Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2025, 2025:25
https://doi.org/10.28919/cmbn/9066
ISSN: 2052-2541

A MODIFIED EXPONENTIATED INVERTED WEIBULL DISTRIBUTION USING MODI FAMILY

YVANA MUHIMPUNDU[1,*], LEO ODIWUOR ODONGO[2], ANANDA OMUTOKOH KUBE[2]

[1]Department of Mathematics, Pan African University Institute for Basic Sciences, Technology and Innovation (PAUSTI), Nairobi, Kenya

[2]Department of Mathematics and Actuarial Science, Kenyatta University (KU), Nairobi, Kenya

Abstract. This paper proposes a new extension of the Exponentiated Inverted Weibull distribution using the Modi family, called the Modi Exponentiated Inverted Weibull (MEIW) distribution that adds an extra shape parameter, allowing for a wider range of shapes for failure rates. Mathematical properties were developed, including hazard rate, survival function, reversed hazard rate, quantile function, moments, order statistics, and Rényi Entropy. Maximum Likelihood Estimation is employed for parameter estimation, with the performance of the estimators assessed through Monte Carlo simulation. The new distribution is fitted to the two real data sets and compared with some existing distributions such as Exponentiated Inverted Weibull (EIW), Inverse Weibull (IW), and Weibull (WE) distributions. The goodness-of-fit statistics and information criteria values demonstrated that the new distribution fits better the two real data sets than the other distributions.

Keywords: exponentiated inverted Weibull; Modi family; maximum Likelihood estimation.

2020 AMS Subject Classification: 62N02.

1. Introduction

The Weibull distribution, introduced by Waloddi Weibull in 1939, is one of the most recognized and widely used probability distributions for analyzing lifetime data. It has found

*Corresponding author

E-mail address: yvana.muhimpundu@students.jkuat.ac.ke

Received December 10, 2024

extensive application across various fields, including engineering, hydrology, biology, and economics, to model phenomena such as material strength, product reliability, and failure times [1]. Numerous studies have investigated the properties and applications of the Weibull distribution. [2] focused on parameter estimation techniques relevant to electric breakdown phenomena, emphasizing both graphical methods and Maximum Likelihood Estimation (MLE).

Despite its popularity, the Weibull distribution has limitations in accurately modeling certain types of data, particularly in contexts like wind speed analysis [3]. To address these shortcomings, researchers have proposed new distributions that offer greater flexibility in modeling hazard rate shapes. One of the extensions is the Exponentiated Weibull distribution, introduced by [4], which accommodates non-monotonic failure rates. The Inverse Weibull distribution was also developed to handle increasing failure rates, with applications in reliability analysis, medical sciences, and mechanical component degradation [5]. However, the Inverse Weibull distribution often struggles to capture more complex hazard rate patterns, such as those that exhibit bathtub curves.

Generalizations of the Inverse Weibull distribution have emerged as a way to overcome these limitations. For instance, the Generalized Inverse Weibull (GIW) distribution, introduced by [6], offers unimodal and decreasing failure rates. The Modified Inverse Weibull (MIWD) distribution enhances the flexibility of failure rate patterns [7]. The Kumaraswamy Modified Inverse Weibull [8] and the Marshall-Olkin Extended Inverse Weibull [9], both of which have been used in modeling complex failure rate patterns. In particular, the Marshall-Olkin Extended Inverse Weibull has demonstrated the capability to model wind speed data more efficiently than traditional Weibull and Inverse Weibull distributions.

Further developments include the Exponentiated Inverted Weibull distribution [10] and the Weibull Exponentiated Inverted Weibull distribution [11]. These advancements illustrate the ongoing evolution of Inverted Weibull models, enhancing their applicability and accuracy in various domains.

Researchers have explored various properties of these distributions, including moments, moment-generating functions, and order statistics. Parameter estimation has been conducted using both maximum likelihood and Bayesian methods.

A key focus in the literature has been on developing more flexible distributions that can be applied to a broader range of data. Many of these models achieve this flexibility by introducing additional parameters, allowing them to capture more complex failure rate behaviors. However, many remain limited to specific hazard rate patterns, such as unimodal or decreasing rates, which can be restrictive when dealing with more complex datasets, particularly those exhibiting increasing or bathtub-shaped hazard rates.

Thus, this paper proposes a new distribution based on the Modi family of distributions, envisaged to offer greater flexibility than the Exponentiated Inverted Weibull distribution. The newly proposed model has been found to capture both monotonic (increasing and decreasing) and non-monotonic hazard rate shapes, that the standard Exponentiated Inverse Weibull distribution cannot overcome.

This paper is organized as follows: Section 2 discusses the Modi family and the baseline distribution, while Section 3 introduces the Modi Exponentiated Inverted Weibull distribution and its cumulative distribution function (CDF), probability density function (PDF), hazard rate function, survival function, cumulative hazard function, reversed hazard rate function, and odds function. Statistical and mathematical properties include quantile function, Skewness and kurtosis, Moments, order statistics, and Rényi Entropy of the proposed distribution are also derived as shown in section 3. The parameter estimation process for the Modi Exponentiated Inverted Weibull distribution is detailed in Section 4. Section 5 exhibits a Monte Carlo simulation to evaluate the performance of the proposed distribution vis-a-vis the traditional models. Section 6 demonstrates the model's goodness-of-fit using two real-world data sets.

2. Modi Family and Baseline Distribution

2.1. Modi Family. The Modi family of distributions was developed and explored by [12], It is adaptable and may be used to model data from a variety of phenomena in several disciplines, such as engineering, economics, and finance.

The cumulative distribution function (CDF) of the Modi generator is defined as :

$$F(x) = \frac{(1+\alpha^{\beta})S(x)}{\alpha^{\beta}+S(x)} \tag{1}$$

while the associated probability density function $f(x)$ is given by:

$$f(x) = \frac{(1+\alpha^{\beta})(\alpha^{\beta} s(x))}{(\alpha^{\beta}+S(x))^2} \tag{2}$$

For all $x > 0$, $\alpha > 0$, and $\beta > 0$, where $S(x)$ and $s(x)$ are respectively the cumulative distribution function (CDF) and the probability density function (PDF) of the distribution to be modified.

2.2. Exponentiated Inverted Weibull distribution. According to [10]. A random variable X has Exponentiated Inverted Weibull (EIW) distribution with two shape parameters if its cumulative distribution function (CDF) is given by:

$$F(x,\delta,\theta) = \left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta} \tag{3}$$

for all $x, \theta, \delta > 0$

Consequently the probability density function $f(x)$ is given by:

$$f(x,\delta,\theta) = \theta\delta x^{-(\delta+1)}\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta} \tag{4}$$

for all $x, \theta, \delta > 0$

The following additional functions are associated with the Exponentiated Inverted Weibull distribution:

Survival function:

$$\begin{aligned} S(x,\delta,\theta) &= 1 - F(x,\delta,\theta), \\ &= 1 - \left(e^{-x^{-\delta}}\right)^{\theta} \end{aligned} \tag{5}$$

Hazard rate function:

$$\begin{aligned} h(x,\delta,\theta) &= \frac{f(x,\delta,\theta)}{1-F(x,\delta,\theta)}, \\ &= \frac{\theta\delta x^{-(\delta+1)}\left(e^{-x^{-\delta}}\right)^{\theta}}{1-\left(e^{-x^{-\delta}}\right)^{\theta}} \end{aligned} \tag{6}$$

Cumulative hazard function:

$$\begin{aligned} H(x,\delta,\theta) &= -\log S(x,\delta,\theta), \\ &= -\log\left[1-\left(e^{-x^{-\delta}}\right)^{\theta}\right] \end{aligned} \tag{7}$$

Reverse hazard rate function:

$$
\begin{aligned}
r(x,\delta,\theta) &= \frac{f(x,\delta,\theta)}{F(x,\delta,\theta)} \\
&= \frac{\theta\delta x^{-(\delta+1)}\left(e^{-x^{-\delta}}\right)^{\theta}}{\left(e^{-x^{-\delta}}\right)^{\theta}} \\
&= \theta\delta x^{-(\delta+1)}
\end{aligned} \tag{8}
$$

odds function:

$$
\begin{aligned}
O(x,\delta,\theta) &= \frac{F(x,\delta,\theta)}{S(x,\delta,\theta)} \\
&= \frac{\left(e^{-x^{-\delta}}\right)^{\theta}}{1-\left(e^{-x^{-\delta}}\right)^{\theta}}
\end{aligned} \tag{9}
$$

3. Modi Exponentiated Inverted Weibull Distribution and Its Properties

In practice, a common approach to modifying a distribution is adding a parameter to increase flexibility. This technique allows for modeling a wider variety of data types, which can be especially useful in data analysis [13].

3.1. Cumulative Distribution Function and Probability Density Function.

From equation (1) and (3), the cumulative distribution function of the Modi Exponentiated Inverted Weibull distribution is defined as:

$$
F(x,\alpha,\beta,\theta,\delta) = \frac{(1+\alpha^{\beta})\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}}{\alpha^{\beta}+\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}} \tag{10}
$$

and its probability density function(PDF) is obtained from equation (10) as follows:

$$
\begin{aligned}
f(x,\alpha,\beta,\theta,\delta) &= \frac{\partial}{\partial x}F(x,\alpha,\beta,\theta,\delta) \\
&= \frac{\partial}{\partial x}\frac{(1+\alpha^{\beta})(e^{-x^{-\delta}})^{\theta}}{\alpha^{\beta}+(e^{-x^{-\delta}})^{\theta}} \\
&= \frac{\theta\delta\alpha^{\beta}x^{-(\delta+1)}\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}(1+\alpha^{\beta})}{\left[\alpha^{\beta}+\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}\right]^{2}}
\end{aligned} \tag{11}
$$

for all $x>0$, $\alpha>0$, $\beta>0$, $\theta>0$, and $\delta>0$.

3.2. Survival Function and Hazard Rate Function. From equation (10), the survival function for Modi Exponentiated Inverted Weibull distribution is given as:

$$S(x,\alpha,\beta,\theta,\delta) = 1-F(x,\alpha,\beta,\theta,\delta)$$
$$= 1-\left[\frac{(1+\alpha^{\beta})(e^{-x^{-\delta}})^{\theta}}{\alpha^{\beta}+(e^{-x^{-\delta}})^{\theta}}\right] \tag{12}$$
$$= \frac{\alpha^{\beta}\left[1-(e^{-x^{-\delta}})^{\theta}\right]}{\left[\alpha^{\beta}+(e^{-x^{-\delta}})^{\theta}\right]}$$

The hazard rate function is derived from equation(11) and (12)

$$h(x,\alpha,\beta,\theta,\delta) = \frac{f(x,\alpha,\beta,\theta,\delta)}{1-F(x,\alpha,\beta,\theta,\delta)}$$
$$= \frac{f(x,\alpha,\beta,\theta,\delta)}{S(x,\alpha,\beta,\theta,\delta)} \tag{13}$$
$$= \frac{\theta\delta x^{-(\delta+1)}\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}(1+\alpha^{\beta})}{\left[\alpha^{\beta}+\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}\right]\left[1-(e^{-x^{-\delta}})^{\theta}\right]}$$

3.3. Some Useful Functions of MEIW distribution. Additional functions related to the MEIW distribution, such as the cumulative hazard function, the reversed hazard rate function calculated as the ratio of the PDF to the CDF, and the odds function defined as the ratio of the CDF to the survival function are provided below:

The Cumulative hazard function:

$$H(x,\alpha,\beta,\theta,\delta) = -\log S(x,\alpha,\beta,\theta,\delta)$$
$$= -\log\frac{\alpha^{\beta}[1-(e^{-x^{-\delta}})^{\theta}]}{\left[\alpha^{\beta}+(e^{-x^{-\delta}})^{\theta}\right]} \tag{14}$$

The reversed hazard rate function:

$$r(x,\alpha,\beta,\theta,\delta) = \frac{f(x,\alpha,\beta,\theta,\delta)}{F(x,\alpha,\beta,\theta,\delta)}$$
$$= \frac{\theta\delta\alpha^{\beta}x^{-(\delta+1)}}{\left[\alpha^{\beta}+\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}\right]} \tag{15}$$

Odds function:

$$O(x,\alpha,\beta,\theta,\delta)=\frac{F(x,\alpha,\beta,\theta,\delta)}{S(x,\alpha,\beta,\theta,\delta)}=\frac{(1+\alpha^{\beta})(e^{-x^{-\delta}})^{\theta}}{\alpha^{\beta}\left[1-(e^{-x^{-\delta}})^{\theta}\right]} \tag{16}$$

Figures 1 and 2 illustrate a range of possible shapes of the MEIW probability density function (PDF) and hazard rate function (HRF) for different parameter settings. The MEIW PDF may display right skewness or symmetry, while the MEIW HRF can exhibit decreasing, increasing, or reversed J-shaped.

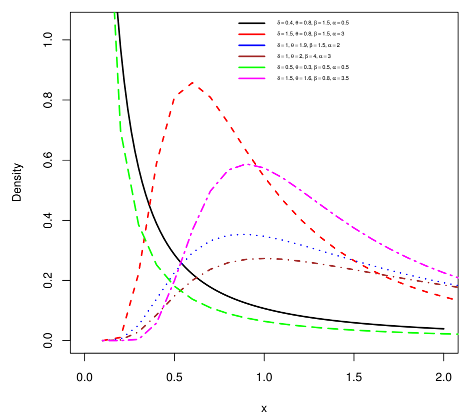

FIGURE 1. Plot of PDF of the MEIW for various values of $\alpha,\beta,\theta,\delta$

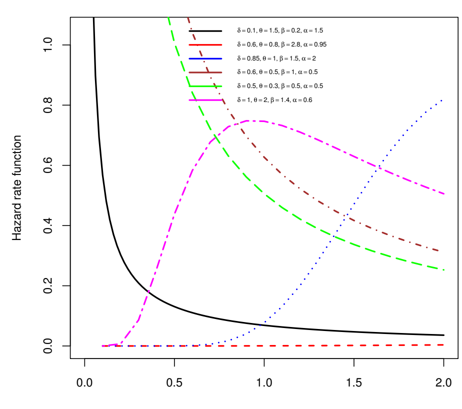

FIGURE 2. Plot of the Hazard rate function of MEIW distribution for some various values of $\alpha,\beta,\theta,\delta$

3.4. Statistical and Mathematical Properties of Modi Exponentiated Inverted Weibull Distribution.

3.4.1. *Quantile Function.*

The quantile function is crucial in simulating random samples from a specified distribution. Additionally, it serves to describe key distribution characteristics, such as the median, skewness, and kurtosis.

Let X be a random variable such that $X \sim \text{MEIW}(\alpha,\beta,\theta,\delta)$. Then the quantile function $Q(u)$ is given by:

$$Q(u) = x_u = F^{-1}(u;\alpha,\beta,\theta,\delta) = \left[-\log\left(\frac{u\alpha^{\beta}}{(1+\alpha^{\beta})-u}\right)^{\frac{1}{\theta}}\right]^{-\frac{1}{\delta}}, \tag{17}$$

where $F^{-1}(\cdot)$ denotes the inverse of the cumulative distribution function (CDF) and $u \sim U(0,1)$ is a random variable uniformly distributed on the interval $(0,1)$.

Proof: The quantile function is derived from the inverse of the Cumulative Distribution Function (CDF) of the Modi Exponentiated Inverted Weibull distribution, defined in equation (10).

Let $F(x;\alpha,\beta,\theta,\delta)$=$u$

$$\frac{(1+\alpha^{\beta})(e^{-x^{-\delta}})^{\theta}}{\alpha^{\beta}+(e^{-x^{-\delta}})^{\theta}} = u$$

$$(1+\alpha^{\beta})(e^{-x^{-\delta}})^{\theta} = u[\alpha^{\beta}+(e^{-x^{-\delta}})^{\theta}]$$

$$(1+\alpha^{\beta})(e^{-x^{-\delta}})^{\theta} = u\alpha^{\beta}+u(e^{-x^{-\delta}})^{\theta}$$

$$(1+\alpha^{\beta})(e^{-x^{-\delta}})^{\theta} - u(e^{-x^{-\delta}})^{\theta} = u\alpha^{\beta}$$

$$(e^{-x^{-\delta}})^{\theta}[(1+\alpha^{\beta})-u] = u\alpha^{\beta}$$

$$(e^{-x^{-\delta}})^{\theta} = \frac{u\alpha^{\beta}}{(1+\alpha^{\beta})-u}$$

$$(e^{-x^{-\delta}}) = \left(\frac{u\alpha^{\beta}}{(1+\alpha^{\beta})-u}\right)^{\frac{1}{\theta}}$$

$$x^{-\delta} = -\log\left(\frac{u\alpha^{\beta}}{(1+\alpha^{\beta})-u}\right)^{\frac{1}{\theta}}$$

$$Q(u) = x_u = F^{-1}(u;\alpha,\beta,\theta,\delta) = \left[-\log\left(\frac{u\alpha^{\beta}}{(1+\alpha^{\beta})-u}\right)^{\frac{1}{\theta}}\right]^{-\frac{1}{\delta}}$$

By applying the quantile function and substituting u by 1/4, 1/2, and 3/4, one can obtain the values of the lower quartile, median, and upper quartile, respectively.

The lower quartile:

$$Q\left(\frac{1}{4}\right) = \left[-\log\left(\frac{\alpha^\beta}{3+4\alpha^\beta}\right)^{\frac{1}{\theta}}\right]^{-\frac{1}{\delta}} \tag{18}$$

The median:

$$Q\left(\frac{1}{2}\right) = \left[-\log\left(\frac{\alpha^\beta}{1+2\alpha^\beta}\right)^{\frac{1}{\theta}}\right]^{-\frac{1}{\delta}} \tag{19}$$

The upper quartile is:

$$Q\left(\frac{3}{4}\right) = \left[-\log\left(\frac{3\alpha^\beta}{1+4\alpha^\beta}\right)^{\frac{1}{\theta}}\right]^{-\frac{1}{\delta}} \tag{20}$$

3.4.2. *Skewness and Kurtosis.* Galton Skewness and Moors Kurtosis of MEIW distribution are defined by:

$$S = \frac{Q\left(\frac{3}{4}\right) + Q\left(\frac{1}{4}\right) - 2Q\left(\frac{1}{2}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \tag{21}$$

$$K = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \tag{22}$$

where the value of the quartile is indicated by $Q(.)$.

3.4.3. *The rth Moments of Modi Exponentiated Inverted Weibull Distribution.* Moments play a crucial role as they allow for calculating key characteristics and properties of a probability distribution, including the mean, variance, skewness, and kurtosis.

A random variable where $x \sim MEIW$ distribution, the r^{th} moment is given by:

$$\mu_r' = E(x^r) = -\theta\alpha^\beta\left(1+\alpha^\beta\right)\sum_{k=0}^{\infty}\binom{-2}{k}(\alpha^\beta)^{-2-k}\frac{\Gamma\left(\frac{-r+\delta}{\delta}\right)}{(\theta+\theta k)^{\left(\frac{-r+\delta}{\delta}\right)}} \tag{23}$$

Proof: The mathematical expression for the r^{th} moment of the MEIW distribution given by:

$$\mu_r' = E(x^r) = \int_0^\infty x^r f(x;\alpha,\beta,\theta,\delta)\,dx \tag{24}$$

where f(x;$\alpha,\beta,\theta,\delta$) is the pdf of the distribution.

By substituting (11) in (24), we get

$$E(x^r) = \int_0^\infty x^r \frac{\theta\delta\alpha^\beta x^{-(\delta+1)}\left(e^{-x^{-\delta}}\right)^\theta\left(1+\alpha^\beta\right)}{\left[\alpha^\beta+\left(e^{-x^{-\delta}}\right)^\theta\right]^2}\,dx$$

$$= \theta\delta\alpha^\beta\left(1+\alpha^\beta\right)\int_0^\infty x^{r-\delta-1}\left(e^{-x^{-\delta}}\right)^\theta\left[\alpha^\beta+\left(e^{-x^{-\delta}}\right)^\theta\right]^{-2}\,dx$$

By using the binomial expansion

$$E(x^r) = \theta\delta\alpha^\beta\left(1+\alpha^\beta\right)\sum_{k=0}^\infty\binom{-2}{k}(\alpha^\beta)^{-2-k}\int_0^\infty x^{r-\delta-1}\left(e^{-x^{-\delta}}\right)^{\theta+\theta k}\,dx \tag{25}$$

To evaluate the integral $\int_0^\infty x^{r-\delta-1}\left(e^{-x^{-\delta}}\right)^{\theta+\theta k}dx$, where r, δ, θ, and k are constants.

$$\int_0^\infty x^{r-\delta-1}\left(e^{-x^{-\delta}}\right)^{\theta+\theta k}dx = \int_0^\infty u^{\frac{-r+\delta+1}{\delta}}\left(e^{-u}\right)^{\theta+\theta k}\left(-\frac{1}{\delta}u^{-1-\frac{1}{\delta}}\right)du$$

$$\int_0^\infty x^{r-\delta-1}\left(e^{-x^{-\delta}}\right)^{\theta+\theta k}dx = -\frac{1}{\delta}\int_0^\infty u^{(-r/\delta)}(e^{-u})^{\theta+\theta k}du$$

By using gamma function the integral becomes:

$$\frac{\Gamma\alpha}{q^\alpha} = \int_0^\infty u^{\alpha-1}e^{-qu}du$$

where $\alpha = (-r/\delta+1)/$ and $q = (1+k)\theta$, get the result

$$\int_0^\infty x^{r-\delta-1}\left(e^{-x^{-\delta}}\right)^{\theta+\theta k}dx = -\frac{1}{\delta}\cdot\frac{\Gamma\left(\frac{-r+\delta}{\delta}\right)}{(\theta+\theta k)^{\left(\frac{-r+\delta}{\delta}\right)}} \tag{26}$$

By combining with equation (25) and (26), we get

$$= -\theta\alpha^\beta\left(1+\alpha^\beta\right)\sum_{k=0}^\infty\binom{-2}{k}(\alpha^\beta)^{-2-k}\frac{\Gamma\left(\frac{-r+\delta}{\delta}\right)}{(\theta+\theta k)^{\left(\frac{-r+\delta}{\delta}\right)}} \tag{27}$$

Table 1 presents the value of the quantile function and Table 2 displays values of the statistical properties of the MEIW distribution for various parameter sets. It reveals how these parameters influence the central tendency (mean), spread (standard deviation and coefficient of variation), and shape (skewness and kurtosis) of the MEIW distribution. The coefficient of skewness (CS) values indicate that distribution can exhibit right skewness, left skewness, and nearly symmetrical while the coefficient of Kurtosis(CK) suggests that the distribution can be leptokurtic, mesokurtic, and platykurtic.

Table 1: Quantiles of the MEIW distribution for some parameter values.

	I	II	III	IV	V
Parameters	(0.6, 0.7, 0.5, 1.5)	(0.8, 0.7, 0.5, 1.5)	(0.6, 0.9, 0.5, 1.5)	(0.6, 0.7, 2.0, 1.5)	(0.6, 0.7, 0.5, 2.5)
0.1	0.04289860	0.05043890	0.07091319	0.4550956	0.03497424
0.2	0.07187697	0.08798341	0.11885597	0.5178220	0.05584017
0.3	0.10874485	0.13773121	0.17976977	0.5742592	0.08099235
0.4	0.16086638	0.21063036	0.26592171	0.6333093	0.11499087
0.5	0.24181114	0.32756888	0.39973284	0.7012598	0.16557358
0.6	0.38254812	0.53739001	0.63237908	0.7864637	0.24980698
0.7	0.67073073	0.98081665	1.10876911	0.9049933	0.41467581
0.8	1.45134103	2.22179455	2.39916052	1.0975965	0.83980822
0.9	5.43572129	8.78485508	8.98557753	1.5269009	2.88919956

TABLE 2. First five moments, skewness, and kurtosis of the MEIW distribution across various parameter values.

	(0.3, 0.5, 1.53, 0.2)	**(2.5, 0.5, 1.53, 0.2)**	**(0.3, 2.5, 1.53, 0.2)**	**(0.3, 0.5, 2.95, 0.2)**	**(0.3, 0.5, 1.53, 1.09)**
M1	0.403893080	0.39693344	0.13902131	0.5439761	0.40010721
M2	0.239531851	0.24248416	0.11703517	0.3952054	0.21105970
M3	0.158565057	0.16428139	0.10040203	0.2975646	0.12729172
M4	0.114279772	0.12044007	0.08751231	0.2314681	0.08569205
M5	0.087747869	0.09362082	0.07730885	0.1853670	0.06273140
SD	0.276409535	0.29142409	0.31258318	0.3151116	0.22577406
CV	0.684363138	0.73418879	2.24845505	0.5792747	0.56428390
CS	0.004883662	0.02463796	1.86512597	-0.8132627	0.17857458
CK	2.179273584	1.99124224	4.62256971	2.3306431	3.00741144

3.4.4. *Order Statistics.* Let x_1, x_2,... x_n be a random sample drawn from a probability density function, and define k^{th} order statistics. $X(1)$ denotes the minimum value, the smallest observation, while $X(n)$ represents the maximum value, the largest observation.

The probability density function (PDF) for the k^{th} order statistic, where $1 \leq k \leq n$ is expressed as follows:

$$f(k;n)(x) = \frac{n!}{(k-1)!(n-k)!} f(x) F^{(k-1)}(x)(1-F(x))^{(n-k)} \tag{28}$$

by substituting Equations (10) and (11) in (28), the expression for the probability density function (PDF) of the k^{th} order statistic for MEIW is as follows:

$$\begin{aligned} f_{(k:n)}(x) =& \frac{n!\theta\delta\alpha^{\beta}x^{-(\alpha+1)}\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}\left(1+\alpha^{\beta}\right)}{(k-1)!(n-k)![\alpha^{\beta}+\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}]^{2}} \left(\frac{(1+\alpha^{\beta})\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}}{[\alpha^{\beta}+\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}]}\right)^{k-1} \times \\ & \left[1-\frac{(1+\alpha^{\beta})\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}}{[\alpha^{\beta}+\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}]}\right]^{n-k} \\ =& \frac{n!\theta\delta\alpha^{\beta}x^{-(\alpha+1)}}{(k-1)!(n-k)![\alpha^{\beta}+\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}]} \left(\frac{(1+\alpha^{\beta})\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}}{\alpha^{\beta}+[1-\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}]}\right)^{k} \times \\ & \left(\frac{\alpha^{\beta}+[1-\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}]}{\alpha^{\beta}+\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}}\right)^{n} \end{aligned} \tag{29}$$

The probability density function (PDF) attains its minimum for the smallest order statistic in the MEIW distribution when k equals 1. This can be represented as:

$$f_{(1:n)}(x) = \frac{n\theta\delta x^{-(\alpha+1)}}{\alpha^{\beta}+\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}} \times \frac{(1+\alpha^{\beta})\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}}{[1-\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}]} \times \left(\frac{\alpha^{\beta}[1-\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}]}{\alpha^{\beta}+\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}]}\right)^{n} \tag{30}$$

The probability density function (PDF) attains its maximum for the largest order statistic in the MEIW distribution when k equals n. This can be represented as:

$$f_{(n:n)}(x) = \frac{n\theta\delta\alpha^{\beta}x^{-(\alpha+1)}}{\alpha^{\beta}+\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}} \times \left(\frac{(1+\alpha^{\beta})\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}}{\alpha^{\beta}+\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta}}\right)^{n} \tag{31}$$

3.4.5. *Rényi Entropy of Modi Exponentiated Inverted Weibull distribution.* If X is a continuous random variable with a probability density function $f(x)$, the Rényi entropy of order α is defined as:

$$R_{\alpha}(x) = \frac{1}{1-\alpha}\log\left(\int_{0}^{\infty} f(x)^{\alpha}\,dx\right) \tag{32}$$

with $\alpha > 0$ and $\alpha \neq 1$ and $f(x)$ is the pdf of the distribution. Then by substituting the pdf of MEIW distribution given in Equation (11) into Equation (32), we have

$$\begin{aligned} R_\alpha(x) =& \frac{1}{1-\alpha}\log\int_0^\infty \left(\frac{\theta\delta\alpha^\beta x^{-(\delta+1)}\left(\mathrm{e}^{-x^{-\delta}}\right)^\theta(1+\alpha^\beta)}{\left[\alpha^\beta+\left(\mathrm{e}^{-x^{-\delta}}\right)^\theta\right]^2}\right)^\alpha dx \\ =& \frac{1}{1-\alpha}\log\left(\left(\theta\delta\alpha^\beta\right)^\alpha\left(1+\alpha^\beta\right)^\alpha\int_0^\infty\left(x^{-(\delta+1)}\left(e^{-x^{-\delta}}\right)^\theta\left[\alpha^\beta+\left(e^{-x^{-\delta}}\right)^\theta\right]^{-2}\right)^\alpha dx\right) \end{aligned} \tag{33}$$

By using the binomial expansion

$$\left[\alpha^\beta+\left(e^{-x^{-\delta}}\right)^\theta\right]^{-2}=\sum_{k=0}^\infty\binom{-2}{k}(\alpha^\beta)^{-2-k}\left(e^{-x^{-\delta}}\right)^{\theta k}$$

substituting in the integral

$$R_\alpha(x)=\frac{1}{1-\alpha}\log\left(\left[\theta\delta\alpha^\beta\left(1+\alpha^\beta\right)\sum_{k=0}^\infty\binom{-2}{k}(\alpha^\beta)^{-2-k}\right]^\alpha\int_0^\infty\left[x^{-(\delta+1)}\left(e^{-x^{-\delta}}\right)^{\theta+\theta k}\right]^\alpha dx\right) \tag{34}$$

Let's make the substitution $u=x^{-\delta}$, so $du=-\delta x^{-\delta-1}dx$. Therefore:

$$dx=-\frac{du}{\delta u^{\frac{\delta+1}{\delta}}}$$

Since

$$x=u^{-\frac{1}{\delta}},\ \text{then}\ x^{-\alpha(\delta+1)}=u^{\frac{\alpha(\delta+1)}{\delta}}$$

$$R_\alpha(x)=\frac{1}{1-\alpha}\log\left(\left[\theta\delta\alpha^\beta\left(1+\alpha^\beta\right)\sum_{k=0}^\infty\binom{-2}{k}(\alpha^\beta)^{-2-k}\right]^\alpha\frac{1}{\delta}\int_0^\infty u^{\frac{\alpha-1}{\delta}(\delta+1)}e^{-\alpha u(\theta+\theta k)}du\right) \tag{35}$$

by using the Gamma function:

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,dt \tag{36}$$

$$z=\frac{\alpha-1}{\delta}(\delta+1)+1$$

$$t=\alpha(\theta+\theta k)u$$

Substitute $u=\frac{t}{\alpha(\theta+\theta k)}$ and $du=\frac{dt}{\alpha(\theta+\theta k)}$ into (35)

$$\begin{aligned} R_\alpha(x)=&\frac{1}{1-\alpha}\log\left(\left[\theta\delta\alpha^\beta\left(1+\alpha^\beta\right)\sum_{k=0}^\infty\binom{-2}{k}(\alpha^\beta)^{-2-k}\right]^\alpha\frac{1}{\delta}\left(\frac{1}{\alpha(\theta+\theta k)}\right)^{\frac{\alpha-1}{\delta}(\delta+1)+1}\int_0^\infty t^{\frac{\alpha-1}{\delta}(\delta+1)}e^{-t}\,dt\right) \\ R_\alpha(x)=&\frac{1}{1-\alpha}\log\left(\left[\theta\delta\alpha^\beta\left(1+\alpha^\beta\right)\sum_{k=0}^\infty\binom{-2}{k}(\alpha^\beta)^{-2-k}\right]^\alpha\cdot\frac{1}{\delta}\left(\frac{1}{\alpha(\theta+\theta k)}\right)^{\frac{\alpha-1}{\delta}(\delta+1)+1}\Gamma\left(\frac{\alpha-1}{\delta}(\delta+1)+1\right)\right) \end{aligned} \tag{37}$$

4. Maximum Likelihood Estimation (MLE)

This section discusses estimating the parameters of MEIW distribution by using the approach called the maximum likelihood method.

Consider a random sample x_1, x_2,... x_n drawn from the MEIW distribution. The corresponding log-likelihood function is given by:

$$l = \log L = \sum_{i=1}^{n} f(x_i, \alpha, \theta, \delta, \beta) = \sum_{i=1}^{n} \log \left(\frac{\theta \delta \alpha^{\beta} x_i^{-(\delta+1)} \left(e^{-x_i^{-\delta}}\right)^{\theta} \left(1+\alpha^{\beta}\right)}{\left[\alpha^{\beta} + \left(e^{-x_i^{-\delta}}\right)^{\theta}\right]^2} \right) \tag{38}$$

$$l = \log L = n\log\theta + n\log\delta + n\log\alpha^{\beta} - (\delta+1)\sum_{i=1}^{n}\log(x_i) + \theta\sum_{i=1}^{n}\log\left(e^{-x_i^{-\delta}}\right)$$
$$+ n\log(1+\alpha^{\beta}) - \sum_{i=1}^{n}\log[\alpha^{\beta} + \left(e^{-x_i^{-\delta}}\right)^{\theta}]^2$$

(39) Hence,

$$\log L = n\log\theta + n\log\delta + n\beta\log\alpha - (\delta+1)\sum_{i=1}^{n}\log(x_i) + \theta\sum_{i=1}^{n}\log\left(e^{-x_i^{-\delta}}\right) +$$
$$n\log(1+\alpha^{\beta}) - 2\sum_{i=1}^{n}\log[\alpha^{\beta} + \left(e^{-x_i^{-\delta}}\right)^{\theta}]$$

By taking partial derivatives of equation (39) with respect to each parameter and setting them equal to zero gives

$$\frac{\partial}{\partial\alpha}(\log L) = \frac{n\beta}{\alpha} + \frac{n\beta\alpha^{\beta-1}}{1+\alpha^{\beta}} - 2\sum_{i=1}^{n}\frac{\beta\alpha^{\beta-1}}{\alpha^{\beta} + \left(e^{-x_i^{-\delta}}\right)^{\theta}} = 0 \tag{40}$$

$$\frac{\partial}{\partial\beta}(\log L) = n\log\alpha + \frac{n\alpha^{\beta}\log\alpha}{1+\alpha^{\beta}} - 2\sum_{i=1}^{n}\frac{\alpha^{\beta}\log\alpha}{\alpha^{\beta} + \left(e^{-x_i^{-\delta}}\right)^{\theta}} = 0 \tag{41}$$

$$\frac{\partial}{\partial\theta}(\log L) = \frac{n}{\theta} + \sum_{i=1}^{n}\log\left(e^{-x_i^{-\delta}}\right) - 2\sum_{i=1}^{n}\frac{\left(e^{-x_i^{-\delta}}\right)^{\theta}\log\left(e^{-x_i^{-\delta}}\right)}{\alpha^{\beta} + \left(e^{-x_i^{-\delta}}\right)^{\theta}} = 0 \tag{42}$$

(43)

$$\frac{\partial}{\partial\delta}(\log L) = \frac{n}{\delta} - \sum_{i=1}^{n}\log(x_i) + \theta\sum_{i=1}^{n}x_i^{-\delta}\log(x_i) - 2\sum_{i=1}^{n}\frac{\theta\left(e^{-x_i^{-\delta}}\right)^{\theta-1}(e^{-x_i^{-\delta}})x_i^{-\delta}\log(x_i)}{\alpha^{\beta} + (e^{-x_i^{-\delta}})^{\theta}} = 0$$

The equations derived from the partial derivatives of the log-likelihood function (log L) cannot be solved analytically. The Broyden Fletcher Goldfarb Shanno (BFGS) [14]–[15]–[16]–[17] algorithm was used to compute the MLE estimates of parameters of Modi Exponentiated Inverted Weibull distribution.

5. Simulation Study

This section presents the results of a Monte Carlo simulation study to evaluate the accuracy and precision of the Maximum Likelihood Estimates (MLEs) for the Modi Exponentiated Inverted Weibull (MEIW) distribution parameters. The study was conducted by generating random samples of varying sizes n= 100,200,300,...900 using the quantile function of the MEIW distribution (equation 17). 1000 iterations were performed for each sample size, and the parameters $\alpha, \beta, \theta, \delta$ were estimated using the MLE method.

To assess the accuracy of the parameter estimates, we computed the Average Bias (AB), and the Root Mean Square Error (RMSE). These metrics were calculated for two different sets of parameter values:

I:$(\alpha, \beta, \theta, \delta)$=(0.7,1.5,2.0,2.5)

II:$(\alpha, \beta, \theta, \delta)$= (1.0, 2.0, 2.5, 3.5).

The formulas for AB and RMSE are as follows as discussed by [24].

$$\text{AB}_{(\mu)} = \frac{1}{N}\sum_{i=1}^{N}(\hat{\mu}_i - \mu) \tag{44}$$

$$\text{RMSE}_{(\mu)} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\mu}_i - \mu)^2} \tag{45}$$

Where:

- μ is the parameter to be estimated which is estimated by $\hat{\mu}$
- $\hat{\mu}_i$ is the estimate at the i^{th} replication for each sample size .
- N is the number of replications.

The following table shows the MLE, AB, and RMSE values of the parameters α, β, θ, and δ for various sample sizes.

The simulation results for Maximum Likelihood Estimates (MLEs), Absolute Biases (ABs), and Root Mean Squared Errors (RMSEs) are presented in Table(3) for the parameters of the Modi Exponentiated Inverted Weibull (MEIW) distribution across various sample sizes. As the sample size increases, the MLEs for all parameters converge toward their true values, indicating improved accuracy. Specifically, the Absolute Biases (ABs) and Root Mean Squared Errors (RMSEs) of the parameter estimators decrease as the sample size (n) increases. This suggests that the MLEs are asymptotically unbiased and consistent [18].

Table 3: Simulation results for Modi Exponentiated Inverted Weibull distribution for set I and set II

Parameters	**n**	**MLE**	**AB**	**RMSE**	**MLE**	**AB**	**RMSE**
α	100	1.573466	0.8762656	1.955696	1.838351	0.844351	1.873193
	200	1.221563	0.5306635	1.463882	1.537638	0.542638	1.512544
	300	1.115144	0.4221442	1.247748	1.405990	0.407990	1.185192
	400	1.003324	0.3075244	0.9860199	1.329766	0.333766	1.098763
	500	0.8602882	0.1665882	0.6968065	1.198620	0.200620	0.838410
	900	0.7927518	0.0962518	0.4467724	1.060381	0.0403812	0.6310727
β	100	3.791234	2.297234	4.749120	5.044935	3.056935	5.917864
	200	3.021905	1.541405	3.923528	4.441930	2.451930	5.484266
	300	2.593628	1.108628	3.267595	4.004234	2.008234	4.999613
	400	2.389260	0.8982599	3.002846	3.504246	1.512246	4.305954
	500	1.918429	0.4319295	1.987725	3.050492	1.054492	3.669937
	900	1.642172	0.1496724	0.8160815	2.074646	0.2346459	2.262524
θ	100	2.188649	0.1966485	1.113958	2.682166	0.197166	1.011578
	200	2.127721	0.1537214	0.9405196	2.664733	0.177233	0.913070
	300	2.075677	0.0956766	0.6981591	2.609982	0.094982	0.693173
	400	2.068445	0.0804446	0.6660829	2.582079	0.072079	0.684903
	500	2.059965	0.0729647	0.6018207	2.572128	0.071128	0.624396
	900	2.006174	0.0161740	0.4077951	2.504760	0.014760	0.405306

Parameters	n	MLE	AB	RMSE	MLE	AB	RMSE
	100	2.629302	0.0393016	0.6401635	3.583366	0.043667	0.752927
	200	2.579676	0.0121763	0.5361607	3.567535	0.034965	0.659687
	300	2.557630	0.0120300	0.4469868	3.541950	0.015195	0.390667
	400	2.537462	0.0116624	0.4171399	3.534192	0.008192	0.019815
δ	500	2.529313	0.0108127	0.3772815	3.531503	0.004497	0.017885
	900	2.500385	0.01038853	0.2774086	3.500217	0.001016	0.011475

6. Application to Real Data Set

This section applied the Modi Exponentiated Inverted Weibull distribution to two real datasets. The distribution's flexibility was compared to well-known distributions, including the Weibull distribution, Inverse Weibull distribution, and Exponentiated Inverted Weibull distribution. The Anderson-Darling, Cramer-von Mises, and Kolmogorov-Smirnov tests were used to evaluate the goodness of fit. The Akaike Information Criterion, Hannan-Quinn Information Criterion, Consistent Akaike Information Criterion, and Bayesian Information Criterion were used to select the best-fitting model.

The distributions to which the MEIW distribution was compared in this section are the Weibull distribution(We)[[19]], Inverted Weibull distribution (IW)[[6]], Exponentiated Inverted Weibull distribution (EIW)[[10]] with the respective pdfs:

$$We : f(x;\alpha,\theta) = \alpha\theta x^{\theta-1}\exp(-\alpha x^{\theta}) \tag{46}$$

$$IW : f(x:\alpha,\beta) = \beta\alpha^{\beta}x^{-(\beta+1)}\exp\left(-\left(\frac{\alpha}{x}\right)^{\beta}\right) \tag{47}$$

$$EIW : f(x,\delta,\theta) = \theta\delta x^{-(\delta+1)}\left(\mathrm{e}^{-x^{-\delta}}\right)^{\theta} \tag{48}$$

6.1. Data set I: Fatigue Life Analysis of 6061-T6 Aluminum Coupons Data Set.

Data set I: This data is from [20] and contains information on the fatigue life of 6061-T6 aluminum coupons cut parallel to the rolling direction and oscillated at 18 cycles per second. 70, 90, 96, 97, 99, 100, 103, 104, 104, 105, 107, 108, 108, 108, 109, 109, 112, 112, 113, 114, 114, 114, 116, 119, 120, 120, 120, 121, 121, 123, 124, 124, 124, 124, 124, 128, 128, 129, 129, 130, 130, 130, 131, 131, 131, 131, 131, 132,132, 132, 133, 134, 134, 134, 134, 136, 136, 137, 138, 138, 138, 139, 139, 141,141, 142, 142, 142, 142, 142, 142, 144, 144, 145, 146, 148, 148, 149, 151, 151,152, 155, 156, 157, 157, 157, 157, 158, 159, 162, 163, 163, 164, 166, 166, 168, 170, 174, 201, and 212.

Figures 3-6 show The TTT, histogram, violin, and box plots for the fatigue life of 6061-T6 aluminum coupons cut parallel to the rolling direction and oscillated at 18 cycles per second data set. The concave-up shape of the TTT plot suggests an increasing hazard rate in the data. The histogram suggests a right-skewed distribution with a concentration of data around the peak and a longer tail to the right. The box plot indicates some outliers and the violin plot illustrates that values are concentrated around the median.

Figure 7 and 8 illustrate the estimated PDF and CDF of the MEIW distribution for the fatigue life of 6061-T6 aluminum coupons cut parallel to the rolling direction and oscillated at 18 cycles per second data set, **Figure 9 and 10** depict the Kaplan-Meier and PP plots. The Kaplan-Meier curve closely approximates the survival function of the model, and the two distributions also exhibit proximity in the PP plot. Furthermore, **Figure 11** shows the plot of fitted pdfs of the distributions considered in this study with the histogram of the observed data.

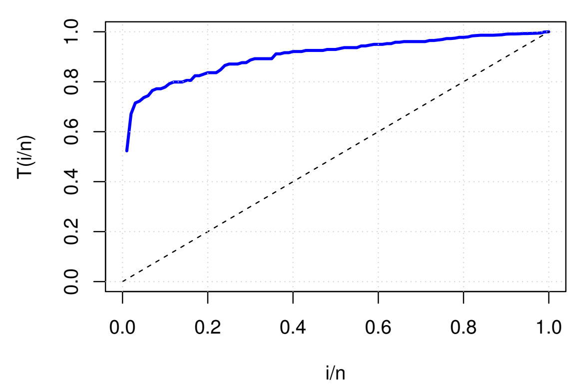

FIGURE 3. TTT Plot for Fatigue Life Analysis of 6061-T6 Aluminum Coupons data set

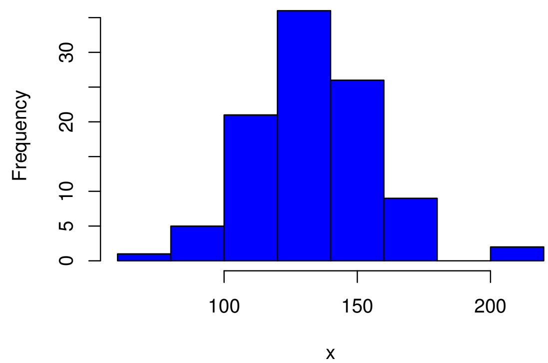

FIGURE 4. Histogram for Fatigue Life Analysis of 6061-T6 Aluminum Coupons data set

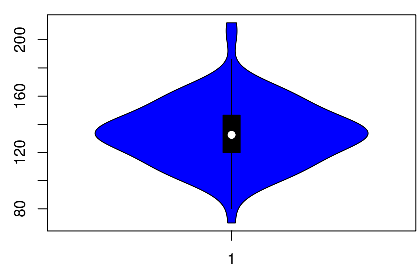

FIGURE 5. Violin plot for Fatigue Life Analysis of 6061-T6 Aluminum Coupons data set

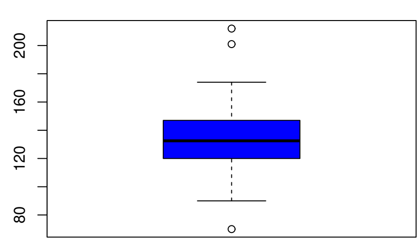

FIGURE 6. Box plot for Fatigue Life Analysis of 6061-T6 Aluminum Coupons data set

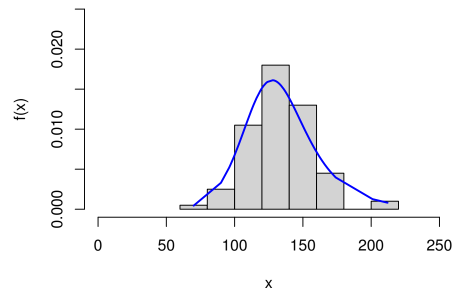

FIGURE 7. Plot of estimated PDF of Fatigue Life Analysis of 6061-T6 Aluminum Coupons data set

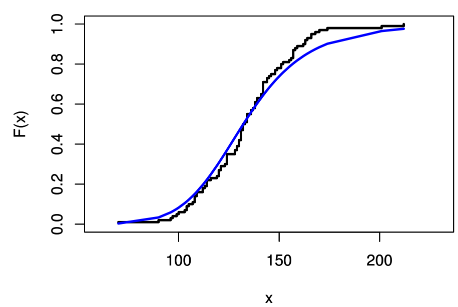

FIGURE 8. Plot of estimated CDF of Fatigue Life Analysis of 6061-T6 Aluminum Coupons data set

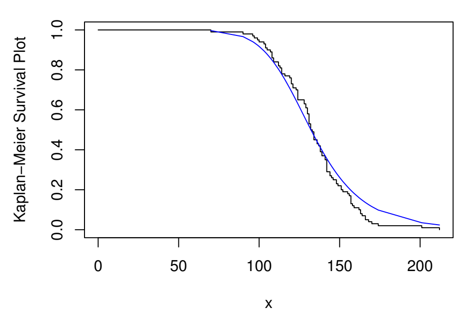

FIGURE 9. Plot of Kaplan-Meier of Fatigue Life Analysis of 6061-T6 Aluminum Coupons data set

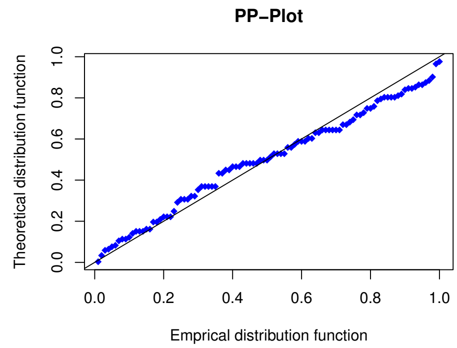

FIGURE 10. PP Plot of Fatigue Life Analysis of 6061-T6 Aluminum Coupons data set

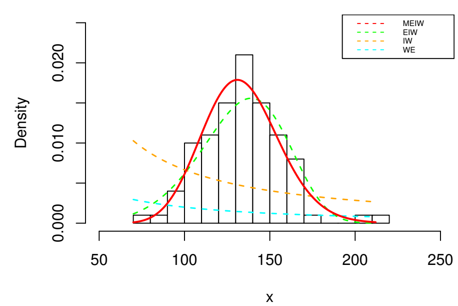

FIGURE 11. Estimated fitted densities of Fatigue Life Analysis of 6061-T6 Aluminum Coupons data set for various distributions

Table 4 offers the descriptive statistics for data set I. The positive skewness (0.3772) indicates a slight rightward asymmetry, and the kurtosis (1.24) suggests a distribution with lighter tails than a normal distribution.

Table 5 and 6 display the AIC, BIC, HQIC, and CAIC values used to evaluate and compare different models. The Modi Exponentiated Inverted Weibull distribution consistently has the lowest values in these criteria, indicating the best fit to the data [21]. It also excels over the other models with the lowest K-S, W*, and A* values, the highest log-likelihood, and the highest P-value for the K-S statistic [22].

TABLE 4. Descriptive statistics of Fatigue Life Analysis of 6061-T6 Aluminum Coupons data set

Minimum	Maximum	Mean	Median	Mode	Variance	Skewness	Kurtosis
70	212	133.8	132.5	142	511.36	0.3772	1.24

TABLE 5. Goodness-of-fit statistics, log-likelihood and maximum likelihood estimates of the model parameters for data set I

Distributions	Estimates (SEs)	Log-Likelihood (l)	w*	A*	K-S (p-value)
MEIW $(\alpha,\beta,\theta,\delta)$	$\alpha = 0.02043(0.0054)$ $\beta = 8.61333(1.9714)$ $\theta = 112.925(10.222)$ $\delta = 0.2487(0.0434)$	-456.21	0.0754	0.4449	0.0863 (0.4456)
EIW (δ,θ)	$\delta = 196.2023(45.947)$ $\theta = 1.11817(0.0513)$	-465.12	0.0863	0.5008	0.10413 (0.21625)
IW (α,β)	$\alpha = 120.7264(2.5438)$ $\beta = 5.0426(0.3265)$	-470.93	0.4134	2.4023	0.1330 (0.05789)
WE (α,θ)	$\alpha = 1.44426(3.8768)$ $\theta = 5.95531(0.9854)$	-562.79	0.5219	3.7653	0.22546 (0.04657)

TABLE 6. Values of Information Criteria for various distributions for data set I

Distributions	AIC	BIC	CAIC	HQIC
MEIW	920.42	921.63	920.54	918.528
EIW	934.24	939.450	934.363	936.348
IW	945.86	951.07	945.983	947.97
WE	1129.58	1134.79	1129.703	1131.68

6.2. Data set II: March Precipitation Observations in Minneapolis/St. Paul (in inches).

Data set II: The second real-life data was first documented by Hinkley (1977). Data consisting of 30 observations of the March precipitation (in inches) in Minneapolis/ St Paul is available in [23]: 0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59 , 0.81 , 2.81 , 1.87 , 1.18 , 1.35, 4.75, 2.48 ,0.96 , 1.89 , 0.90 , and 2.05.

Figures 12-15 depict the TTT, histogram, violin, and box plots for March precipitation (in inches) in the Minneapolis/St. Paul dataset. The TTT plot indicates an increasing hazard rate, while the histogram reveals a right-skewed distribution of precipitation data. The box plot suggests the presence of outliers, and the violin plot illustrates a concentration of values around the median.

Figures 16 and 17 illustrate the estimated probability density function (PDF) and cumulative distribution function (CDF) of the Modi Exponentiated Inverted Weibull (MEIW) distribution for March precipitation (in inches) in the Minneapolis/St. Paul dataset. **Figures 18 and 19** depict the Kaplan-Meier and PP plots. The Kaplan-Meier curve closely approximates the model's survival function, and the two distributions also exhibit proximity in the PP plot. **Figure 20** shows a plot of the fitted PDFs of the distributions considered in this study, overlaid with the histogram of the observed data.

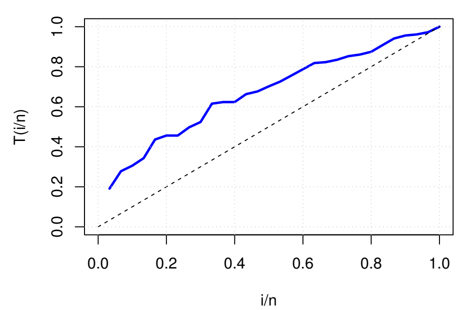

FIGURE 12. TTT plot of March precipitation data set

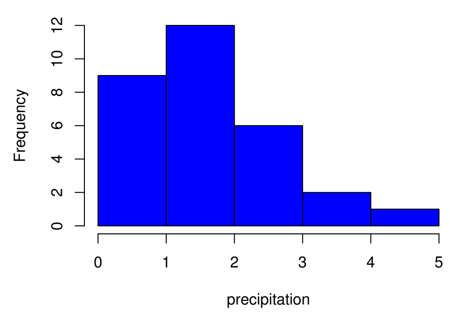

FIGURE 13. Histogram of March precipitation data set

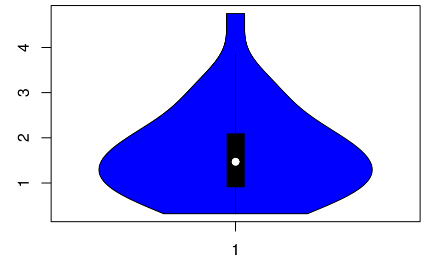

FIGURE 14. Violin plot of March precipitation data set

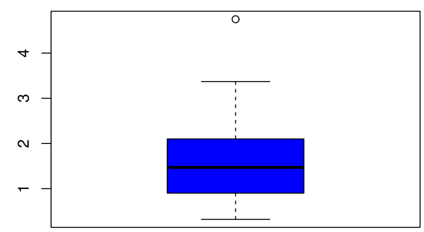

FIGURE 15. Box plot of March precipitation data set

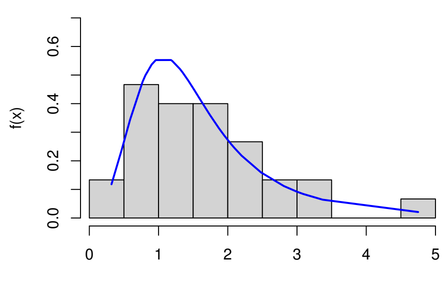

FIGURE 16. Plot of estimated PDF of March precipitation data set

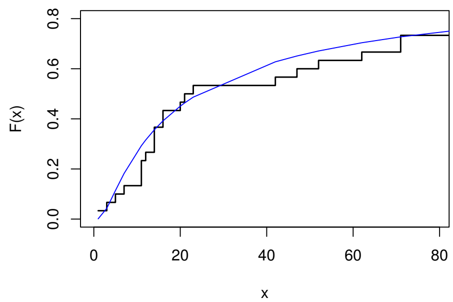

FIGURE 17. Plot of estimated CDF of March precipitation data set

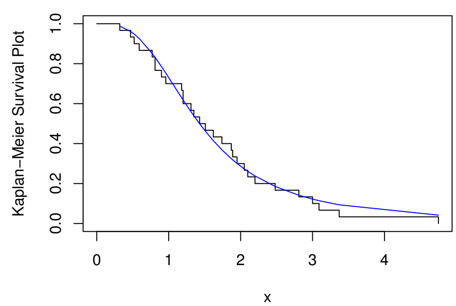

FIGURE 18. Box plot of March precipitation data set

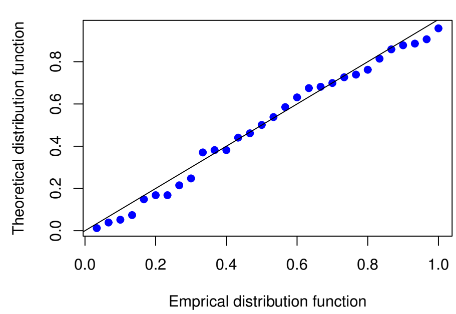

FIGURE 19. PP Plot of March precipitation data set

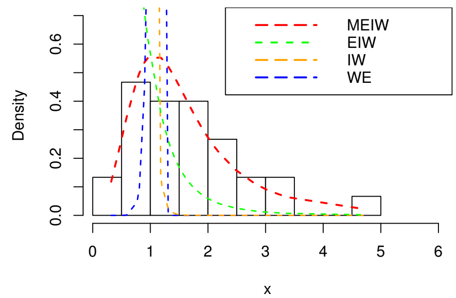

FIGURE 20. Estimated fitted densities of March precipitation (in inches) data set for various distributions

Table 7 offers the descriptive statistics for data set II. A skewness value is 1.145 which indicates that the distribution is positively skewed and the kurtosis of 1.665 suggests that the distribution is platykurtic.

Tables 8 and 9 present the AIC, BIC, HQIC, and CAIC values used to evaluate and compare different models. The Modi Exponentiated Inverted Weibull distribution consistently demonstrates the lowest values in these criteria, suggesting the best fit to the data [11]. Additionally, it outperforms the other models with the lowest K-S, W, and A* values, the highest log-likelihood, and the highest P-value for the K-S statistic.

TABLE 7. Descriptive statistics of the March precipitation

Minimum	**Maximum**	**Mean**	**Median**	**Mode**	**Variance**	**Skewness**	**Kurtosis**
0.32	4.75	1.675	1.47	0.81	1.001233	1.145	1.665

TABLE 8. Goodness-of-fit statistics, log-likelihood and maximum likelihood estimates of the model parameters for data set II

Distributions	Estimates (SEs)	Log-Likelihood (l)	w*	A*	K-S (p-value)
MEIW $(\alpha,\beta,\theta,\delta)$	$\alpha = 0.13943(0.3201)$ $\beta = 15.71258(0.00751)$ $\theta = 31.95127(39.473)$ $\delta = 0.08877(0.112101)$	-39.1796	0.0393	0.2524	0.0751 (0.9958)
EIW (δ,θ)	$\delta = 1.02525(0.1978)$ $\theta = 1.54959(0.2026)$	-42.9170	0.0417	0.2628	0.1506 (0.6556)
IW (α,β)	$\alpha = 1.016224(0.1272412)$ $\beta = 1.549595(0.2026504)$	-42.9203	0.1260	0.7721	0.1523 (0.4893)
WE (α,θ)	$\alpha = 0.315464(0.0906148)$ $\theta = 1.808877(0.2491126)$	-43.92	0.1419	0.8693	0.2412 (0.4567)

TABLE 9. Values of Information Criteria for various distributions for data set II

Distributions	AIC	BIC	CAIC	HQIC
MEIW	86.35935	91.96412	87.95935	88.15237
EIW	89.83402	92.63641	90.27846	90.73050
IW	89.84060	92.64299	90.28504	90.73711
WE	91.84000	94.64200	92.28400	92.73600

7. CONCLUSION

This paper applies a new four-parameter model known as the Modi Exponentiated Inverted Weibull distribution(MEIW) to two real datasets. We explore the mathematical and statistical properties of this distribution, deriving expressions for its cumulative distribution function, probability density function, survival function, hazard rate function, Cumulative hazard function, reversed hazard rate function, Odds function, quantile function, moment, order statistics, and Rényi Entropy. To estimate the parameters of the MEIW distribution, we utilize maximum likelihood estimation (MLE). Monte Carlo simulations are employed to evaluate the performance of the MLEs. Our study finds that the MLEs are both accurate and reliable in estimating model parameters. As the sample size increases, the MLEs converge to the true parameter values, as evidenced by the reduction in average biases (ABs). Additionally, root mean square errors (RMSEs) decrease with larger sample sizes. The analysis reveals that the MEIW distribution surpasses existing distributions in modeling the datasets used in this study. Moreover, various plots show that the MEIW distribution can adapt to different shapes, highlighting its flexibility in fitting diverse data sets.

Acknowledgment

The authors express their gratitude to the Pan African University, Institute for Basic Sciences, Technology, and Innovation (PAUSTI) for their valuable support in facilitating this study.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] A.J. Hallinan, A Review of the Weibull Distribution, J. Qual. Technol. 25 (1993), 85–93. https://doi.org/10.1080/00224065.1993.11979431.

[2] G. Stone, R. Van Heeswijk, Parameter Estimation for the Weibull Distribution, IEEE Trans. Electr. Insul. EI-12 (1977), 253–261. https://doi.org/10.1109/TEI.1977.297976.

[3] A. Marshall, A New Method for Adding a Parameter to a Family of Distributions with Application to the Exponential and Weibull Families, Biometrika 84 (1997), 641–652. https://doi.org/10.1093/biomet/84.3.641.

[4] M.M. Nassar, F.H. Eissa, On the Exponentiated Weibull Distribution, Commun. Stat. - Theory Methods 32 (2003), 1317–1336. https://doi.org/10.1081/STA-120021561.

[5] C. Tojeiro, F. Louzada, M. Roman, P. Borges, The Complementary Weibull Geometric Distribution, J. Stat. Comput. Simul. 84 (2014), 1345–1362. https://doi.org/10.1080/00949655.2012.744406.

[6] F.R.S. De Gusmão, E.M.M. Ortega, G.M. Cordeiro, The Generalized Inverse Weibull Distribution, Stat. Papers 52 (2011), 591–619. https://doi.org/10.1007/s00362-009-0271-3.

[7] M.S. Khan, R. King, Modified Inverse Weibull Distribution, J. Stat. Appl. Probab. 1 (2012), 115.

[8] G. Aryal, I. Elbatal, Kumaraswamy Modified Inverse Weibull Distribution: Theory and Application, Appl. Math. Inf. Sci. 9 (2015), 651-660.

[9] R.M. Pakungwati, Y. Widyaningsih, D. Lestari, Marshall-Olkin Extended Inverse Weibull Distribution and Its Application, J. Phys.: Conf. Ser. 1108 (2018), 012114. https://doi.org/10.1088/1742-6596/1108/1/012114.

[10] A. Flaih, H. Elsalloukh, E. Mendi, M. Milanova, The Exponentiated Inverted Weibull Distribution, Appl. Math. Inf. Sci. 6 (2012), 167–171.

[11] J.O. Braimah, J.A. Adjekukor, N. Edike, S.O. Elakhe, A New Weibull Exponentiated Inverted Weibull Distribution for Modelling Positively-Skewed Data, Glob. J. Pure Appl. Sci. 27 (2021), 43–53. https://doi.org/10.4314/gjpas.v27i1.6.

[12] Kanak Modi, A New Family of Distribution with Application on Two Real Datasets on Survival Problem, Sci. Technol. Asia 25 (2020), 110. https://doi.org/10.14456/SCITECHASIA.2020.1.

[13] A.D. Ndayisaba, L.O. Odongo, A. Ngunyi, The Modi Exponentiated Exponential Distribution, J. Data Anal. Inf. Process. 11 (2023), 341–359. https://doi.org/10.4236/jdaip.2023.114017.

[14] C.G. Broyden, A New Method of Solving Nonlinear Simultaneous Equations, Computer J. 12 (1969), 94–99. https://doi.org/10.1093/comjnl/12.1.94.

[15] R. Fletcher, A Class of Methods for Nonlinear Programming with Termination and Convergence Properties, in: Integer and Nonlinear Programming, North-Holland, Amsterdam, Holland, 1970.

[16] D. Goldfarb, A Family of Variable-Metric Methods Derived by Variational Means, Math. Comput. 24 (1970), 23–26.

[17] D.F. Shanno, Conditioning of Quasi-Newton Methods for Function Minimization, Math. Comput. 24 (1970), 647–656.

[18] G. Casella, R. Berger, Statistical Inference, CRC Press, 2024.

[19] S.J. Almalki, S. Nadarajah, Modifications of the Weibull Distribution: A Review, Rel. Eng. Syst. Saf. 124 (2014), 32–55. https://doi.org/10.1016/j.ress.2013.11.010.

[20] S. Lee, Y. Noh, Y. Chung, Inverted Exponentiated Weibull Distribution with Applications to Lifetime Data, Commun. Stat. Appl. Methods 24 (2017), 227–240. https://doi.org/10.5351/CSAM.2017.24.3.227.

[21] S.O. Bleed, R.A.-E. Attwa, R.F.M. Ali, T. Radwan, On Alpha Power Transformation Generalized Pareto Distribution and Some Properties, J. Appl. Math. 2024 (2024), 6270350. https://doi.org/10.1155/2024/6270350.

[22] A. Niyoyunguruza, L.O. Odongo, E. Nyarige, A. Habineza, A.H. Muse, Marshall-Olkin Exponentiated Frechet Distribution, J. Data Anal. Inf. Process. 11 (2023), 262–292. https://doi.org/10.4236/jdaip.2023.113014.

[23] R.M. Usman, M. Ahsan Ul Haq, The Marshall-Olkin Extended Inverted Kumaraswamy Distribution: Theory and Applications, J. King Saud Univ. - Sci. 32 (2020), 356–365. https://doi.org/10.1016/j.jksus.2018.05.021.

[24] S.Y. Lee, H.T. Zhu, Maximum Likelihood Estimation of Nonlinear Structural Equation Models, Psychometrika 67 (2002), 189–210. https://doi.org/10.1007/BF02294842.