Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2025, 2025:41
https://doi.org/10.28919/cmbn/9120
ISSN: 2052-2541

# STABILITY ANALYSIS OF THE TWO-DIMENSIONAL ADVECTION-DIFFUSION EQUATION FOR PARTICLE DISTRIBUTION BY THE CRANK NICOLSON-ALTERNATING DIRECTION IMPLICIT METHOD

ROSSALINA ALBASATIN[1,*], SUNARSIH[2], SUSILO HARIYANTO[2]

[1]Magister of Mathematics, Diponegoro University, Semarang, Indonesia

[2]Department of Mathematics, Diponegoro University, Semarang, Indonesia



**Abstract**. Particle distribution that occurs in waste stabilization ponds is a phenomenon that can be formulated using mathematics. The proposed model is a two-dimensional advection and diffusion model. We develop the model using differential equations that consider the $x$-axis and $y$-axis. We will solve the equation with the crank nicolson-alternating direction implicit finite difference method. the goal is to show the stability analysis of the method used to solve the proposed model. The results obtained show that the method used obtains unconditional stability.

**Keywords:** advection-diffusion model; partial differential equation; stability analysis; particle distribution.

**2020 AMS Subject Classification:** 35Q49, 35Q49.

## 1. INTRODUCTION

Environmental pollution in densely populated settlements is a matter of concern. The denser a settlement, the more environmental pollution increases, especially household waste or domestic

---

*Corresponding author
E-mail address: rossalinaalbasatin@gmail.com
Received January 13, 2025

liquid waste. Water is the most important aspect of life [1]. Three percent of the water on Earth is freshwater, but only about (0.01%) of this pure water can be used by humans [2]. This forces humans to treat polluted water and reuse it to meet needs [3]. Therefore, wastewater treatment can be used as a way to reduce water scarcity [4]–[7]. One example of the application of treated municipal wastewater is agricultural irrigation [4]. Farmers can use water sources that come from wastewater that has been fully or partially treated [8]. Improper wastewater treatment methods can be harmful to crops, soil, and the health of farmers and consumers [9], [10].

Improperly treated wastewater when continuously discharged into the soil results in changes in soil quality due to bacterial pollution, organic matter, and mineral pollution and structural degradation [11], [12]. In addition, the amount of salt, chlorine, and nitrate in groundwater increases as a result of long-term irrigation with untreated or poorly treated sewage [13].

Improving water quality requires proper wastewater treatment procedures so that the water can be used for agriculture [14], [15]. It is important to prevent possible environmental pollution and protect public health from related hazards [16]–[18]. One way to control water pollution is to create wastewater ponds that are useful for treating domestic wastewater [19]–[21]. In Indonesia, there are several cities that have provided sewage ponds, such as Yogyakarta, located in Sewon, Bantul. This wastewater treatment system is called a wastewater treatment plant (WWTP), and it is built to treat domestic wastewater by utilizing pipelines [22].

Domestic wastewater that flows daily to the WWTP may contain particles harmful to the environment. Therefore, research on the movement of Biochemical Oxygen Demand (BOD) particles in the pond needs to be done. BOD is an important parameter in assessing water quality. The BOD parameter is used to understand and determine the effect of wastewater conditions on the degradation process of organic matter [22].

Various studies have been conducted related to sewage stabilization ponds, including research by [22], which monitors water quality by measuring BOD concentration using machine learning methods such as random forest (RF), support vector regression (SVR), and multilayer perceptron (MLP) to get the best model in predicting BOD pollutants. A novel system is proposed to integrate microbial fuel cells (MFCs) with algae for organic matter degradation, nutrient removal, and bioenergy production simultaneously. This study also utilized mathematical models for simulation related to the performance of the proposed system [23]. Research on the analysis of secondary purification of wastewater treatment plants (WWTP) through activated sludge with the aim of knowing the settling velocity and describing the relationship between factors that affect the settling velocity. This study developed and predicted a one-dimensional model (1-D model) based on a complete experimental study conducted at the WWTP [24]. Numerical simulation of the advection-diffusion mechanism of BOD concentration used as an effluent indicator in only one flow direction of the effluent stabilization pond (1-dimensional (1-D)). This model is represented in a partial differential equation of order 2. The numerical method used to solve this model is the finite difference method with the Forward Time Central Space scheme [25]. Research on wastewater treatment with the help of bacteria has been conducted [26]. In this study, the development of WWTP water treatment was carried out, namely with a rapid infiltration system through three stages. The mathematical model of the rapid infiltration system was developed and calibrated by incorporating a biokinetic model so as to predict changes in water quality.

Changes in water quality can be assessed by the distribution of BOD particles. In [27], the BOD particle distribution model is described as an advection-diffusion mechanism that represents a partial differential equation. This study used the finite difference method to solve the 1-dimensional to 3-dimensional model. Research conducted by [28] used a mathematical model to observe the interaction between Dissolved Oxygen (DO) and BOD in open flow. The proposed model is represented by partial differential equations solved by the finite volume method. Research [29] presents a mathematical model related to water pollution by organic matter, DO, and BOD, which are important indicators in monitoring water quality. This study presents a two-dimensional mathematical model through a system of non-linear differential equations with Holling type III kinetic reaction between DO and BOD solved by the central finite difference method for space discretization and the Crank-Nicolson method for time discretization.

The exact solution of partial differential equations is difficult, so numerical methods are used to solve these equations [30]. In [31] and [32], numerical methods such as the finite element method were used to solve partial differential equations. Research [30] used fine particle hydrodynamics and a one-dimensional finite difference method to model the distribution of chemical pollutants. A higher-order time discretization method for solving stochastic partial differential equations was studied by [33]. The solution of the groundwater flow diffusion equation using a modified center finite difference method with a backward finite difference was conducted by [34]. Research on pollutant distribution based on the advection-diffusion equation using the forward time center space (FTCS) finite difference method was conducted by [27]. [35] used an implicit scheme with the alternating direction implicit (ADI) method to solve the two-dimensional fractional diffusion equation.

Implicit methods have the advantage of producing stable and convergent solutions, this is used to overcome the weaknesses of explicit methods that have conditions on their stability [36]. Solving the solution of differential equations with implicit schemes in two-dimensional space takes a long time. One way to overcome this shortcoming is to use a solver method. This method is known as the ADI method. The ADI method is one of the implicit methods that can be used to solve partial differential equations that are parabolic, hyperbolic, and elliptic. Using the ADI method, the differential equation will be broken down into a simpler structure to solve it efficiently with the tridiagonal matrix algorithm.

Based on the quality of wastewater that is affected by BOD concentration, research will be conducted on the distribution of BOD particles in the effluent stabilization pond. Particles are represented by BOD concentrations measured horizontally and vertically with respect to changes in time and space. Changes in BOD concentration over time and space can be represented by partial differential equations. The proposed model considers a two-dimensional advection-diffusion process that will be solved using the Crank Nicolson-Alternating Direction Implicit (CN-ADI) method. This study aims to describe the phenomenon of BOD particle distribution in the effluent stabilization pond.

## 2. ADVECTION-DIFFUSION MODEL

In this study, we propose a particle distribution model for a waste stabilization pond. Our model describes the distribution in one dimension and two dimensions. The particle transfer equation takes into account the inflow, outflow, and motion of particles in the water. To clarify the description of the phenomenon that occurs, it can be seen in Figure 1.

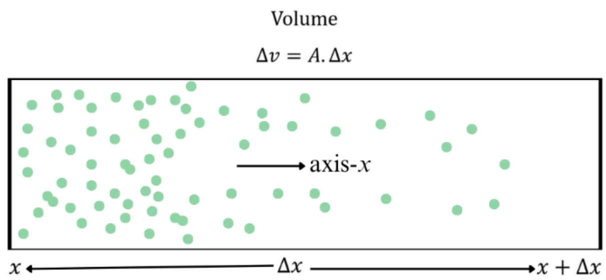


FIGURE 1. Particle distribution in dimension one

In Figure 1, wastewater flows into the pond with a flow velocity u in a straight direction in the $x$-axis. We assume that the length of the pond is $\Delta x$, so the following mathematical formulation is obtained

(1)
$$\frac{\partial C}{\partial t} = -u\frac{\partial C}{\partial x} + D_{mx}\frac{\partial^2 C}{\partial x^2}$$

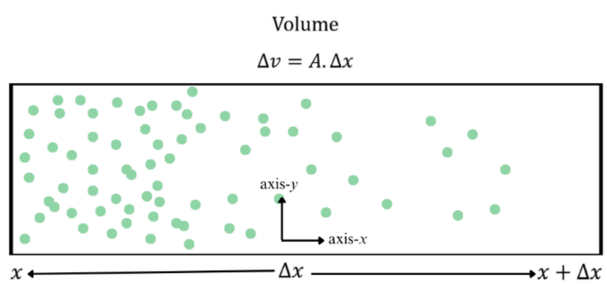


FIGURE 2. Particle distribution in dimension two

Figure 2 shows that the particle displacement moves along the $x$-axis and $y$-axis, with the same pool length $\Delta x$. Therefore, the mathematical formulation of the particle distribution in the second dimension is obtained as follows

(2)
$$\frac{\partial C}{\partial t} = -u\frac{\partial C}{\partial x} - v\frac{\partial C}{\partial y} + D_{mx}\frac{\partial^2 C}{\partial x^2} + D_{my}\frac{\partial^2 C}{\partial y^2}$$

Where

$u$ : Advection coefficient for $x$-axis

$v$ : Advection coefficient for $y$-axis

$D_{mx}$ : Diffusion coefficient for $x$-axis

$D_{my}$ : Diffusion coefficient for $y$-axis

## 3. DISCRETIZATION OF THE MODEL

To solve the advection-diffusion equations in equations (1) and (2) we use the Crank Nicolson method to discretize time and the Alternating Direction Implicit (ADI) method to discretize space. The numerical scheme of the two-dimensional advection-diffusion equation using the ADI method is carried out by performing a second-order central difference approach. The two-dimensional advection-diffusion equation is given as follows

$$\frac{\partial C}{\partial t} = -u\frac{\partial C}{\partial x} - v\frac{\partial C}{\partial y} + D_{mx}\frac{\partial^2 C}{\partial x^2} + D_{my}\frac{\partial^2 C}{\partial y^2} \tag{3}$$

The main idea of the ADI method in solving the two-dimensional diffusion advection equation is to divide the scheme into two steps. The first step aims to obtain the values from $t$ to $t + \frac{\Delta t}{2}$ by discretizing the derivative of $C$ with respect to $x$ through an implicit central difference approach, while the derivative of $C$ with respect to $y$ through an explicit central difference approach.

The first step approach in equation (3) is as follows

$$\frac{C_{i,j}^{n+\frac{1}{2}} - C_{i,j}^{n}}{\frac{\Delta t}{2}} = -u\left(\frac{C_{i+1,j}^{n+\frac{1}{2}} - C_{i-1,j}^{n+\frac{1}{2}}}{2\Delta x}\right) - v\left(\frac{C_{i,j+1}^{n} - C_{i,j-1}^{n}}{2\Delta x}\right) + D_{mx}\left(\frac{C_{i+1,j}^{n+\frac{1}{2}} - 2C_{i,j}^{n+\frac{1}{2}} + C_{i-1,j}^{n+\frac{1}{2}}}{(\Delta x)^2}\right) + D_{my}\left(\frac{C_{i,j+1}^{n} - 2C_{i,j}^{n} + C_{i,j-1}^{n}}{(\Delta y)^2}\right) \tag{4}$$

$$C_{i,j}^{n+\frac{1}{2}} - C_{i,j}^{n} = -u\frac{\Delta t}{4\Delta x}\left(C_{i+1,j}^{n+\frac{1}{2}} - C_{i-1,j}^{n+\frac{1}{2}}\right) - v\frac{\Delta t}{4\Delta x}\left(C_{i,j+1}^{n} - C_{i,j-1}^{n}\right) + D_{mx}\frac{\Delta t}{2(\Delta x)^2}\left(C_{i+1,j}^{n+\frac{1}{2}} - 2C_{i,j}^{n+\frac{1}{2}} + C_{i-1,j}^{n+\frac{1}{2}}\right) + D_{my}\frac{\Delta t}{2(\Delta y)^2}\left(C_{i,j+1}^{n} - 2C_{i,j}^{n} + C_{i,j-1}^{n}\right) \tag{5}$$

To simplify Equation (5), the following is done:

We assume $q_x = u\frac{\Delta t}{4\Delta x}$; $q_y = u\frac{\Delta t}{4\Delta y}$; $\lambda_x = D_{mx}\frac{\Delta t}{(\Delta x)^2}$; $\lambda_y = D_{my}\frac{\Delta t}{(\Delta y)^2}$, then equation (5) becomes

$$C_{i,j}^{n+\frac{1}{2}} - C_{i,j}^{n} = -q_x\left(C_{i+1,j}^{n+\frac{1}{2}} - C_{i-1,j}^{n+\frac{1}{2}}\right) - q_y\left(C_{i,j+1}^{n} - C_{i,j-1}^{n}\right) + \lambda_x\left(C_{i+1,j}^{n+\frac{1}{2}} - 2C_{i,j}^{n+\frac{1}{2}} + C_{i-1,j}^{n+\frac{1}{2}}\right) + \lambda_y\left(C_{i,j+1}^{n} - 2C_{i,j}^{n} + C_{i,j-1}^{n}\right) \quad (6)$$

The discretization results in the first step are as follows

$$(1+2\lambda_x)C_{i,j}^{n+\frac{1}{2}} + (q_x - \lambda_x)C_{i+1,j}^{n+\frac{1}{2}} + (-q_x - \lambda_x)C_{i-1,j}^{n+\frac{1}{2}} = \left(1-2\lambda_y\right)C_{i,j}^{n} + \left(-q_y + \lambda_y\right)C_{i,j+1}^{n} + \left(q_y + \lambda_y\right)C_{i,j-1}^{n} \quad (7)$$

Furthermore, the second step discretization of the ADI method aims to obtain values from $t + \frac{\Delta t}{2}$ to $t+1$ by discretizing the derivative of $C$ with respect to $y$ through an implicit central difference approach, while the derivative of $C$ with respect to $x$ through an explicit central difference approach.

The second step approach in equation (3) is as follows

$$\frac{C_{i,j}^{n+1} - C_{i,j}^{n+\frac{1}{2}}}{\frac{\Delta t}{2}} = -u\left(\frac{C_{i+1,j}^{n+\frac{1}{2}} - C_{i-1,j}^{n+\frac{1}{2}}}{2\Delta x}\right) - v\left(\frac{C_{i,j+1}^{n+1} - C_{i,j-1}^{n+1}}{2\Delta x}\right) + D_{mx}\left(\frac{C_{i+1,j}^{n+\frac{1}{2}} - 2C_{i,j}^{n+\frac{1}{2}} + C_{i-1,j}^{n+\frac{1}{2}}}{(\Delta x)^2}\right) + D_{my}\left(\frac{C_{i,j+1}^{n+1} - 2C_{i,j}^{n+1} + C_{i,j-1}^{n+1}}{(\Delta y)^2}\right) \quad (8)$$

With similar steps, the second discretization for equation (8) is obtained as follows

$$\left(1+2\lambda_y\right)C_{i,j}^{n+1} + \left(q_y - \lambda_y\right)C_{i,j+1}^{n+1} + \left(-q_y - \lambda_y\right)C_{i,j-1}^{n+1} = (1-2\lambda_x)C_{i,j}^{n+\frac{1}{2}} + (-q_x + \lambda_x)C_{i+1,j}^{n+\frac{1}{2}} + (q_x + \lambda_x)C_{i-1,j}^{n+\frac{1}{2}} \quad (9)$$

## 4. Stability Analysis with Von Neumann

The focus of this study is to demonstrate the stability of the CN-ADI method in solving the two-dimensional diffusion advection equation. The stability of a numerical method can be tested using the Von Neumann technique. An equation will be said to be stable if the value of $\alpha \leq 1$, by replacing $C_{i,j}^n = \alpha^n e^{i\beta\Delta ij}$ in equation (7) obtained

(10) $$(1+2\lambda_x)\alpha^{n+\frac{1}{2}}e^{i\beta_1 ij} + (q_x-\lambda_x)\alpha^{n+\frac{1}{2}}e^{i\beta_1(i+1)j} + (-q_x-\lambda_x)\alpha^{n+\frac{1}{2}}e^{i\beta_1(i-1)j}$$
$$= \left(1-2\lambda_y\right)\alpha^n e^{i\beta_1 ij} + \left(-q_y+\lambda_y\right)\alpha^n e^{i\beta_1 i(j+1)}$$
$$+ \left(q_y+\lambda_y\right)\alpha^n e^{i\beta_1 i(j-1)}$$

To obtain the value of $\alpha$, the second equation segment is multiplied by $\frac{1}{\alpha^n e^{i\beta_1 ij}}$, so that we get

(11) $$(1+2\lambda_x)\alpha^{\frac{1}{2}} + (q_x-\lambda_x)\alpha^{\frac{1}{2}}e^{i\beta_1 j} + (-q_x-\lambda_x)\alpha^{\frac{1}{2}}e^{-i\beta_1 j}$$
$$= \left(1-2\lambda_y\right) + \left(-q_y+\lambda_y\right)e^{i\beta_1 i} + \left(q_y+\lambda_y\right)e^{-i\beta_1 i}$$

(12) $$q_x\alpha^{\frac{1}{2}}\left(e^{i\beta_1 j}-e^{-i\beta_1 j}\right) - \lambda_x\alpha^{\frac{1}{2}}\left(e^{i\beta_1 j}+e^{-i\beta_1 j}\right) + (1+2\lambda_x)\alpha^{\frac{1}{2}}$$
$$= \left(1-2\lambda_y\right) - q_y\left(e^{i\beta_1 i}-e^{-i\beta_1 i}\right) + \lambda_y\left(e^{i\beta_1 i}+e^{-i\beta_1 i}\right)$$

Equation (12) contains the following Euler formula:
$e^{i\beta x} = \cos(\beta x) + \sin(\beta x)$
$e^{-i\beta x} = \cos(\beta x) - \sin(\beta x)$
To simplify equation (12), we substitute Euler's formula into equation (12), thus obtaining

(13) $$q_x\alpha^{\frac{1}{2}}\left(2i\,sin(\beta_1 j)\right) - \lambda_x\alpha^{\frac{1}{2}}(2\cos(\beta_1 j)) + (1+2\lambda_x)\alpha^{\frac{1}{2}}$$
$$= \left(1-2\lambda_y\right) - q_y(2i\sin(\beta_1 i)) + \lambda_y(2\cos(\beta_1 i))$$

Equation (13) contains trigonometry and it is known that $\cos(\alpha) = \left(1-2sin^2\left(\frac{\alpha}{2}\right)\right)$. By substituting the trigonometric function in equation (13), we get

(14) $$q_x\alpha^{\frac{1}{2}}\left(2i\,sin(\beta_1 j)\right) - \lambda_x\alpha^{\frac{1}{2}}(2\cos(\beta_1 j)) + (1+2\lambda_x)\alpha^{\frac{1}{2}}$$
$$= \left(1-2\lambda_y\right) - q_y(2i\sin(\beta_1 i)) + \lambda_y(2\cos(\beta_1 i))$$

(15) $$\alpha_1 = \left(\frac{1-4\lambda_y sin^2\left(\frac{\beta_1 i}{2}\right) - 2q_y i\sin(\beta_1 i)}{1+4\lambda_x sin^2\left(\frac{\beta_1 j}{2}\right) + 2q_x i\,sin(\beta_1 j)}\right)^2$$

The same steps are taken to obtain the value of $\alpha$ in the second discretization, by replacing $C_{i,j}^{n} = \alpha^{n} e^{i\beta\Delta ij}$ in equation (9), we obtain

$$(1+2\lambda_y)C_{i,j}^{n+1} + (q_y - \lambda_y)C_{i,j+1}^{n+1} + (-q_y - \lambda_y)C_{i,j-1}^{n+1} = (1-2\lambda_x)C_{i,j}^{n+\frac{1}{2}} + (-q_x + \lambda_x)C_{i+1,j}^{n+\frac{1}{2}} + (q_x + \lambda_x)C_{i-1,j}^{n+\frac{1}{2}} \tag{16}$$

$$(1+2\lambda_y)\alpha^{n+1}e^{i\beta_2 ij} + (q_y - \lambda_y)\alpha^{n+1}e^{i\beta_2 i(j+1)} + (-q_y - \lambda_y)\alpha^{n+1}e^{i\beta_2 i(j-1)} = (1-2\lambda_x)\alpha^{n+\frac{1}{2}}e^{i\beta_2 ij} + (-q_x + \lambda_x)\alpha^{n+\frac{1}{2}}e^{i\beta_2 (i+1)j} + (q_x + \lambda_x)\alpha^{n+\frac{1}{2}}e^{i\beta_2 (i-1)j} \tag{17}$$

In the same way, equation

(17) is multiplied by $\frac{1}{\alpha^{n+\frac{1}{2}}e^{i\beta_1 ij}}$ to obtain the value of $\alpha$

$$q_y\alpha^{\frac{1}{2}}\left(e^{i\beta_2 i} - e^{-i\beta_2 i}\right) - \lambda_y\alpha^{\frac{1}{2}}\left(e^{i\beta_2 i} + e^{-i\beta_2 i}\right) + (1+2\lambda_y)\alpha^{\frac{1}{2}} = (1-2\lambda_x) - q_x\left(e^{i\beta_2 j} - e^{-i\beta_2 j}\right) + \lambda_x\left(e^{i\beta_2 j} + e^{-i\beta_2 j}\right) \tag{18}$$

Substitute the euler value into equation

(18), we get

$$q_y\alpha^{\frac{1}{2}}(2i\sin(\beta_2 i)) - \lambda_y\alpha^{\frac{1}{2}}(2\cos(\beta_2 i)) + (1+2\lambda_y)\alpha^{\frac{1}{2}} = (1-2\lambda_x) - q_x(2i\sin(\beta_2 j)) + \lambda_x(2\cos(\beta_2 j)) \tag{19}$$

Since $\cos(\alpha) = \left(1 - 2sin^2\left(\frac{\alpha}{2}\right)\right)$, we obtain $\alpha$ is

$$\alpha_2 = \left(\frac{1 - 2q_x i\sin(\beta_2 j) - 4\lambda_x \sin^2\left(\frac{\beta_2 j}{2}\right)}{2q_y i\sin(\beta_2 i) + 4\lambda_y \sin^2\left(\frac{\beta_2 i}{2}\right) + 1}\right)^2 \tag{20}$$

Based on equation

(15) and equation (20), it can be seen that $\alpha$ is never negative and $\alpha \leq 1$, so it can be concluded that the ADI scheme for the two-dimensional advection-diffusion equation is always unconditionally stable.

## 5. CONCLUSIONS

A particle distribution process can be represented by a mathematical formula with a advection-diffusion model. The advection- diffusion model can be solved by the Crank Nicolson-Alternating Direction Implicit method with respect to the x-axis and y-axis. The Crank Nicolson-Alternating Direction Implicit method is a numerical method that needs to be sought for stability. to prove the stability of the method we use the Von Neumann method. The results show that the two-dimensional advection-diffusion model with the Crank Nicolson-Alternating Direction Implicit method is unconditionally stable

## ACKNOWLEDGEMENTS

Thanks to the Leaders and staff of WWTP Sewon Bantul, Yogyakarta who has given permission to conduct research and collect data at WWTP Sewon Bantul. Thanks to Mrs. Zani Anjani Rafsanjani as a lecturer from the Faculty of Science and Mathematics, Diponegoro University, who has guided and provided direction during the research process.

## CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

## REFERENCES

[1] A. Azizullah, M.N.K. Khattak, P. Richter, et al. Water Pollution in Pakistan and Its Impact on Public Health — A Review, Environ. Int. 37 (2011), 479–497. https://doi.org/10.1016/j.envint.2010.10.007.

[2] D. Hinrichsen, H. Tacio, The Coming Freshwater Crisis Is Already Here, in: The Linkages Between Population and Water, Woodrow Wilson International Center for Scholars, Washington, (2002), pp. 1–26.

[3] E.T. Sayed, M. Obaid, A.G. Olabi, et al. Recent Progress on the Application of Capacitive Deionization for Wastewater Treatment, J. Water Process Eng. 56 (2023), 104379. https://doi.org/10.1016/j.jwpe.2023.104379.

[4] H. Yu, F. Chen, J. Ma, et al. Comparative Evaluation of Groundwater, Wastewater and Canal Water for Irrigation on Toxic Metal Accumulation in Soil and Vegetable: Pollution Load and Health Risk Assessment, Agric. Water Manag. 264 (2022), 107515. https://doi.org/10.1016/j.agwat.2022.107515.

[5] G.K.C. Ding, Wastewater Treatment, Reused and Recycling – A Potential Source of Water Supply, in:

Encyclopedia of Sustainable Technologies, Elsevier, 2024: pp. 676–693. https://doi.org/10.1016/B978-0-323-90386-8.00062-0.

[6] H.E. Al-Hazmi, A. Mohammadi, A. Hejna, et al. Wastewater Reuse in Agriculture: Prospects and Challenges, Environ. Res. 236 (2023), 116711. https://doi.org/10.1016/j.envres.2023.116711.

[7] O.I. Areguamen, P. Ekwumengbo, I. Omoniyi, et al. Evaluation of the Source, Distribution and Risk of Metal Contaminated Stream Sediment, Case Stud. Chem. Environ. Eng. 8 (2023), 100429. https://doi.org/10.1016/j.cscee.2023.100429.

[8] M. Qadir, D. Wichelns, L. Raschid-Sally, et al. The Challenges of Wastewater Irrigation in Developing Countries, Agric. Water Manag. 97 (2010), 561–568. https://doi.org/10.1016/j.agwat.2008.11.004.

[9] C. Becerra-Castro, A.R. Lopes, I. Vaz-Moreira, et al. Wastewater Reuse in Irrigation: A Microbiological Perspective on Implications in Soil Fertility and Human and Environmental Health, Environ. Int. 75 (2015), 117–135. https://doi.org/10.1016/j.envint.2014.11.001.

[10] A. Christou, P. Karaolia, E. Hapeshi, et al. Long-Term Wastewater Irrigation of Vegetables in Real Agricultural Systems: Concentration of Pharmaceuticals in Soil, Uptake and Bioaccumulation in Tomato Fruits and Human Health Risk Assessment, Water Res. 109 (2017), 24–34. https://doi.org/10.1016/j.watres.2016.11.033.

[11] M. Jaramillo, I. Restrepo, Wastewater Reuse in Agriculture: A Review about Its Limitations and Benefits, Sustainability 9 (2017), 1734. https://doi.org/10.3390/su9101734.

[12] T.A. Bauder, R.M. Waskom, P.L. Sutherland, et al. Irrigation Water Quality Criteria, Crop Series/Irrigation, Fact Sheet No. 0.506, Colorado State University Extension Publication, 2011.

[13] V.E. Emongor, G.M. Ramolemana, Treated Sewage Effluent (Water) Potential to Be Used for Horticultural Production in Botswana, Phys. Chem. Earth, Parts A/B/C 29 (2004), 1101–1108. https://doi.org/10.1016/j.pce.2004.08.003.

[14] W. Chen, S. Lu, W. Jiao, M. Wang, A.C. Chang, Reclaimed Water: A Safe Irrigation Water Source?, Environ. Dev. 8 (2013), 74–83. https://doi.org/10.1016/j.envdev.2013.04.003.

[15] H. Karimi-Maleh, R. Darabi, F. Karimi, et al. State-of-Art Advances on Removal, Degradation and Electrochemical Monitoring of 4-Aminophenol Pollutants in Real Samples: A Review, Environ. Res. 222 (2023), 115338. https://doi.org/10.1016/j.envres.2023.115338.

[16] H. Liu, M. Baghayeri, A. Amiri, et al. A Strategy for As(III) Determination Based on Ultrafine Gold

Nanoparticles Decorated on Magnetic Graphene Oxide, Environ. Res. 231 (2023), 116177. https://doi.org/10.1016/j.envres.2023.116177.

[17] P. Abdollahiyan, M. Hasanzadeh, F. Seidi, et al. An Innovative Colorimetric Platform for the Low-Cost and Selective Identification of Cu(II), Fe(III), and Hg(II) Using GQDs-DPA Supported Amino Acids by Microfluidic Paper-Based (μPADs) Device: Multicolor Plasmonic Patterns, J. Environ. Chem. Eng. 9 (2021), 106197. https://doi.org/10.1016/j.jece.2021.106197.

[18] S.A.A.A.N. Almuktar, S.N. Abed, M. Scholz, Wetlands for Wastewater Treatment and Subsequent Recycling of Treated Effluent: A Review, Environ. Sci. Pollut. Res. 25 (2018), 23595–23623. https://doi.org/10.1007/s11356-018-2629-3.

[19] Sunarsih, Sutrisno, Implementation of Fuzzy Optimization Approach to Facultative Wastewater Stabilization Ponds Problem Considering Fuzzy Parameters, J. Math. Comput. Sci. 11 (2021), 1193-1205. https://doi.org/10.28919/jmcs/5325.

[20] F.G.G. Ibrahim, V.A. Gómez, R.M. Torre, et al. Scale-down of High-Rate Algae Ponds Systems for Urban Wastewater Reuse, J. Water Process Eng. 56 (2023), 104342. https://doi.org/10.1016/j.jwpe.2023.104342.

[21] M. Liu, Q. Lian, Y. Zhao, M. Ni, J. Lou, J. Yuan, Treatment Effects of Pond Aquaculture Wastewater Using a Field-Scale Combined Ecological Treatment System and the Associated Microbial Characteristics, Aquaculture 563 (2023), 739018. https://doi.org/10.1016/j.aquaculture.2022.739018.

[22] K.S. Ooi, Z. Chen, P.E. Poh, et al. BOD5 Prediction Using Machine Learning Methods, Water Supply 22 (2022), 1168–1183. https://doi.org/10.2166/ws.2021.202.

[23] S. Luo, Z.W. Wang, Z. He, Mathematical Modeling of the Dynamic Behavior of an Integrated Photo-Bioelectrochemical System for Simultaneous Wastewater Treatment and Bioenergy Recovery, Energy 124 (2017), 227–237. https://doi.org/10.1016/j.energy.2017.02.039.

[24] Z. Bakiri, S. Nacef, A Mathematical Model to Predict the Performance of the Secondary Clarifier of a Municipal Wastewater Treatment Plants, J. Water Process Eng. 49 (2022), 103106. https://doi.org/10.1016/j.jwpe.2022.103106.

[25] G.A. Putri, Sunarsih, S. Hariyanto, Numerical Simulation of Advection-Diffusion on Flow in Waste Stabilization Ponds (1-Dimension) with Finite Difference Method Forward Time Central Space Scheme, Environ. Eng. Res. 23 (2018), 442–448. https://doi.org/10.4491/eer.2017.031.

[26] Y. Shen, X. Lu, J. Jiang, et al. Mathematical Modelling of a Three-Stage Constructed Rapid Infiltration System for Wastewater Plant Tailwater Treatment, J. Water Process Eng. 64 (2024), 105533. https://doi.org/10.1016/j.jwpe.2024.105533.

[27] S. Sunarsih, D.P. Sasongko, S. Sutrisno, et al. Analysis of Wastewater Facultative Pond Using Advection-Diffusion Model Based on Explicit Finite Difference Method, Environ. Eng. Res. 26 (2021), 190496. https://doi.org/10.4491/eer.2019.496.

[28] S. Hasadsri, M. Maleewong, Finite Element Method for Dissolved Oxygen and Biochemical Oxygen Demand in an Open Channel, Procedia Environ. Sci. 13 (2012), 1019–1029. https://doi.org/10.1016/j.proenv.2012.01.095.

[29] D.C. Guaca, E.C.C. Poletti, Modeling and Numerical Simulation of Dissolved Oxygen and Biochemical Oxygen Demand Concentrations with Holling Type III Kinetic Relationships, Appl. Math. Comput. 415 (2022), 126690. https://doi.org/10.1016/j.amc.2021.126690.

[30] J. Pei, R. Imin, Study and Application of Pollutant Diffusion Based on SPH Method, Eng. Anal. Bound. Elements 155 (2023), 789–802. https://doi.org/10.1016/j.enganabound.2023.07.012.

[31] J. Tian, M. He, P. Sun, Energy-Stable Finite Element Method for a Class of Nonlinear Fourth-Order Parabolic Equations, J. Comput. Appl. Math. 438 (2024), 115576. https://doi.org/10.1016/j.cam.2023.115576.

[32] J. Fang, Z. Shen, X. Cui, A Coupled Nonlinear Finite Element Scheme for Anisotropic Diffusion Equation with Nonlinear Capacity Term, J. Comput. Appl. Math. 438 (2024), 115512. https://doi.org/10.1016/j.cam.2023.115512.

[33] Y. Li, L. Vo, G. Wang, Higher Order Time Discretization Method for a Class of Semilinear Stochastic Partial Differential Equations with Multiplicative Noise, J. Comput. Appl. Math. 437 (2024), 115442. https://doi.org/10.1016/j.cam.2023.115442.

[34] Y. Song, W. Liu, G. Fan, Second-Order Efficient Algorithm for Coupled Nonlinear Model of Groundwater Transport System, J. Math. Anal. Appl. 531 (2024), 127847. https://doi.org/10.1016/j.jmaa.2023.127847.

[35] G. Gao, Z. Sun, Two Alternating Direction Implicit Difference Schemes with the Extrapolation Method for the Two-Dimensional Distributed-Order Differential Equations, Comput. Math. Appl. 69 (2015), 926–948. https://doi.org/10.1016/j.camwa.2015.02.023.