,\lambda,\alpha \mid x_1,x_2,\ldots,x_n) = \sum_{i=1}^{n} \log\left[\frac{\theta\lambda(\log\alpha)e^{-\lambda x_i}[1+e^{-\lambda x_i}]\alpha^{\left(1-\frac{e^{-\lambda x_i}}{e^{1-e^{-\lambda x_i}}}\right)}}{(\alpha-1)e^{1-e^{-\lambda x_i}}\left[\theta+(1-\theta)(\alpha-1)^{-1}\left(\alpha^{\left(1-\frac{e^{-\lambda x_i}}{e^{1-e^{-\lambda x_i}}}\right)}-1\right)\right]^2}\right]$$

Then the logarithm of the likelihood function of MOAPTE$_{Ex}$ given as

$$\begin{aligned}
&\mathscr{L}(\theta,\lambda,\alpha \mid x_1,x_2,\ldots,x_n)\\
&= n\log(\theta)+n\log(\lambda)+n\log(\log\alpha)-\lambda\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\log(1+e^{-\lambda x_i})+\log(\alpha)\sum_{i=1}^{n}\left(1-\frac{e^{-\lambda x_i}}{e^{1-e^{-\lambda x_i}}}\right)\\
&-\sum_{i=1}^{n}\left[\log(\alpha-1)+\left(1-e^{-\lambda x_i}\right)\right]-\sum_{i=1}^{n}\left[2\log\left(\theta+(1-\theta)(\alpha-1)^{-1}\left(\alpha^{\left(1-\frac{e^{-\lambda x_i}}{e^{1-e^{-\lambda x_i}}}\right)}-1\right)\right)\right]
\end{aligned} \tag{22}$$

The derivative of $\mathscr{L}(\theta,\lambda,\alpha \mid x_1,x_2,\ldots,x_n)$ with respect to θ, α and λ given as

$$\frac{\partial\mathscr{L}}{\partial\theta} = \frac{n}{\theta} - 2\sum_{i=1}^{n}\frac{\alpha-\alpha^{\left(1-\frac{e^{-\lambda x_i}}{e^{1-e^{-\lambda x_i}}}\right)}}{\theta(\alpha-1)-(\theta-1)\left(\alpha^{\left(1-\frac{e^{-\lambda x_i}}{e^{1-e^{-\lambda x_i}}}\right)}-1\right)} \tag{23}$$

$$\frac{\partial\mathscr{L}}{\partial\alpha} = \frac{n}{\alpha\log(\alpha)} - \frac{n}{\alpha-1} + \frac{n}{\alpha} - \frac{1}{\alpha}\left(\sum_{i=1}^{n} e^{e^{-\lambda x_i}-1}e^{-\lambda x_i}\right) \tag{24}$$

$$-2\sum_{i=1}^{n}\frac{(\theta-1)\alpha^{e^{e^{-\lambda x_i}-1}e^{-\lambda x_i}}\left(\alpha^{1-\frac{e^{-\lambda x_i}}{e^{1-e^{-\lambda x_i}}}}-1\right)+(\alpha-1)(\theta-1)(e^{e^{-\lambda x_i}-1}e^{-\lambda x_i}-1)}{\left[\theta(\alpha-1)-(\theta-1)\left(\alpha^{1-\frac{e^{-\lambda x_i}}{e^{1-e^{-\lambda x_i}}}}-1\right)\right](\alpha-1)\alpha^{e^{e^{-\lambda x_i}-1}e^{-\lambda x_i}}}$$

$$\begin{aligned}\frac{\partial\mathscr{L}}{\partial\lambda}&=\frac{n}{\lambda}-\frac{\mathrm{e}^{-\lambda n}}{\mathrm{e}^{\lambda}-1}-\sum_{i=1}^{n}\frac{x_i}{\mathrm{e}^{\lambda x_i}+1}-\frac{n(n+1)}{2}-\frac{\mathrm{e}^{\lambda}}{\left(\mathrm{e}^{\lambda}-1\right)^2}+\frac{\mathrm{e}^{-\lambda(n-1)}}{\left(\mathrm{e}^{\lambda}-1\right)^2}+\frac{\mathrm{e}^{-\lambda n}(n+1)}{\mathrm{e}^{\lambda}-1}\\&+\log(\alpha)\left(\left(\sum_{x=1}^{n}x_i\,\mathrm{e}^{\mathrm{e}^{-\lambda x_i}-\lambda x_i-1}\right)+\left(\sum_{i=1}^{n}x_i\,\mathrm{e}^{\mathrm{e}^{-\lambda x_i}-2\lambda x_i-1}\right)\right)\\&+\frac{2\log(\alpha)(\theta-1)}{\alpha-1}\sum_{x=1}^{n}\frac{x\,\alpha^{1-e^{e^{-\lambda x}-\lambda x-1}}\left(e^{e^{-\lambda x}-\lambda x-1}+e^{e^{-\lambda x}-2\lambda x-1}\right)}{\theta-(\theta-1)\dfrac{\alpha^{1-e^{e^{-\lambda x}-\lambda x-1}}-1}{\alpha-1}}\end{aligned}\tag{25}$$

Since the maximum likelihood estimates (MLEs) of the parameters in the MOAPTE$_{Ex}$ distribution, as derived from equation (23) to (25) , do not have closed-form solutions, the resulting system of equations from setting the partial derivatives to zero might not have a simple analytical solution. In such cases, numerical optimization methods are used to find the MLEs. One of the most effective methods for this purpose is the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm.

4.2. Least Squares (LS) method. The OLS objective function is defined as the sum of squared differences between the theoretical CDF $F(x_i;\theta,\alpha,\lambda)$ and the empirical CDF:

$$S(\theta,\alpha,\lambda)=\sum_{i=1}^{n}\left(F(x_i;\theta,\alpha,\lambda)-\frac{i}{n+1}\right)^2$$

The OLS function for the parameters θ, α, and λ based on the provided CDF $F(x;\theta,\alpha,\lambda)$ is defined as:

$$S(\theta,\alpha,\lambda)=\sum_{i=1}^{n}\left(\frac{\alpha^{\left(1-\frac{e^{-\lambda x_i}}{e^{1-e^{-\lambda x_i}}}\right)}-1}{(\alpha-1)\left(\theta+(1-\theta)(\alpha-1)^{-1}\left(\alpha^{\left(1-\frac{e^{-\lambda x_i}}{e^{1-e^{-\lambda x_i}}}\right)}-1\right)\right)}-\frac{i}{n+1}\right)^2.$$

The function above can be minimized to obtain the estimates for the parameters θ, α and λ.

Let

$$f_i(\alpha,\theta,\lambda) = \frac{\alpha^{\left(1-\frac{e^{-\lambda x_i}}{e^{1-e^{-\lambda x_i}}}\right)}-1}{(\alpha-1)\left(\theta+(1-\theta)(\alpha-1)^{-1}\left(\alpha^{\left(1-\frac{e^{-\lambda x_i}}{e^{1-e^{-\lambda x_i}}}\right)}-1\right)\right)} - \frac{i}{n+1},$$

then, this estimates can be obtained by solving the nonlinear function bellow

$$\frac{\partial \mathscr{S}(\theta,\alpha,\lambda)}{\partial \theta} = \sum_{i=1}^{n} f_i(\alpha,\theta,\lambda)\frac{\partial f_i(\alpha,\theta,\lambda)}{\partial \theta} = 0$$
$$\frac{\partial \mathscr{S}(\theta,\alpha,\lambda)}{\partial \alpha} = \sum_{i=1}^{n} f_i(\alpha,\theta,\lambda)\frac{\partial f_i(\alpha,\theta,\lambda)}{\partial \alpha} = 0$$
$$\frac{\partial \mathscr{S}(\theta,\alpha,\lambda)}{\partial \lambda} = \sum_{i=1}^{n} f_i(\alpha,\theta,\lambda)\frac{\partial f_i(\alpha,\theta,\lambda)}{\partial \lambda} = 0$$

4.3. Maximum Product Spacing (MPS) method. Given a sample $x_1, x_2, \ldots, x_n$ from MOAPTE$_{Ex}$ distribution with $F(x_i;\theta,\alpha,\lambda)$ where θ, α, and λ are the parameters of the distribution, the MPS method as introduced by [18] and [19] and later used by [20], and [21] by maximizing the following function with respect to θ, α, and λ

$$D(\theta,\alpha,\lambda) = \left(\prod_{i=1}^{n+1}\left(F_\theta(X_{(i:n)}) - F_\theta(X_{(i-1):n})\right)\right)^{\frac{1}{n+1}} = \frac{1}{n+1}\sum_{i=1}^{n+1}\ln\left(F_\theta(X_{(i:n)}) - F_\theta(X_{(i-1):n})\right)$$

$$D(\phi) = \frac{1}{n+1}\sum_{i=1}^{n+1}\ln\left(\frac{(\alpha-1)^{-1}\left[\alpha^{\left(1-\frac{e^{-\lambda x_{(i+1)}}}{e^{1-e^{-\lambda x_{(i+1)}}}}\right)}-1\right]}{\left(\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{\left(1-\frac{e^{-\lambda x_{(i+1)}}}{e^{1-e^{-\lambda x_{(i+1)}}}}\right)}-1\right)\right)} - \frac{(\alpha-1)^{-1}\left[\alpha^{\left(1-\frac{e^{-\lambda x_i}}{e^{1-e^{-\lambda x_i}}}\right)}-1\right]}{\left(\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{\left(1-\frac{e^{-\lambda x_i}}{e^{1-e^{-\lambda x_i}}}\right)}-1\right)\right)}\right)$$

For, $\phi = (\theta,\alpha,\lambda)$, $x_{(0)} = -\infty$ and $F(x_{(0)};\theta,\alpha,\lambda) = 0$, and $x_{(n+1)} = \infty$, $F(x_{(n+1)};\theta,\alpha,\lambda) = 1$, $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ are ordered sample values.

Maximizing $D(\phi)$, by finding the partial derivatives with respect to each parameter and setting them equal to zero, yields:

$$\frac{\partial D(\phi)}{\partial \theta} = 0, \quad \frac{\partial D(\phi)}{\partial \alpha} = 0, \quad \frac{\partial D(\phi)}{\partial \lambda} = 0$$

Since the resulting system of equations is typically complex and cannot be solved explicitly due to the lack of a closed-form solution, the MPS function is maximized by numerical optimization techniques.

5. Simulation Study

The simulation was conducted using a Monte Carlo approach, which involves repeatedly generating random samples from a specified distribution and evaluating the performance using different estimation methods such as Maximum Likelihood Estimation (MLE), Maximum Product Spacing (MPS), and Least Squares (LS). The random variables were generated using defined quantile function with the sample size of n= 30, 300, 600, 800, and 1000 taken into consideration and each sample size was repeated 1000 times. True parameter values for simulation were set to $\theta = 0.6$, $\alpha = 0.5$, $\lambda = 0.1$ and $\theta = 0.6$, $\alpha = 0.5$, $\lambda = 0.1$. Based on the simulation results, MLE is the preferred method for estimating the parameters of the MOAPTE$_{Ex}$ distribution, particularly for larger sample sizes. MPS may be considered as an alternative for smaller sample sizes or when computational efficiency is a concern.

i. The results confirmed the expected trend: as sample size increased, bias and MSEs for λ, θ, and α decreased.
ii. The parameter α is more sensitivity to sample size, with substantial reductions in bias and MSEs observed for larger samples.
iii. The MLEs demonstrated overall unbiasedness, consistency, and efficiency, indicating their reliability in providing precise parameter estimates for the MOAPTE$_{Ex}$ distribution.
iv. MPS tends to perform better for smaller sample sizes, but its performance deteriorates as the sample size increases. LS shows consistent performance across different sample sizes but is generally outperformed by MLE.

TABLE 4. Simulation results for different methods of parameter estimation (MLE, MPS, and LS

		$\theta = 0.6,\quad \alpha = 0.5,\quad \lambda = 0.1$								
		$\hat{\theta}$			$\hat{\alpha}$			$\hat{\lambda}$		
n	**Methods**	**Bias**	**Variance**	**RMSE**	**Bias**	**Variance**	**RMSE**	**Bias**	**Variance**	**RMSE**
30	MLE	0.3227	1.0781	1.0383	8.7214	7149.1731	84.5528	0.0370	0.0096	0.0982
	MPS	-0.0182	0.5536	0.7441	4.9185	2412.3705	49.1159	-0.0279	0.0060	0.0772
	LS	0.3127	1.0454	1.0224	6.0610	1333.8247	36.5216	0.0212	0.0108	0.1037
300	MLE	0.0632	0.1123	0.3351	0.1225	0.1848	0.4299	-0.0043	0.0007	0.0270
	MPS	-0.0039	0.1158	0.3403	-0.1009	0.1888	0.4345	-0.0303	0.0024	0.0490
	LS	0.0978	0.1391	0.3730	-0.0015	0.1837	0.4286	-0.0135	0.0011	0.0338
600	MLE	0.0347	0.0653	0.2555	0.1157	0.1710	0.4135	-0.0035	0.0004	0.0208
	MPS	-0.0128	0.0842	0.2902	-0.0758	0.1666	0.4082	-0.0243	0.0018	0.0421
	LS	0.0603	0.0754	0.2746	-0.0034	0.1747	0.4180	-0.0137	0.0008	0.0281
800	MLE	0.0420	0.0516	0.2270	0.0917	0.1520	0.3898	-0.0022	0.0003	0.0177
	MPS	0.0031	0.0830	0.2881	-0.0512	0.1546	0.3932	-0.0187	0.0015	0.0384
	LS	0.0587	0.0574	0.2395	-0.0457	0.1590	0.3988	-0.0144	0.0007	0.0267
1000	MLE	0.0424	0.0497	0.2230	0.0813	0.1496	0.3867	-0.0031	0.0002	0.0156
	MPS	0.0136	0.0739	0.2718	-0.0824	0.1412	0.3758	-0.0182	0.0013	0.0367
	LS	0.0533	0.0536	0.2316	-0.0217	0.1609	0.4012	-0.0131	0.0007	0.0264
		$\theta = 0.4,\quad \alpha = 0.6,\quad \lambda = 0.5$								
		$\hat{\theta}$			$\hat{\alpha}$			$\hat{\lambda}$		
n	**Methods**	**Bias**	**Variance**	**RMSE**	**Bias**	**Variance**	**RMSE**	**Bias**	**Variance**	**RMSE**
30	MLE	0.3972	1.0371	1.0184	6.2579	4027.0886	63.4593	0.2480	0.3605	0.6004
	MPS	0.0625	0.3950	0.6285	4.8199	2105.0213	45.8805	-0.1280	0.1837	0.4286
	LS	0.2768	0.7953	0.8918	7.1306	2004.0386	44.7665	0.2048	0.5351	0.7315
300	MLE	0.0707	0.0626	0.2502	0.0493	0.1691	0.4113	-0.0159	0.0237	0.1539
	MPS	0.0269	0.0559	0.2363	-0.1776	0.2109	0.4593	-0.1476	0.0592	0.2434
	LS	0.1015	0.0853	0.2920	-0.0006	2.4671	1.5707	-0.0654	0.0345	0.1857
600	MLE	0.0525	0.0299	0.1728	0.0347	0.1576	0.3970	-0.0228	0.0137	0.1170
	MPS	0.0153	0.0426	0.2065	-0.1592	0.1855	0.4307	-0.1319	0.0483	0.2198
	LS	0.0595	0.0431	0.2075	-0.0636	1.0132	1.0066	-0.0806	0.0234	0.1531
800	MLE	0.0491	0.0275	0.1657	0.0266	0.1505	0.3879	-0.0143	0.0094	0.0968
	MPS	0.0132	0.0368	0.1919	-0.1302	0.1700	0.4123	-0.1000	0.0364	0.1909
	LS	0.0554	0.0329	0.1812	-0.1063	0.1783	0.4222	-0.0780	0.0202	0.1422
1000	MLE	0.0475	0.0269	0.1640	0.0164	0.1469	0.3833	-0.0199	0.0075	0.0869
	MPS	0.0119	0.0385	0.1962	-0.1549	0.1676	0.4094	-0.0984	0.0361	0.1899
	LS	0.0615	0.0329	0.1813	-0.1212	0.1805	0.4248	-0.0810	0.0196	0.1402

6. Applications

The MOAPTE$_{Ex}$ distribution was applied to two data sets in order to compare its performance and the other models, the results shows MOAPTE$_{Ex}$ distribution outperforming other competing models based on all selection criteria and goodness-of-fit tests, making it the most suitable model, the first data set used has been obtained by [22], which represents the strength of single carbon fibers tested at a gauge length of 1mm, this data have been previously used by [4]in their research. The second dataset used by [23]in Comparative analysis of the GAPIE distribution using strengths of glass fibres data. The MOAPTE$_{Ex}$ was compared to Marshall–Olkin exponential distribution [7], exponential (Ex) distributions, Exponentiated Exponential (EEx) [24], Alpha Power Exponential distribution [4] and transmuted generalized exponential (TGEx) [25] which are summarized below.

Function	Equation	Conditions
$f_{MOEx}(x;\theta,\lambda)$	$\frac{\theta\lambda e^{(-\lambda x)}}{\left[1-(1-\theta)e^{(-\lambda x)}\right]^2}$	$\theta,\lambda,x>0$
$f_{GEx}(x;\alpha,\lambda)$	$\alpha\lambda e^{(-\lambda x)}\left[1-e^{(-\lambda x)}\right]^{\alpha-1}$	$\alpha,\lambda>0$
$f_{APEx}(x;\alpha,\lambda,\beta)$	$\frac{\log(\alpha)\lambda e^{(-\lambda x)}\alpha^{1-e^{(-\lambda x)}}}{(\alpha-1)}$	$\alpha,\lambda>0,\ \alpha\neq 1$
$F_{TGEx}(x;\alpha,\lambda,\theta)$	$\alpha\lambda e^{(-\lambda x)}\left[1-e^{(-\lambda x)}\right]^{\alpha-1}\left\{1+\theta-2\theta\left[1-e^{(-\lambda x)}\right]^{\alpha}\right\}$	$\alpha,\lambda>0,\lvert\theta\rvert<1$
$f_{Ex}(x;\lambda)$	$1-e^{(-\lambda x)}$	$x>0,\lambda>0$

6.1. The strength for the single carbon fibers dataset. The skewness value of 0.07602 of single carbon fibers dataset as shown in table 5 and the figure 5 shows the dataset is nearly normally distributed and the distribution of the fiber strength measurements is almost symmetric. Figure 7 show a concave TTT plot which suggests that the dataset follows an increasing failure rate. Table6 and 7 as the results shows MOAPTE$_{Ex}$ distribution as the best fit model for the strength for the single carbon fibers dataset based on various selection criteria and goodness-of-fit assessments. It outperformed other competing models, in terms of log-likelihood values. Additionally, the MOAPTE$_{Ex}$ model demonstrated acceptable K-S p-values and favorable A* and w* statistics, indicating a superior fit to this type of dataset.

TABLE 5. Summary Statistics of the Strength for Single Carbon Fibers

N	Mean	Skew	Kurtosis	Min	Max	1st Quartile	3rd Quartile	CV
56	4.261	0.07602	-0.003403	2.247	6.06	3.728	4.683	0.1927

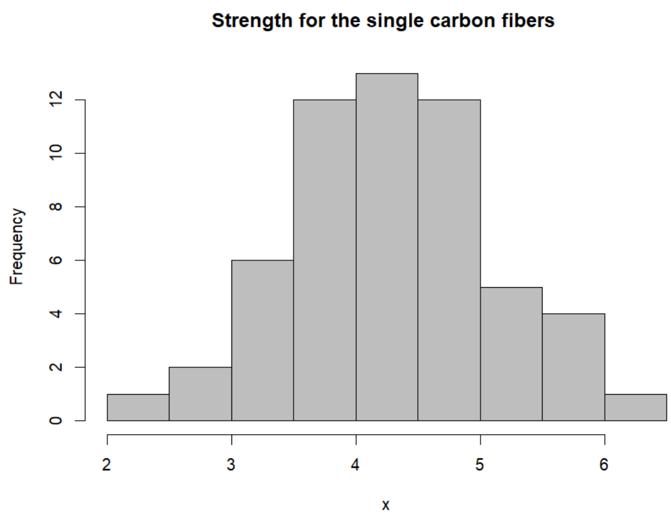


FIGURE 5. Histogram plot of single carbon fibers dataset

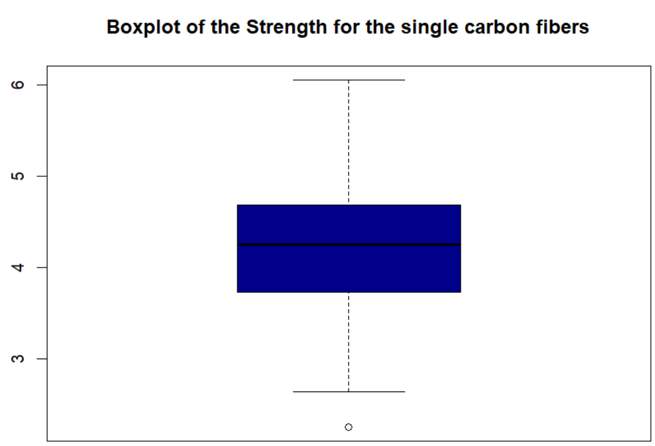


FIGURE 6. Box plot of single carbon fibers dataset

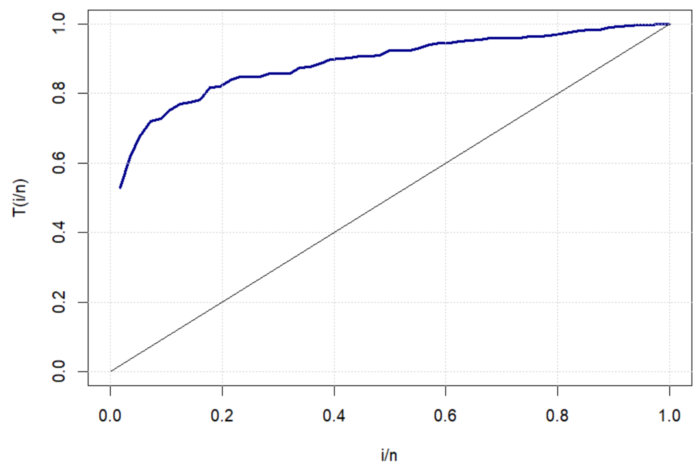


FIGURE 7. TTT plot of single carbon fibers data set

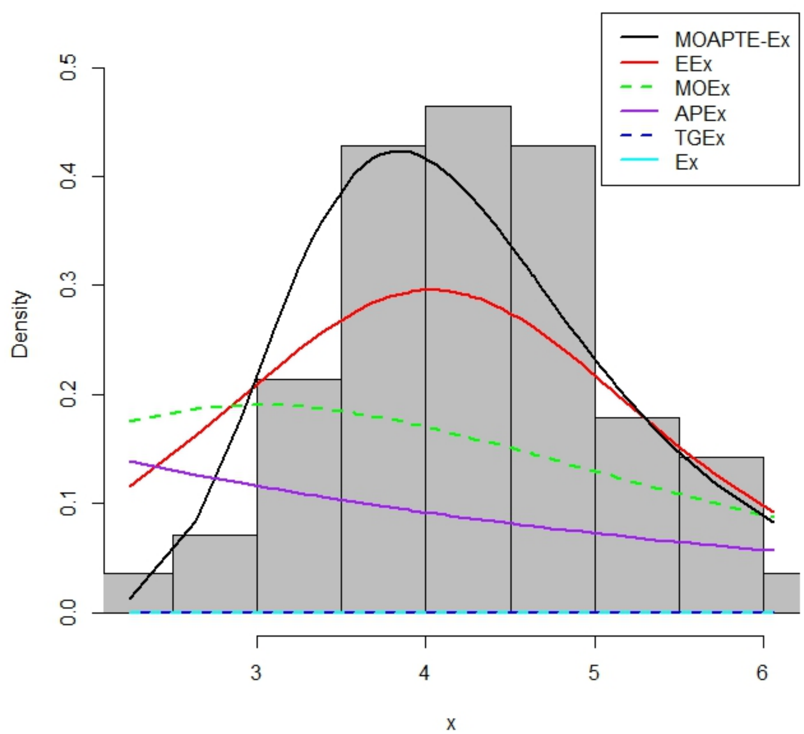


FIGURE 8. The Fitted densities plot for the single carbon fibers dataset

TABLE 6. Values of Selection Criteria for various competing distributions for the single carbon fibers dataset

Model	$\log(l)$	AIC	CAIC	BIC	HQIC
MOAPTE-Ex	**-69.18**	**146.1458**	**146.6073**	**152.2218**	**148.5015**
EEx	-72.25	148.4926	148.7190	152.5433	150.0630
MOE-Ex	-80.64	165.5955	165.8219	169.6462	167.1660
APEx	-105.07	214.1473	214.3738	218.1980	215.7178
TGEx	-129.45	258.895	258.971	259.971	258.251
Exponential	-137.17	276.3327	276.4068	278.3581	277.1179

TABLE 7. Estimates of the parameters and goodness-of-fit tests for the single carbon fibers dataset

Model	Estimates (Std. Error)			K–S (p-value)	A*	w*
	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$			
MOAPTE-Ex	1138.3	489.36	0.2743	**0.085863** (**0.7713**)	**0.15395**	**0.02106**
	(489.36)	(1.0112e-05)	(0.0516)			
EEx	-	80.68967	1.1444	0.1046 (0.5376)	0.1609	0.08144
	-	(28.9522)	(0.0982)			
MOE-Ex	111.3425	-	1.183O5	0.2165 (0.0088)	0.1779	0.1206
	(36.16408416)	-	(0.08417492)			
APEx	-	104.5916	0.5121	0.3269 (7.898e-06)	0.1882	0.2744
	-	(43.9717)	(0.0402)			
TGEx	87.789	127.527	0.0934	0.3942 (7.772e-08)	0.1929	0.32971
	(11.8621)	(21.1235)	(0.0314)			
Ex	-	-	0.2347	0.4633 (1.591e-11)	0.2363	0.52971
	-	-	(0.0314)			

6.2. The glass fibre strengths. The data on 1.5 cm strengths of glass fibres appears to be slightly left-skewed with the skewness value of -0.922 as shown in table 8 and the histogram in figure 9 shows the dataset is skewed to the left. Figure 11 show a concave TTT plot which suggests that the dataset follows an increasing failure rate. Table9 and 10 as the results shows MOAPTE$_{Ex}$ distribution as the best fit model for The glass fibre strengths dataset based on various selection criteria and goodness-of-fit assessments. It outperformed other competing models, in terms of log-likelihood values. Additionally, the MOAPTE$_{Ex}$ model demonstrated acceptable K-S p-values and favorable A* and w* statistics, indicating a superior fit to this type of dataset.

TABLE 8. Summary Statistics of the Strengths of glass Fibres

N	Mean	Skew	Kurtosis	Min	Max	1st Quartile	3rd Quartile	CV
63	1.507	-0.922	1.103	0.55	2.24	1.375	1.685	0.2151

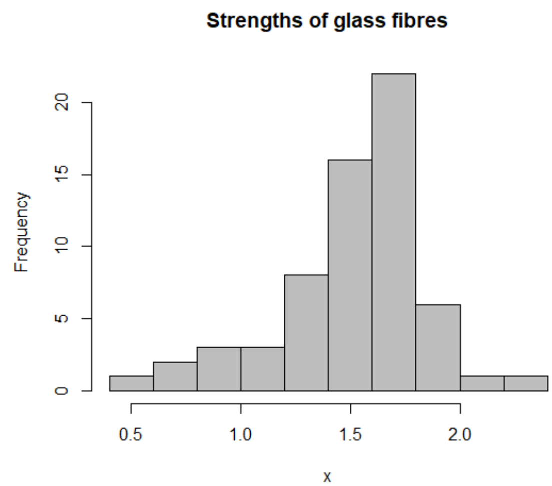


FIGURE 9. Histogram plot of glass fibre strengths dataset

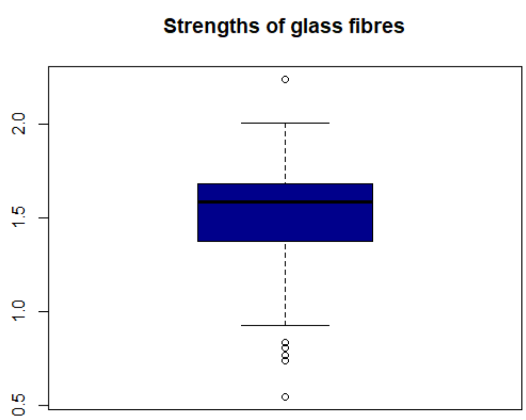


FIGURE 10. Box plot of glass fibre strengths dataset

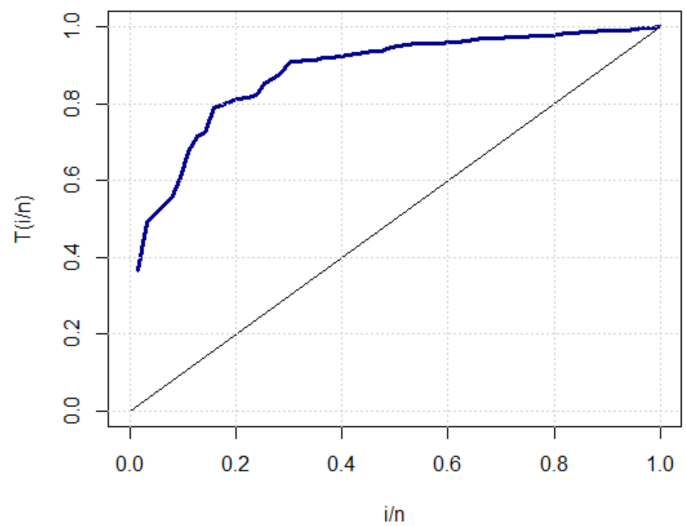


FIGURE 11. TTT plot of single carbon fibers data set

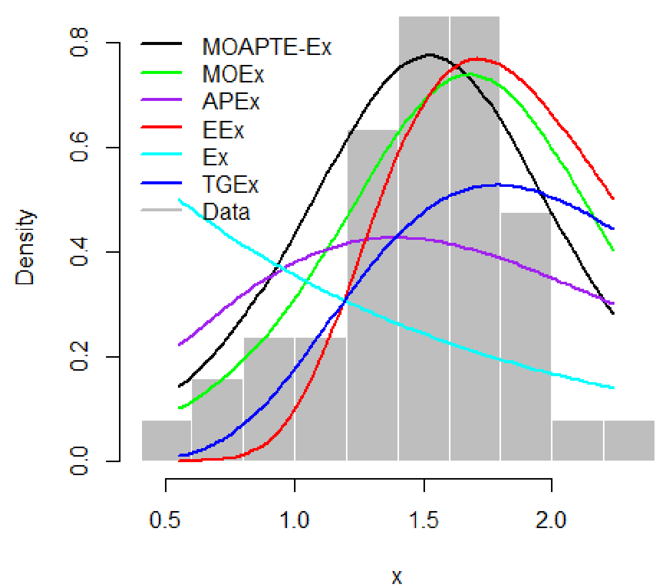


FIGURE 12. The Fitted densities plot for the glass fibre strengths dataset

TABLE 9. Values of Selection Criteria for various competing distributions for the glass fibre strengths dataset

Model	$\log(l)$	AIC	CAIC	BIC	HQIC
MOAPTE-Ex	**-16.65**	**42.2835**	**51.7129**	**48.7129**	**44.8122**
MOE-Ex	-25.97	55.4981	61.7844	59.7844	57.1839
EEx	-31.38	66.7669	73.0532	71.0532	68.4528
APEx	-53.88	111.755	118.0413	116.0413	113.4409
TGEx	-88.83	179.6606	182.8038	181.8038	180.5035
Ex	-95.63	187.257	189.687	188.687	187.786

TABLE 10. Estimates of the parameters and goodness-of-fit tests for the glass fibre strengths dataset

Model	Estimates (Std. Error)			K–S (p-value)	A*	w*
	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$			
MOAPTE-Ex	140.0067	5.40168	3.0574	**0.1043** (**0.4085**)	**0.1187**	**0.09235**
	(108.761)	(33.145)	(1.2135)			
MOE-Ex	139.0027	-	2.9341	0.2901 (0.3805)	0.1264	0.1834
	(55.356)	-	(0.13545)			
EEx	-	33.924	2.0556	0.3264(0.3276)	0.1434	0.2562
	-	(8.7106)	(0.1251)			
APEx	-	140.534	1.1593	0.3945 (0.2017)	0.1839	0.3034
	-	(78.6311)	(0.09564)			
TGEx	0.20667	11.00169	1.2743	0.4734 (0.1156)	0.2229	0.3568
	(0.1356)	(8.7642)	(0.1961)			
Ex	-	-	0.757	0.5213 (2.435e-3)	0.2957	0.4078
	-	-	(0.2479)			

7. CONCLUSION

This article proposed the Marshall-Olkin Alpha Power Transformed Extended Exponential distribution as a robust three-parameter univariate model that enhances the flexibility of exponential distribution with two shape parameters, it serves as a strong alternative in many cases.

Some mathematical properties, including the quantile function, moments, Rényi entropy, skewness, order statistics, and kurtosis, are derived. Parameter estimation is performed using Maximum Likelihood Estimation (MLE), Maximum Product Spacing (MPS), and Least Squares (LS) methods, with MLE demonstrating the most reliability, particularly for larger sample sizes. The MOAPTE_{Ex} was effectively used in two real datasets, consistently outperforming competing models in terms of the evaluation of goodness of fit and the values of the selection criteria. The MOAPTE_{Ex} distribution provides better fit for the assessed datasets based on the numerical results, so its potential use in several fields, including long-term sustainability and reliability research, where hazard rates may be declining, rising, or showing unimodal patterns.

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DATA AVAILABILITY STATEMENT

The authors declare that all data used in this study are as indicated in the manuscript and, where necessary, appropriate citations have been made.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] S. Nadarajah, S. Kotz, The Beta Exponential Distribution, Reliab. Eng. Syst. Saf. 91 (2006), 689–697. https://doi.org/10.1016/j.ress.2005.05.008.

[2] Y. Lan, L.M. Leemis, The Logistic–Exponential Survival Distribution, Naval Res. Logist. 55 (2008), 252–264. https://doi.org/10.1002/nav.20279.

[3] G.M. Cordeiro, M. De Castro, A New Family of Generalized Distributions, J. Stat. Comput. Simul. 81 (2011), 883–898. https://doi.org/10.1080/00949650903530745.

[4] A. Mahdavi, D. Kundu, A New Method for Generating Distributions with an Application to Exponential Distribution, Commun. Stat. Theory Methods. 46 (2017), 6543–6557. https://doi.org/10.1080/03610926.2015.1130839.

[5] M. Kilai, G.A. Waititu, W.A. Kibira, R. Aldallal, M.E. Bakr, Y.A. Tashkandy, F.H. Riad, Information Approach for Change Point Analysis of EGGAPE Distribution and Application to COVID-19 Data, Math. Probl. Eng. 2022 (2022), 9924902. https://doi.org/10.1155/2022/9924902.

[6] M. Kargbo, A.W. Gichuhi, A.K. Wanjoya, A Novel Extended Inverse-exponential Distribution and Its Application to COVID-19 Data, Eng. Rep. 6 (2024), e12828. https://doi.org/10.1002/eng2.12828.

[7] A. Marshall, A New Method for Adding a Parameter to a Family of Distributions with Application to the Exponential and Weibull Families, Biometrika 84 (1997), 641–652. https://doi.org/10.1093/biomet/84.3.641.

[8] G.M. Cordeiro, A.J. Lemonte, On the Marshall–Olkin Extended Weibull Distribution, Stat. Papers 54 (2013), 333–353. https://doi.org/10.1007/s00362-012-0431-8.

[9] E. Krishna, K.K. Jose, T. Alice, M.M. Ristić, The Marshall-Olkin Fréchet Distribution, Commun. Stat. Theory Methods 42 (2013), 4091–4107. https://doi.org/10.1080/03610926.2011.648785.

[10] F. Willayat, N. Saud, M. Ijaz, A. Silvianita, M. El-Morshedy, Marshall–Olkin Extended Gumbel Type-II Distribution: Properties and Applications, Complexity 2022 (2022), 2219570. https://doi.org/10.1155/2022/2219570.

[11] R.P. Tharu, S. Pahari, G.P. Sedhai, G.P. Dhungana, Exponentiated Marshall-Olkin Exponential Distribution: Application of COVID-19 Second Wave in Nepal, Nepal J. Math. Sci. 2 (2021), 43–56. https://doi.org/10.3126/njmathsci.v2i2.40078.

[12] J. Gillariose, L. Tomy, F. Jamal, C. Chesneau, A Discrete Kumaraswamy Marshall-Olkin Exponential Distribution, J. Iran. Stat. Soc. 20 (2021), 129–152. https://doi.org/10.52547/jirss.20.2.129.

[13] M. Nassar, D. Kumar, S. Dey, G.M. Cordeiro, A.Z. Afify, The Marshall–Olkin Alpha Power Family of Distributions with Applications, J. Comput. Appl. Math. 351 (2019), 41–53. https://doi.org/10.1016/j.cam.2018.10.052.

[14] L. Handique, S. Chakraborty, T.A.N. De Andrade, The Exponentiated Generalized Marshall–Olkin Family of Distribution: Its Properties and Applications, Ann. Data Sci. 6 (2019), 391–411. https://doi.org/10.1007/s40745-018-0166-z.

[15] W. Bo, Z. Ahmad, A.R.A. Alanzi, A.I. Al-Omari, E.H. Hafez, S.F. Abdelwahab, The Current COVID-19 Pandemic in China: An Overview and Corona Data Analysis, Alexandria Eng. J. 61 (2022), 1369–1381. https://doi.org/10.1016/j.aej.2021.06.025.

[16] A.L. Bowley, Elements of Statistics, P.S. King, 1926.

[17] J.J.A. Moors, A Quantile Alternative for Kurtosis, J. R. Stat. Soc. Ser. D 37 (1988), 25–32. https://doi.org/10.2307/2348376.

[18] R.C.H. Cheng, N.A.K. Amin, Estimating Parameters in Continuous Univariate Distributions with a Shifted Origin, J. R. Stat. Soc. Ser. B 45 (1983), 394–403. https://doi.org/10.1111/j.2517-6161.1983.tb01268.x.

[19] B. Ranneby, The Maximum Spacing Method. An Estimation Method Related to the Maximum Likelihood Method, Scand. J. Stat. 11 (1984), 93–112. https://www.jstor.org/stable/4615946.

[20] K. Bagci, N. Erdogan, T. Arslan, H.E. Celik, Alpha Power Inverted Kumaraswamy Distribution: Definition, Different Estimation Methods, and Application, Pak. J. Stat. Oper. Res. 18 (2022), 13–25. https://doi.org/10.18187/pjsor.v18i1.3327.

[21] M. Nassar, A.Z. Afify, M. Shakhatreh, Estimation Methods of Alpha Power Exponential Distribution with Applications to Engineering and Medical Data, Pak. J. Stat. Oper. Res. 16 (2020), 149–166. https://doi.org/10.18187/pjsor.v16i1.3129.

[22] M.G. Bader, A.M. Priest, Statistical Aspects of Fibre and Bundle Strength in Hybrid Composites, in: Progress in Science and Engineering Composites, pp. 1129–1136, Tokyo, 1982.

[23] J.T. Eghwerido, S.C. Zelibe, E. Efe-Eyefia, Gompertz-Alpha Power Inverted Exponential Distribution: Properties and Applications, Thailand Statistician 18 (2020), 319–332.

[24] R.D. Gupta, Debasis. Kundu, Generalized Exponential Distribution: Different Method of Estimations, J. Stat. Comput. Simul. 69 (2001), 315–337. https://doi.org/10.1080/00949650108812098.

[25] M.S. Khan, R. King, I.L. Hudson, Transmuted Generalized Exponential Distribution: A Generalization of the Exponential Distribution with Applications to Survival Data, Commun. Stat. Simul. Comput.. 46 (2017), 4377–4398. https://doi.org/10.1080/03610918.2015.1118503.