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PREDATOR CONTROL POLICY FOR PREY-PREDATOR SYSTEM WITH GROUP DEFENCE IN PREY AND ALLEE EFFECT IN PREDATOR

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Abstract. The present work discusses a simple two-species model of a predator-prey with group defense for prey species and a weak Allee effect. The local dynamics revealed that the system could undergo Hopf bifurcation as well as have an extinction domain. Combinations of different control methods have been thoroughly researched, and the policy reduces the predator population and increases the prey population simultaneously. Finally, some numerical illustrations are provided to validate our analytical results using various parameter sets. According to the findings, increasing the parameter value of γ (aggregation efficiency) under the optimal control strategy does not affect on predator or prey biomass.

Keywords: Allee effect; group defense; Saddle node bifurcation; bistability; Hopf bifurcation; optimal control policy.

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1. INTRODUCTION

Mammal carnivores like red fox Vulpes vulpes, the wildcat Felis catus, and the wild dog Canis lupus have been identified as dominant species present in the ecosystem because of their wide range of impacts on agronomy, cultivation, and native species [1]. According to Dickman [2], feral cats have been considered the primary reason for the extinction of a few bird species in New Zealand. Implementation of several carnivore control strategies including shooting, trapping, fencing, and poisoning has been studied previously [3]. Even though predator control can have significant benefits, the toxicant used may be detrimental to the survival of other wildlife [4]. Excessive use of toxins can lead to different types of pollution such as soil pollution, water pollution, as well as skin and health problems. Our primary goal is to reduce the effects of toxicants on the species. However, predicting or quantifying these particular species is not easy. Apart from this, some biological control strategies can hinder fertilization via releasing sex pheromones or by removing the target species at a certain rate [5]. Boukal and Berec [6], describe two strategies commonly considered in biological control: one being massive pheromone release and the other being constant and consistent efforts leading to the culling effect. They have described the applicability of combining the component Allee effect with mate limitation, inducing multiple Allee effects by releasing enemies in general which can assail the pest population using functional response of type II [7] and a mass introduction of individuals that are sterile within the population [8]. They used a population model including the pest population to predict that complete eradication of the pest could be achieved if the number of males or females or both sexes were brought below a certain value known as threshold density. According to Allee [9], the idea behind the Allee effect is characterized by defining a positive correlation between the population density and per capita growth rate when the population density is low. The Allee effect happens in species because of the complexity of mate discovery, reproduction enhancement, predation, environmental conditioning, and suppression of inbreeding, etc. Populations such as plants [10], birds, and mammals [11] are examples of existential justifications for the Allee effect. Allee effects can be broadly divided into two groups: strong and weak [12, 13]. In the case of strong Allee effects, there is a population threshold at which species below that threshold population density become extinct. On the other hand, weak Allee effects occur when growth rates decline but remain positive at low population densities [14], [15], [16], [17]. Since it might be responsible for the extinction of a species, it is a crucial topic for research. Studying the combination of Allee effects on the dynamics of the predator-prey population may help us better comprehend the diversity of species.

Previously, a lot of work has been done using a prey-predator system with allee effect some of which we discussed above. Some considerable contributions based on mathematical modelling of predator-prey models under distinct conditions can be found in the works of [18],[19], [20]. Saha et. all[21] in their work, proposed a predator-prey model in which the species of prey exhibited anti-predator characteristics to defend themselves from being attacked. They introduced a strong allee effect so that the model becomes more realistic with respect to the environment. Various other researchers have worked on different predator-prey models and have studied their bifurcation analysis under different conditions [22, 23, 24, 25, 26]. Sen et. all [27] in their work discussed the allee effect induced in prey and did a comparative analysis of that with the hunting cooperation on the predator. Using their results, they developed the conditions of stability for the coexistence of the two species. Numerous other researchers used the classical Leslie Gover predator-prey system and discussed various kinds of allee effects to study the dynamics of the system in detail [28, 29, 30]. Emergence of oscillatory Turing patterns induced by cross diffusion in a predator–prey system was studied by [31] and their findings gave more insights and better clarity about the subject of pattern formation in ecology. Similar work was done by [32] focusing on the pattern selection in spatiotemporal system. Sasmal et.all [33] in their work discussed the impact of fear on the efficiency of searching for prey on a predator-free system exhibiting weak allee effect. Han et. all [34] conducted a rigorous analysis of the spatiotemporal pattern selection in a predator-prey model involving a hunting cooperation strategy and allee effect amongst the prey population. Wang [35] also discussed the dynamics of patterns in predator-prey systems with cross-diffusion experiencing an allee effect and employing a general Holling type IV functional response. Significant contributions in the domain of studying the dynamics of predator-prey systems under different forms of allee effects can also be found in the works of [36, 37, 38, 39, 40].

Prey-predator interaction has developed in bioecological modeling following the classical work done by Lotka and Volterra, which was later explored by Holling by introducing several functional responses. Holling types I, II, and III, were the classifications of these functional responses developed through an elaborate analysis of field data. Recently, researchers have made significant contributions to the modeling of more complex situations, with the goal of improving knowledge of reality [41]. In this article, the authors consider a two-species model in which individuals from one population congregate in herds to forage for food and to defend themselves. Similar types of defense mechanisms have been studied in [42, 43]. In scenarios when separate members of a group have different tasks that are distinctly defined, group defenses are normally described in terms of the behavioral responses of specific individuals of the population [44]. The objective of the article is to assess the behavior of the population under the influence of the Allee effect in predator growth rate with the group defense mechanism in prey. According to the author, it is the first attempt to study the impact of the weak Allee effect on a population model of two-species prey and predator along with the group defense mechanism in prey.

The novelty of our work lies in the fact that we are working with a unique predator-prey model with group defense and a weak allee effect. A lot of work has been done in the literature previously with regards to various types of prey-predator models experiencing allee effects [45, 46, 47]. However, no such work can be found that primarily focuses on group defense mechanisms adopted by the prey species under a weak allee effect which highlights the novelty of our work. We observe that this type of problem of optimal control theory for the preservation of endangered species has never been discussed before and hence it can be considered to be a completely new approach. We have done an in-depth discussion and analysis of the combination of different control strategies that will help in controlling the predator population and increase the prey population at the same time. The results that we obtain from our findings suggest that increasing the parameter value of aggregation efficiency under the optimal control strategy won't have any impact on the prey or predator biomass. So our findings can be extremely useful for ecologists and epidemiologists working in this field to establish some new policies and regulations in order to save the endangered species from the risk of extinction.

The physical field quantities used as our control parameters here are primarily the allee effect and the power of the prey for group defense. Some illustrations may include the usage of sterile males to control the population of food flies as they tend to be a major threat to agricultural crops globally. This is one practical example of allee effect and it can be commonly seen in various ecosystems involving a prey-predator interaction. One of the major advantage of this strategy is that It can lower the expansion speed in a given range which will further help in preventing any form of biological invasion. The most evident symptom of the allee effect in any plant or animal species is the limitation in mating which kind contains the predator population thereby, solving the problem of prey extinction to a certain extent. Coming to the next significant control parameter is group defense in the prey population. It can be considered as an anti-predator adaptation technique adopted by the prey population to safeguard themselves. Not much work has been done in literature previously implementing this group defense strategy to control the predator population previously which kind of gives a new dimension to Our research problem. Individual defensive structures have been discussed by researchers in their ability to defend the prey but group defense as a whole has not been widely discussed. In the group defense strategy, we can observe a dilution effect amongst the prey population which kinds of minimizes their risk of being attacked. Shoaling of fish, Camargue horses in Southern France, and water striders are some common examples of prey that employ group defense mechanisms to minimize the attack on themselves and become a greater challenge for the predator population. We have organized our article in the following manner: We have developed our model of study in Section (2); in Section (3), we discuss the model population dynamics. The optimal control policy for the predator is discussed in Section (4). In Section (5), we have done the Numerical simulations followed by a final discussion in Section (6).

2. MODEL FORMULATION

When there are no predators present, we can study the population dynamics of the prey species as described below:

$$\frac{dS}{dt} = aS(1-bS). \tag{1}$$

Here the population of prey is given by S; a describes the birth rate of prey and the prey population carrying capacity is given by $\frac{1}{b}$. When the predators are present in the population, g(S) is the rate at which they consume the prey. g(S) can be represented using the Holling-Type I or II or III functional response.

We can use the following system of nonlinear differential equations to demonstrate a two-species prey-predator model with the predator population exhibiting weak Allee effect [48, 49, 50, 51, 52]:

$$\begin{aligned} \frac{dS}{dt} &= aS(1-bS)-g(S)P \\ \frac{dP}{dt} &= P(cg(S)\frac{P}{\theta+P}-d) \end{aligned} \tag{2}$$

Here, d denotes the predator's natural death rate and $c \in (0;1]$ the rate of conversion of prey biomass into the predator biomass.

In this article, the authors focus on the case when $g(S) = \beta S^{\gamma}$. Venturino and Petrovskii [53], defined a functional response using γ where $\gamma \in (0,1)$ denotes the prey power, thereby describing the group defense mechanism of the prey species. The fundamental contention is that the prey that inhabit the outermost locations in the group are primarily affected by the interactions between predators and prey. In terms of modeling, the assumption has the effect of replacing the traditional term representing mass action found in the Lotka-Volterra model with an unorthodox component in a non-linear form that holds a power law of the prey density. Using this assumption, we can modify the system (2) as:

$$\begin{aligned} \frac{dS}{dt} &= aS(1-bS)-\beta S^{\gamma}P \\ \frac{dP}{dt} &= P(cPS^{\gamma}\frac{\beta}{\theta+P}-d) \\ &= P(S^{\gamma}\frac{\alpha P}{\theta+P}-d) \end{aligned} \tag{3}$$

$S(0) > 0, P(0) > 0$.

We have defined our model in such a manner that the group defense mechanism in prey is used as a significant control strategy and the functional response, denoted by γ is defined by $\gamma \in (0,1)$ to describe the power of the prey. The physical significance with respect to the application of the form of the model proposed for our study lies in the fact that the prey occupying the outermost regions of the prey group are the primary targets of the predators and hence are most prone to be attacked and face the consequences of this predator-prey interaction. Keeping in mind this arrangement and the key role played by the prey inhabiting the boundary positions we have developed our model using a nonlinear function for the power of prey which is exclusive from the classical predator-prey model descriptions. Thus, it helps the researchers and ecologists in a much better analysis of the subject under study. So under those physical circumstances where one needs to control the predator-prey population employing a weak allee effect and group defense technique in prey, this model proposition that we have used for our study becomes extremely vital and relevant.

3. MATHEMATICAL ANALYSIS OF THE SYSTEM

The author first studies the system (3) with $\gamma = 1$. So we can write the system (3) as

$$\begin{aligned} \frac{dS}{dt} &= aS(1-bS) - \beta SP \\ \frac{dP}{dt} &= P(\frac{\alpha}{\theta+P}PS - d) \end{aligned} \tag{4}$$

The boundary equilibrium points obtained for the above system are as follows:

$$E_0 = (0,0),\ E_1 = (\frac{1}{b},0).$$

An interior attractor $E^* = (S^*, P^*)$ is obtained for the system (4). The values of S^* and P^* are the positive roots obtained from the system of equations

$$\begin{aligned} &\beta P^* = a(1-bS^*) \\ &\alpha S^* P^* = d(\theta + P^*). \end{aligned} \tag{5}$$

We can determine P^* by solving the equation $\frac{\alpha\beta}{a}P^2 + p(bd-\alpha) + bd\theta = 0$. If the condition $S^* < \frac{1}{b}$, $bd - \alpha < 0$ and $(bd-\alpha)^2 > \frac{4bd\alpha\theta\beta}{a}$ is satisfied then the interior equilibrium point will exist.

Proposition[Local stability of the equilibrium points for the Model (4)**]:**

In Table (1) we can see the summarized results describing the local stability of the equilibrium points for our given model.

Proof. For the model (4) the Jacobian matrix at any given equilibrium point (S_*, P_*) can be represented in the following manner:

Equilibrium points	Existence Condition	Stability Condition
E_0	Always exists	Saddle
E_1	Always exists	Always stable
E^*	$S^* < \frac{1}{b}$, $\alpha > bd$ and $(bd-\alpha)^2 > 4bd\theta\frac{\alpha\beta}{a}$	locally asymptotically stable if $ab > \frac{\alpha\theta P^*}{(\theta+P^*)^2}$

TABLE 1. The local stability of equilibrium points for the Model (4)

$$J|_{(S_*,P_*)} = \begin{bmatrix} a-2abS_*-\beta P_* & -\beta S_* \\ \frac{\alpha P_*^2}{\theta+P_*} & \frac{\alpha S_* P_*(2\theta+P_*)}{(\theta+P_*)^2} - d \end{bmatrix}$$

On substituting $(S_*,P_*) = E_i, i = 0,1$ into (6), we can find the eigenvalues for the respective equilibrium points.

(1) The eigen values for the equilibrium point $E_0 = (0,0)$ obtained from the equation (6) are

$$\lambda_1 = a\ (>0),\ \lambda_2 = -d\ (<0).$$

This shows that $E_0 = (0,0)$ is always unstable.

(2) The eigen values for the equilibrium point $E_1 = (\frac{1}{b},0)$ obtained from the equation (6) are

$$\lambda_1 = -a\ (<0),\ \lambda_2 = -d(<0).$$

This shows that $E_1 = (\frac{1}{b},0)$ is locally asymptotically stable.

Now for the interior equilibrium point given by $E^* = (S^*,P^*)$, the Jacobian matrix of our model (4) under study can be represented as below:

$$J|_{E^*=(S^*,P^*)} = \begin{bmatrix} -abS^* & -\beta S^* \\ \frac{\alpha P^{*2}}{\theta+P^*} & \frac{\alpha\theta S^* P^*}{(\theta+P^*)^2} \end{bmatrix}$$

The characteristic equation of the above matrix is given by:

$$\lambda^2 + (abS^* - \frac{\alpha\theta S^* P^*}{(\theta+P^*)^2})\lambda + \frac{\alpha\beta S^* P^*}{\theta+P^*}P^* = (\lambda_1-\lambda)(\lambda_2-\lambda) = 0. \tag{6}$$

where λ_i, $i = 1,\ 2$ are roots of equation (8). If the solutions $\lambda_i, i = 1,2$ have negative real parts, then we have

$$\begin{aligned} \Sigma_{i=1}^{2}\lambda_i &= (abS^* - \tfrac{\alpha\theta S^* P^*}{(\theta+P^*)^2}) > 0, \\ \Pi_{i=1}^{2}\lambda_i &= \tfrac{\alpha\beta S^* P^*}{\theta+P^*}P^* > 0. \end{aligned}$$

Performing some mathematical simplification we can establish the locally asymptotic stability of the interior equilibrium provided the condition stated below holds true:

$$ab > \tfrac{\alpha\theta P^*}{(\theta+P^*)^2}. \tag{7}$$

□

We shall consider the values of the parameters as follows: $\beta = 0.60$; $d = 0.1$; $a = 0.60$; $\alpha = 0.60$; $\theta = 0.1$; $b = 1.66$. For these specified values of the parameters above, the system (4) under study possesses two interior equilibrium of which one is a stable focus $E^* = (0.19, 0.68)$ another is unstable given by $E_1^* = (0.58, 0.02)$. This is clear from the fact that all the trajectories that start within the attractive region tend to converge to the equilibrium point $E^* = (0.19, 0.68)$ (Figure (1)). Various other sets of initial values are considered as $[0.57, 0.05]$, $[0.25, 0.90]$, $[0.7, 0.6]$ and the corresponding phase plane portrait diagram of the system (4) is displayed in Fig.1.

3.1. Saddle node bifurcation: The two equilibrium points E^* and E_1^* will exist if $(bd-\alpha)^2 > 4bd\theta\frac{\alpha\beta}{a}$. For $(bd-\alpha)^2 < 4bd\theta\frac{\alpha\beta}{a}$, E_1^* and E^* vanish. The existence of the two equilibrium points depends on the value of θ. The saddle-node bifurcation can be observed between E_1^* and E^* if the given condition holds

$$\frac{(bd-\alpha)^2}{4bd\theta\frac{\alpha\beta}{a}} = \theta.$$

3.2. Bistability: The phenomena describing the convergence of the system to two distinct equilibrium points within a common parametric region is known as Bi-stability. It depends on variation in the initial conditions. In the existence of the two interior equilibrium points E^* and E_1(see Figure (2)) we can clearly see that the system (4) experiences bi-stability. Whenever the interior equilibrium points exist, all trajectories that rise from the interior region of R_+^2 will either converge towards the equilibrium point E^* and E_1. This non-delayed model described in the system (4) possesses two attractors of which one is E^* while the other is E_1 which is locally

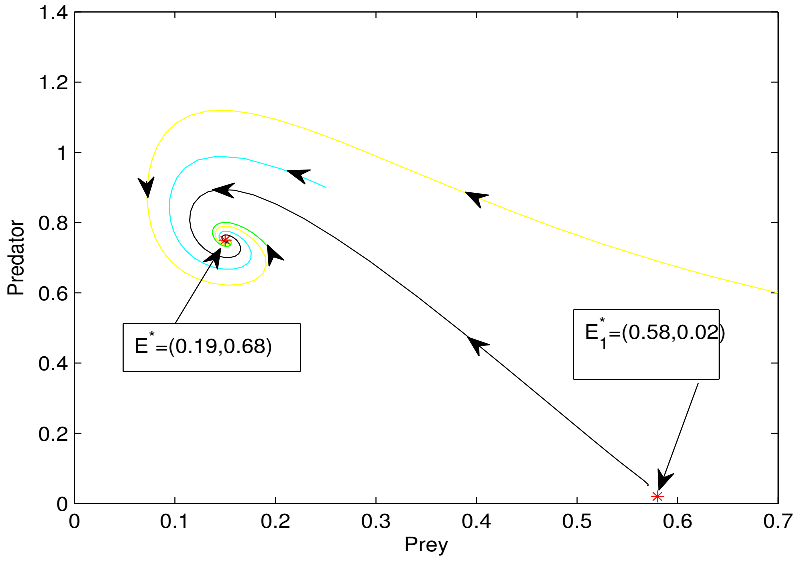

FIGURE 1. Stability of the interior equilibrium for the model (4), when $\beta = 0.60$, $b = 1.66$, $d = 0.10$, $a = 0.60$, $\alpha = 0.60$ and $\theta = 0.1$, with different initial conditions.

asymptotically stable. The region of bi-stability for the two interior equilibria E^* and $E_1$is displayed in Figure (4) when we vary the two parameters θ and d. The bistability of this type (see Figure (2)) is quite significant as depending on the given initial conditions it determines whether the predator will become extinct or both the species will coexist.

3.3. Hopf bifurcation: With reference to θ i.e. the Allee parameter, the Hopf bifurcation description for the system (4) is displayed in Figure (3). (the author omits the proof here as a similar kind of theorem and proof is given in (3.2)). We can see at $\theta = 0.62$, the Hopf bifurcation takes place. On further increasing θ, a limit point is obtained at $\theta = 3.32$. We can observe that there is an eigenvalue having the real part zero and the equilibrium point has a turning point such that it is stable on one side and unstable on the other. A phenomenon known as hysteresis or bistability is observed in Figure (3) which is an important characteristic of our model under study.

Figure (3) describes the Hopf bifurcation scenario of the system (4) with respect to Allee parameter θ (the author omits the proof here as a similar kind of theorem and proof is given in (**??**)). The Hopf bifurcation occurs at $\theta = 0.62$. Further increment of θ, results in a limit point

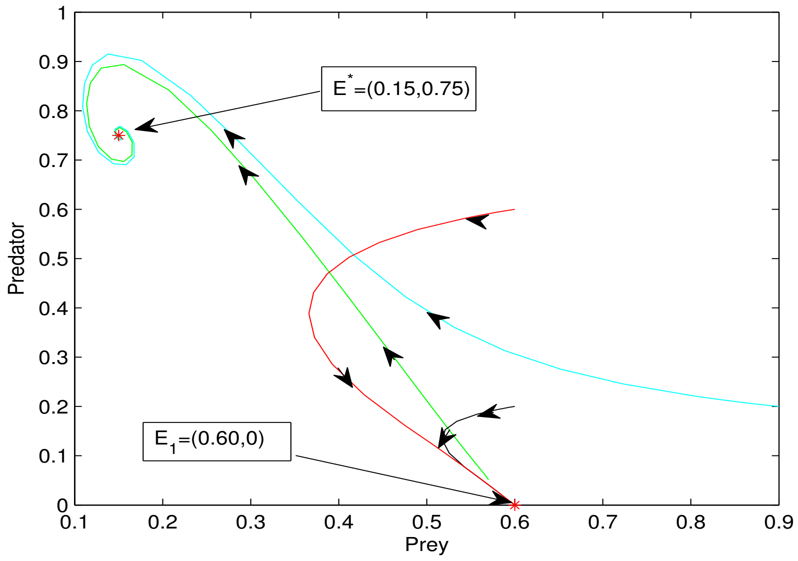

FIGURE 2. Bi-stability of the interior equilibrium E^* and E_1 for the model (4).

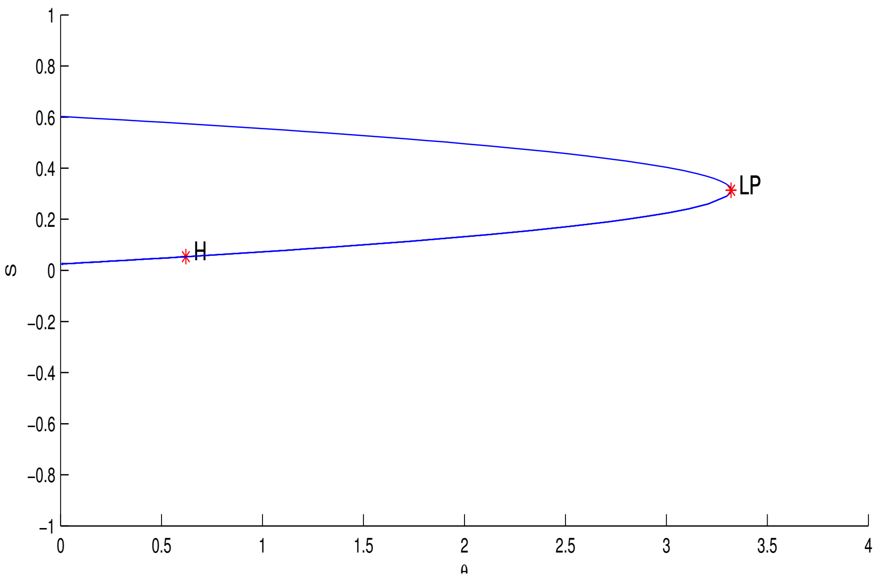

FIGURE 3. Bifurcation diagram with respect to θ for the model (4), when $\beta = 1$, $\alpha = 4$, $a = 0.60$ and $b = 1.66$.

at $\theta = 3.32$. Here one eigenvalue has zero real part and the curve of equilibria has a turning point, on one side it is stable, and on the other unstable. Figure (3) also showed a phenomenon called hysteresis or bistability. This is an important feature of our model (4).

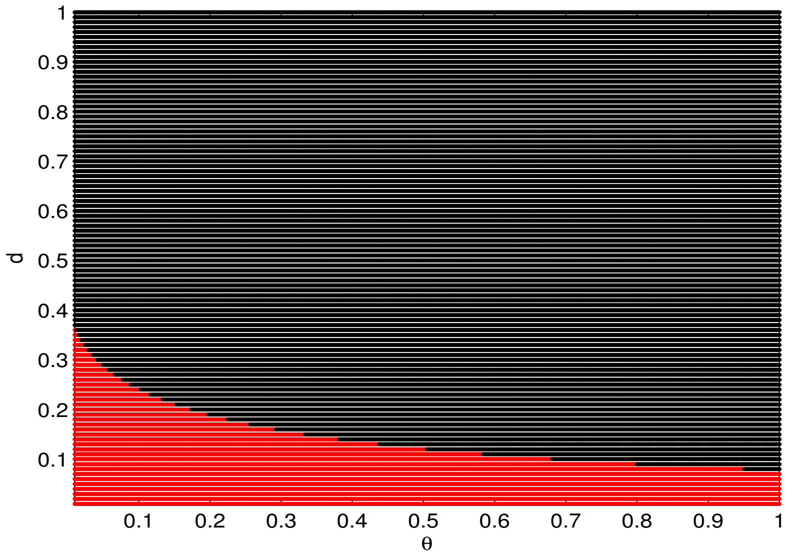

FIGURE 4. Bi-stability region of the interior equilibrium E^* and E_1 for the model (4), when $\beta = 0.60$, $a = 0.60$, $\alpha = 0.60$ and $b = 1.66$. In the red region, both the equilibrium points E^* and E_1 are locally asymptotically stable, while in the black region only E_1 is locally asymptotically stable but E^* is unstable.

Next, the author draws a bifurcation phase plane diagram of two parameters θ and d for the system (4). Matcont identified two points, which correspond to codim 2 bifurcations known as Bogdanov-Takens (BT) in Figure (5). At each of these BT points the system possesses an equilibrium point having a double zero eigenvalue.

3.4. Mathematical analysis of the full system (3)**:** In this section, the local stability of the equilibrium points for the differential equation (3) is studied and analyzed in detail.

The system (3) has the following boundary equilibrium points:

$$E_0^* = (0,0),\ E_1^* = (\frac{1}{b},0).$$

The system (3) has interior attractor $E_2^* = (S_2^*, P_2^*)$, where S_2^* and P_2^* are obtained by calculating the positive roots of the following equation:

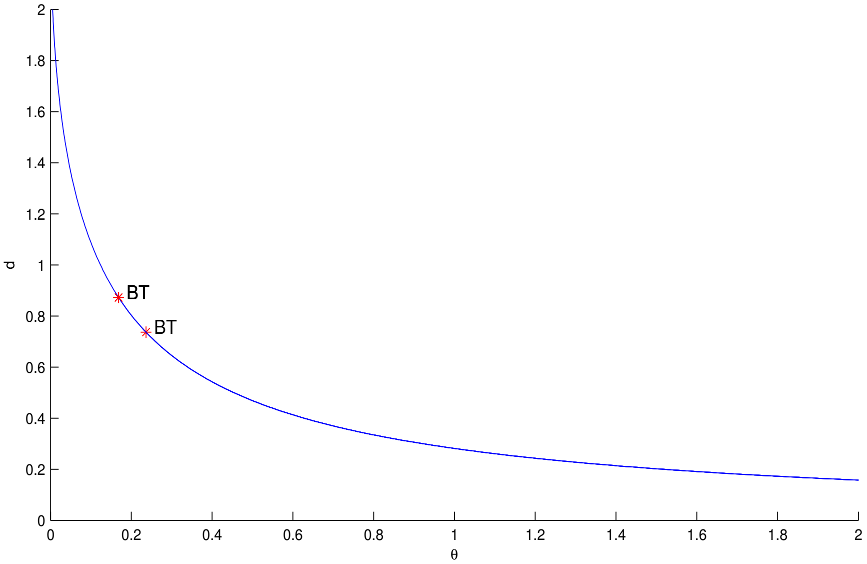

FIGURE 5. Two parameter bifurcation diagram with respect to θ and d for the model (4), when $\beta = 1$, $\alpha = 4$, $a = 0.60$ and $b = 1.66$.

$$\begin{aligned} \beta(S_2^*)^\gamma P_2^* &= aS_2^*(1-bS_2^*) \\ \alpha(S_2^*)^\gamma P_2^* &= d(\theta+P_2^*). \end{aligned} \tag{8}$$

The interior equilibrium exists if $\frac{d}{\alpha} < (S_2^*)^\gamma$ and $S_2^* < \frac{1}{b}$.

By analyzing the eigenvalues obtained from the Jacobian matrix given below we can very easily determine the nature of the local stability of the equilibrium points of E_1^* and E_2^*.

$$J = \begin{bmatrix} a - 2abS_* - \gamma\beta P_* S_*^{(\gamma-1)} & -\beta S_*^\gamma \\ \gamma \frac{\alpha P_*^2}{\theta+P_*} S_*^{(\gamma-1)} & \frac{\alpha S_*^\gamma P_*(2\theta+P_*)}{(\theta+P_*)^2} - d \end{bmatrix}$$

Because of the singularity property of the Jacobian matrix J, it cannot be used to establish the stability conditions for ${E_0}^*$. But despite that, we can use the concept of [43], to study the singularity dynamics near the origin of our proposed model (3) in the following manner:

Theorem 1. *Consider* $0 < \gamma < 1$. *let us take the initial conditions in the form* $S_0 = S(0), P_0 = P(0)$ *for the system of equations* (3). *Also, let us consider* G *according to the definition:* $G = (S,P), S > 0, P > P'(S)$, *to be a necessary part of the phase plane of our system under study, where* $P'(S) = \frac{d+(1-\gamma)a}{\beta(1-\gamma)} S^{1-\gamma}(S)$, *where* $P'(S) = d + (1-\gamma)a\beta(1-\gamma)S(1-\gamma)$.*Then for*

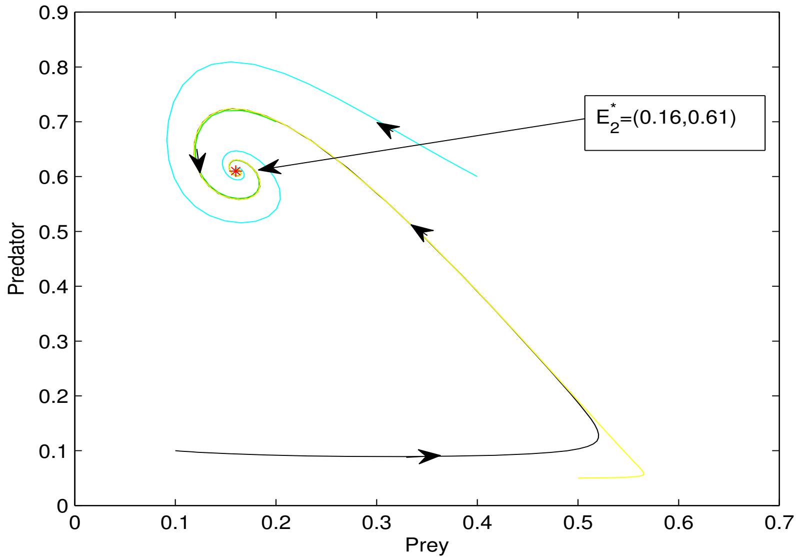

FIGURE 6. Stability of the interior equilibrium for the model (3), when $\beta = 0.60$, $b = 1.66$, $d = 0.10$, $a = 0.60$, $\gamma = 0.90$, $\alpha = 0.60$ and $\theta = 0.10$, with different initial conditions.

any $(S_0, P_0) \in G$ *the trajectory of the system intersects the axis* P *at a specified time* $t^1 < \infty$ *implying the extinction of the prey over a finite period of time.*

Proof. The proof is the same as in [53].From (3.1), it can be observed that $E_0^* = (0,0)$ behaves like a saddle point [53, 54] for the system (3). The boundary equilibrium point E_1^* is always stable and E_2^* is stable if $ab(S_2^*)^{(2-\gamma)} - S_2^*\alpha\frac{P_2^*\theta}{(P_2^*+\theta)^2} - \beta(1-\gamma)P_2^* > 0$.

Now let us take the following set of values for our parameter: $\beta = 0.60$; $d = 0.1$; $a = 0.60$; $\alpha = 0.60$; $\theta = 0.1$; $b = 1.66; \gamma = 0.90$. The system (3) for this particular set of parameter values possesses a stable interior equilibrium point; $E_2^* = (0.16, 0.61)$ is a stable focus since we can observe for all the trajectories that are initiating within this particular region of attraction eventually tend to converge to $E_2^* = (0.16, 0.61)$ (Figure (6)). We have taken various different sets of initial values as follows: $[0.57, 0.05]$, $[0.25, 0.90]$, $[0.7, 0.6]$. We have then drawn their respective phase plane portrait with respect to the system (3) in Figure (6).

$\square$

3.5. The impact of the Allee effect: The system (4) in the absence of the Allee effect can be modeled as follows:

$$\begin{aligned} \frac{dS}{dt} &= aS(1-bS)-\beta SP \\ \frac{dP}{dt} &= P(S\alpha - d) \end{aligned} \tag{9}$$

The above-defined system (12) possesses two equilibrium points in the boundary which are (0,0), $(\frac{1}{b},0)$. Another equilibrium point $(\frac{d}{\alpha},\frac{a(1-b\frac{d}{\alpha})}{\beta})$ is found in the interior and it is unique. The existence of the unique equilibrium point depends on if $\frac{1}{b} > \frac{d}{\alpha}$ and by nature it is asymptotically stable whenever it exists (Figure (7)). '

Similarly, in the absence of Allee effect the proposed system (3) also has a unique interior equilibrium point $((\frac{d}{\alpha})^{\frac{1}{\gamma}}, a(\frac{d}{\alpha})^{\frac{1-\gamma}{\gamma}}(1-b(\frac{d}{\alpha})^{\frac{1}{\gamma}}))$ and asymptotically stable if $a\frac{1-\gamma}{2-\gamma} < (\frac{d}{\alpha})^{\frac{1}{\gamma}}$ (see Figure (8)).

Theorem 2. *Assume that* $a\frac{1-\gamma}{2-\gamma} < (\frac{d}{\alpha})^{\frac{1}{\gamma}}$*, then model (3) without Allee effect does not have any nontrivial periodic orbits in* $\mathbb{R}^2_+$*. Moreover whenever* $a\frac{1-\gamma}{2-\gamma} > (\frac{d}{\alpha})^{\frac{1}{\gamma}}$ *, then model (3) without Allee effect has possesses exactly one single limit cycle which is stable in nature.*

Proof. The proof is the same as in [54]. □

The system (4), in the absence of the Allee effect, comprises just one interior equilibrium point and this is stable in nature whenever it exists. On the contrary, when the Allee effect is present in the system, the existence of either no equilibrium point or exactly two interior equilibrium points is observed based on the value of θ which is the Allee parameter. Through the above description of the dynamics of the system, we can deduce a significant result that the Allee effect plays a vital role in the generation or destruction of interior attractors. Also, the system can be destabilized because of the non-linearity which may be induced in the presence of the Allee effect.

3.6. Boundedness of the solution of the system (3): Here, we have established the conditions under which the solutions of the proposed system (3) will be bounded. The author denotes by $\mathbb{R}^2_+ = \{(S,P) \in \mathbb{R}^2 : S \geq 0, P \geq 0\}$ the non-negative quadrant and by $int(\mathbb{R}^2_+) = \Big\{(S,P) \in \mathbb{R}^2 : S > 0, P > 0\Big\}$.

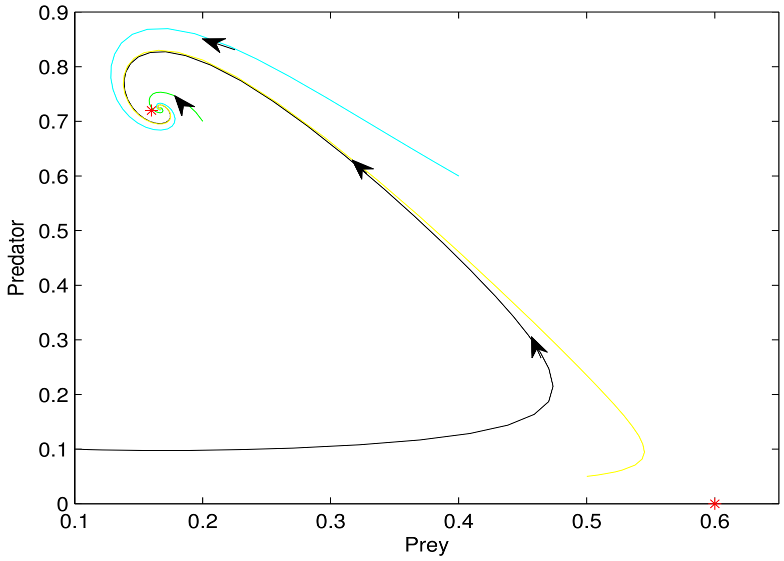

FIGURE 7. Existence of stable interior equilibrium for the model (9), when $\beta = 0.60$, $b = 1.66$, $d = 0.10$, $a = 0.60$ and $\alpha = 0.60$ with different initial conditions.

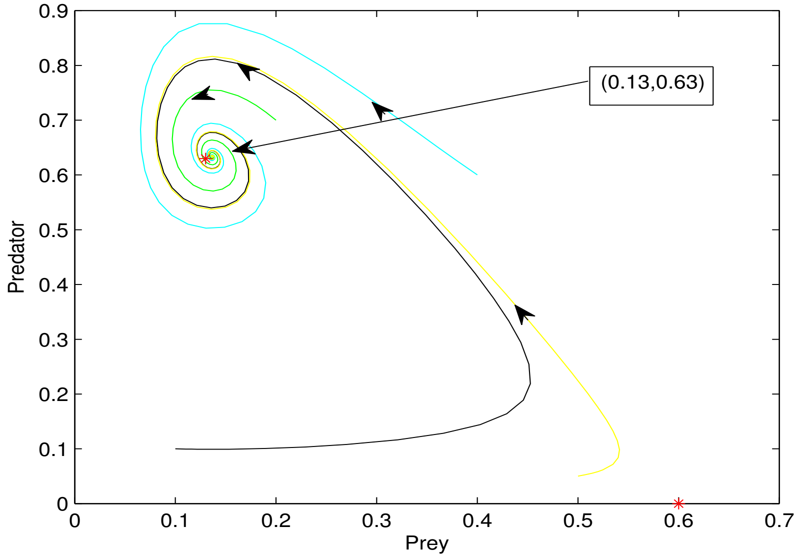

FIGURE 8. Existence of stable interior equilibrium for the model (3) with out Allee effect, when $\beta = 0.60$, $b = 1.66$, $d = 0.10$, $a = 0.60$ and $\alpha = 0.60$ with different initial conditions.

The first equation of the system of equations (3) can be rewritten as

$$\frac{dS}{S} = aI - bS - \beta PS^{\gamma-1}dt$$

Taking the lower and upper limits 0 and t respectively and integrating them, we shall get,

$$S(t) = S(0)\exp\int_0^t Sal - bS - \beta PS^{\gamma-1}ds.$$

In a similar manner using the second equation from the system of equations (3) we can obtain,

$$P(t) = P(0)\exp\left\{\int_0^t \alpha S^{\gamma}\frac{P(s)}{\theta+P} - dds\right\}.$$

The initial conditions being $S(0) = S_0 > 0$ and $P(0) = P_0 > 0$. Thus, S(t) and P(t) are both strictly positive.

Lemma 3. *The solutions of the system (3) that begin in the region* $int(\mathbb{R}^2_+)$ *are bounded uniformly having M as an ultimate bound.*

Proof. Rewriting the first equation of the model (3) as

$$\frac{dS}{dt} = Sa1 - bS - \beta PS^{\gamma-1}.$$

Thus,

$$\frac{dS}{dt} \leq aS(1-bS)$$
$$\therefore \limsup_{t\to\infty} S(t) \leq \frac{1}{b}.$$

Let us consider a function V defined as follows: $V = \frac{1}{\beta}S(t) + \frac{1}{\alpha}P(t)$. Differentiating V with respect to the time t in line with the solution of the system of equations (3) we get,

$$\begin{aligned}\dot{V} &\leq \frac{a}{\beta}S(t)(1-bS(t) - \frac{d}{\alpha}P(t),\\ &\leq \frac{a}{b^2\beta} - \min\{a,d\}V.\end{aligned}$$

So, we obtain,

$$\lim_{t\to\infty} V(t) \leq M_1.$$
$$\text{Where } M_1 = \frac{\frac{a}{b^2\beta}}{\min\{a,d\}}.$$

Here M1 is given by $M_1 = \frac{\frac{a}{b^2\beta}}{\min\{a,d\}}$
and we have taken $M = \max\{\frac{1}{b}, \alpha M_1\}$.

□

4. Optimal Control

4.1. Optimal control policy. The dynamical characteristics of the system of equations (3) have been discussed in detail in the above section. Following this, in order to curb the predator population, the author employs certain control parameters which include the inverse of individual searching efficiency or the Allee parameter θ and nonselective culling with a per capita rate of $h(t) > 0$ as a control parameter to reduce predator population. The time interval within which the controls are employed is [0; T], with T denoting the final or end time.

Now the modified non-linear system of differential equations describing the impacts of the various controls on our standard model (3) is represented in the following manner:

$$\begin{aligned} \frac{dS}{dt} &= aS(1-bS) - \beta S^{\gamma} P \\ \frac{dP}{dt} &= P(S^{\gamma}\frac{\alpha P}{\theta(t)+P} - d - h(t)). \end{aligned} \tag{10}$$

For our optimal control problem under consideration, we can write the objective functional as follows:

$$J(E) = \int_0^T (AP(t) + B\theta^2(t) + Ch^2(t))dt,$$

which is subject to the system of equations (13). The primary objective behind this problem is to obtain θ_0 and h_0 i.e. the optimal controls such that

$$J(\theta_0, h_0) = \min_{(\theta,h)\in U} J(E)$$

, U being the set of controls which we define in the following way,

$$U = \{(\theta, h) : (\theta, h) \; is \; measurable \; and \; 0 \leq \theta \leq \theta_{max} \; and \; 0 \leq h \leq h_{max}\}.$$

Here, A, B, and C are the respective weights for the predator population, the Allee parameter, and the culling effect.

Let, $H = AP(t) + B\theta^2 + Ch^2 + \lambda_S[aS1 - bS - \beta S^{\gamma} P] + \lambda_P[\alpha PS^{\gamma}\frac{P}{\theta+P} - dP]$.

The adjoint variables in the objective function are λ_S, λ_P and the conditions for transversality are $\lambda_S(T) = 0$, $\lambda_P(T) = 0$.

Theorem 4. *An optimal control denoted by* (θ_0, h_0) *exists for any* $t \in [0, T]$ *which satisfies*

$$J(S(t), P(t), \theta_0, h_0) = \min_{(\theta,h)\in U} J(S(t), P(t), \theta, h)$$

which is subject to the system of differential equations (13). Also, adjoint variables λ_S, λ_P exists with the transversality conditions as $\lambda_S(T) = 0$, $\lambda_P(T) = 0$.

4.2. Characterization of the Optimal Control. We have used Pontryagin's maximum principle in order to deduce the necessary conditions for optimal control (13). Here the state equations are:

$x(t) = H(x,\theta,h,\lambda)(t)$.

The conditions for optimality are

$0 = H_\theta(x,\theta,h,\lambda)(t)$, and $0 = H_h(x,\theta,h,\lambda)(t)$.

The adjoint equation is given by,

$-\frac{d\lambda}{dt}(t) = H_x(x,\theta,h,\lambda)(t) + \chi_{[0\ T-\tau]}\lambda(t+\tau)H_{x_\tau}(x,x_\tau,\theta,\lambda)(t)$,

Here H_θ, H_x and H_{x_τ} respectively denote the derivatives with respect to θ, x and x_τ, respectively. $\widehat{\theta}$ being the positive root of the equation given below, one can obtain the optimal control conditions using the equation

$$(\frac{\theta}{P})^3 + (\frac{\theta}{P})^2 + (\frac{\theta}{P}) - \frac{\alpha S^\gamma \lambda_P}{BP} = 0.$$

The constant term being negative guarantees that at least one positive root will exist.

At any given time t, the optimal Allee parameter is obtained by,

$$\theta_0 = \begin{cases} 0, \ \frac{dH}{d\theta} < 0 \\ \widehat{\theta}, \ \frac{dH}{d\theta} = 0 \\ \theta_{\max}, \ \frac{dH}{d\theta} > 0 \end{cases} \tag{11}$$

Also,

$$\widehat{h} = \frac{\lambda_P P}{2C}.$$

The optimal culling effect at any time t is given by

$$h_0 = \begin{cases} 0, \ \frac{dH}{dh} < 0 \\ \widehat{h}, \ \frac{dH}{dh} = 0 \\ h_{\max}, \ \frac{dH}{dh} > 0 \end{cases} \tag{12}$$

The adjoint equations are

$$
\begin{aligned}
\frac{d\lambda_S}{dt} &= -\lambda_S(t)a(1-2bS) - \beta\gamma S^{\gamma-1}P - \lambda_P(t)\alpha\gamma S^{\gamma-1}P\tfrac{P}{\theta+P}, \\
\frac{d\lambda_P}{dt} &= -A - \lambda_P(t)(\alpha S^{\gamma}\tfrac{P^2+2P\theta}{(\theta+P)^2} - d - h) + \lambda_S(t)\beta S^{\gamma}.
\end{aligned}
\tag{13}
$$

5. NUMERICAL SIMULATION

We have obtained the numerical solution of our optimal control problem using the fourth-order Runge-Kutta forward - backward numerical scheme.

For the numerical illustration, the values of the parameters chosen are as follows: $\beta = 0.60$; $d = 0.10$; $a = 0.60$; $\alpha = 0.60$; $b = 1.66$; $\gamma = 0.90$; $T = 50$; $A = 1$; $B = 1$;$C = 1$; $S(0) = 0.20$; $P(0) = 0.70$ and assume that $0.20 \leq \theta \leq 0.50$, $0 \leq h \leq 0.20$.

5.1. System (3) with Allee parameter but without culling effect.

In figures (9) & (10) the solution curves are represented for the two-state variables when the Allee control parameter is present but without the culling effect. Without the Allee parameter or any control, the system (13) is stable at (0.13,0.63) (see Figure (8)). At $t = 600$ units of time, the predator population reaches its minimum value and the prey population reaches its maximum value but after that predator population increases, and the prey population decreases. Figure (11) shows the variation of optimal Allee parameter θ.

5.2. System (3) with culling effect but without Allee Parameter.

Under this strategy, for optimization of our objective function J, the culling effect (h) is employed while the Allee parameter $\theta = 0$. Comparing this scenario with that when no control parameters were present in the system (see Figure (8)), one can clearly see that the prey population (Figure (12)) increases and the predator population (Figure (13)) decreases. At t=425 units of time predator population is extinct from the system and the prey population reaches its maximum value. The control profile is shown in Figure (14). It can be observed that the optimal culling effect reaches its upper bound at t=200 units and it converges to its lower bound at t= 650 units of time. Through this, we can conclude that the use of the culling effect requires low effort.

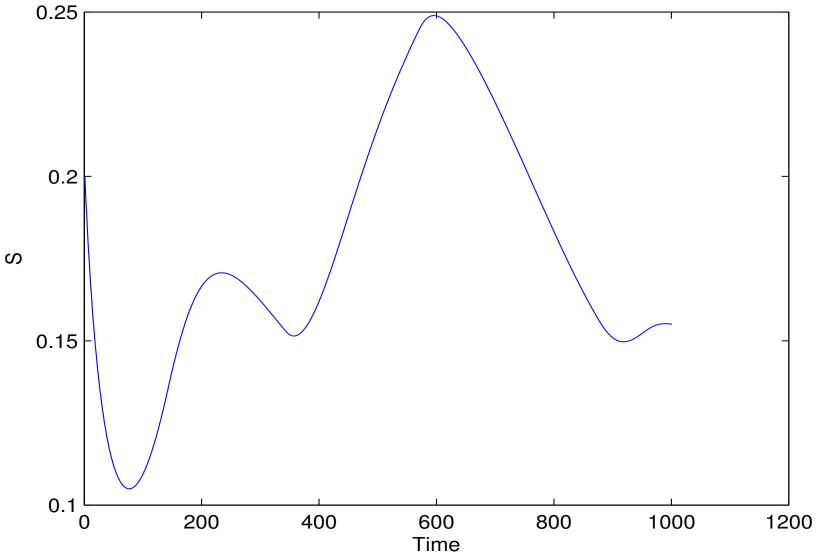

FIGURE 9. Diagram for the prey population with Allee parameter but without culling effect

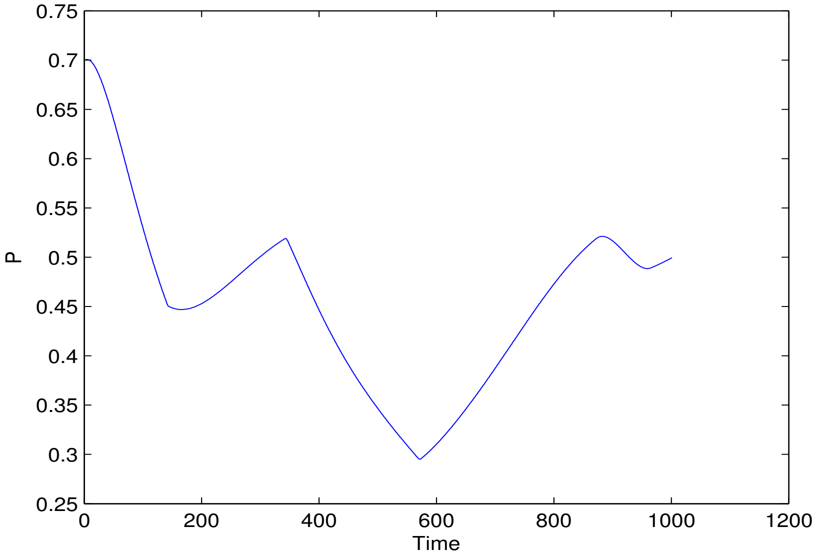

FIGURE 10. Diagram for the predator population with Allee parameter but without culling effect

5.3. System (3) with both the control parameters (Allee parameter and culling effect).

Under this strategy, in order to optimize the objective function J both the controls θ and h have been used. The solution curves depicted in Figures (15) and (16) represent the solution for these

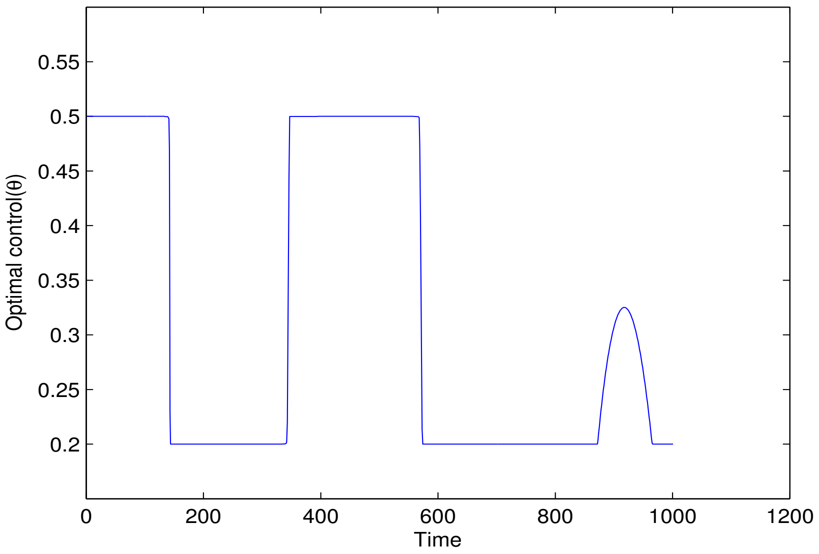

FIGURE 11. Variation of the θ with respect to time in the absence of a culling effect.

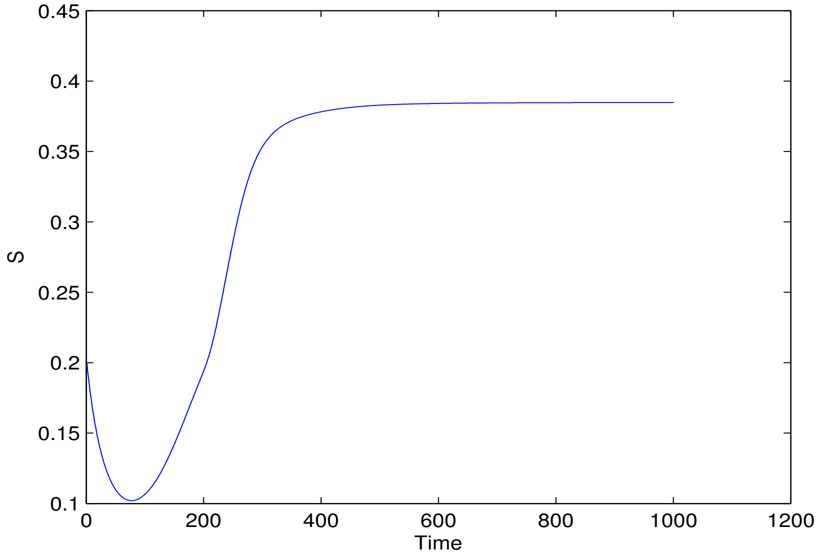

FIGURE 12. Diagram for the prey population with culling effect but without Allee Parameter

two state variables when the two control parameters are present in the system. On applying the optimal control we can observe that comparatively quite a significant number of predators are reduced than that in the absence of any control. Without any control, the system (13) is stable at (0.13,0.63) (see Figure (8)). At $t = 200$ units of time, it is clearly evident that the predator

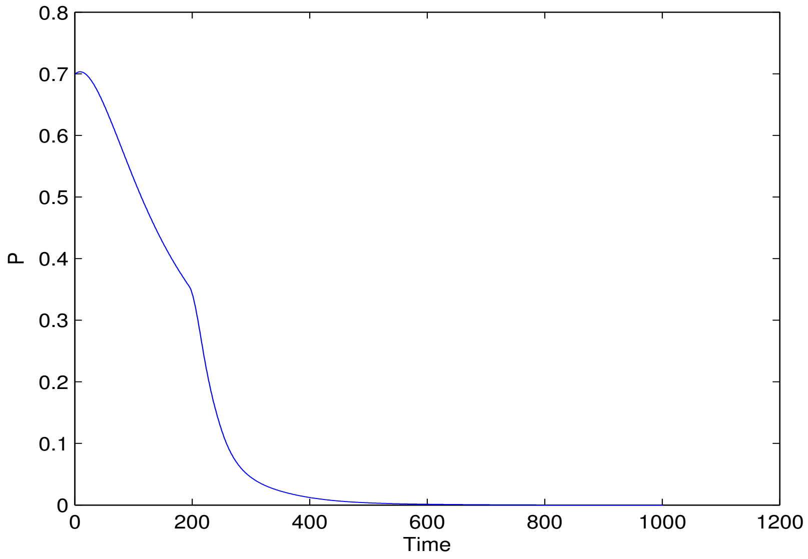

FIGURE 13. Diagram for the predator population with culling effect but without Allee Parameter

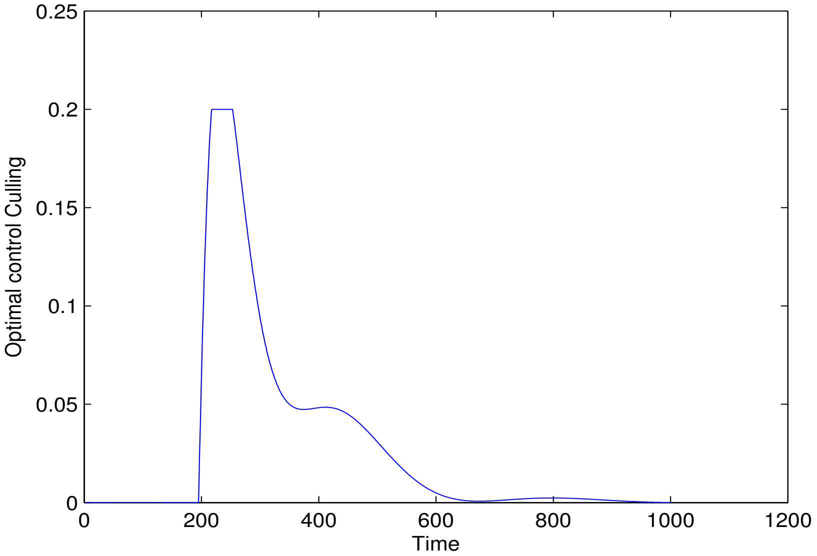

FIGURE 14. Variation of the *h* with respect to time in the absence of Allee parameter

population has become extinct. As an application of optimal control, it may be concluded that optimal control plays a significant role in not just reducing the predator population in the system but it simultaneously increases the population of the prey. Figures (19) and (20) represent the variation of optimal control and in figures (17) & (18), the author describes how the adjoint

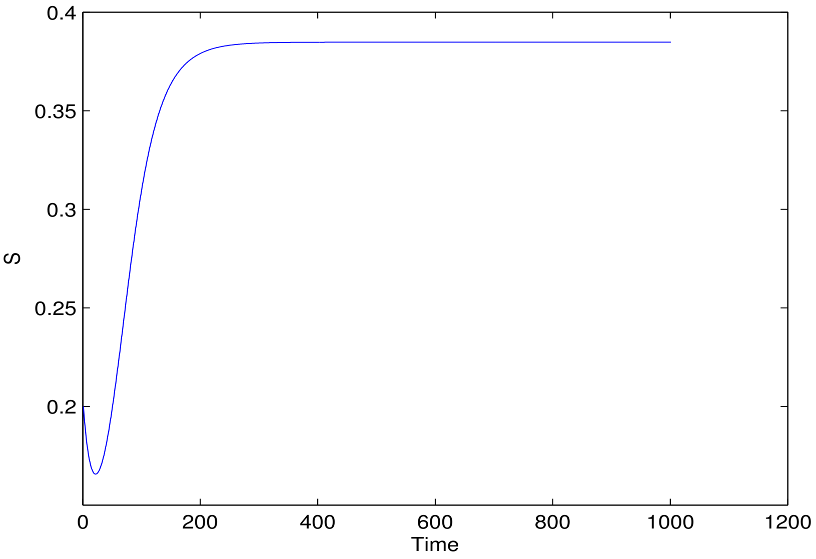

FIGURE 15. Diagram for the prey population with both the controls

variables vary when the control parameters are present. It is observed that the optimal condition for the two controls, the Allee parameter, and the culling effect is reached if they can be used at their highest level for approximately the first 990 units of time and 620 units of time respectively. Increasing the parameter value of γ, under the same control strategy as discussed earlier is not going to change the predator or prey biomass. If we increase aggregation efficiency (γ), first predator biomass decreases as a result prey biomass increase but after t=200 unit of time, all the solutions of the system (13) converges to the same value (Figure (21) & (22)).

6. DISCUSSION

This paper focuses on the impact of the weak Allee effect on predators in the prey-predator model. On the basis of model (3), the author provides with results about the local stability of the equilibrium points. The existence of saddle-node bifurcation between the equilibrium points E_1^* and E^* has also been discussed. On the basis of model (4), the existence of Hopf bifurcation has been studied in detail together with the notion of bistability between the interior equilibrium points. The standard model (4) shows that the origin is a saddle point, whereas, for the model (3), the behavior of the solution near the origin equilibrium is singular in nature. It is clearly evident that the model (3) encounters Hopf-bifurcation as the prey density $< a\frac{1-\gamma}{2-\gamma}$.

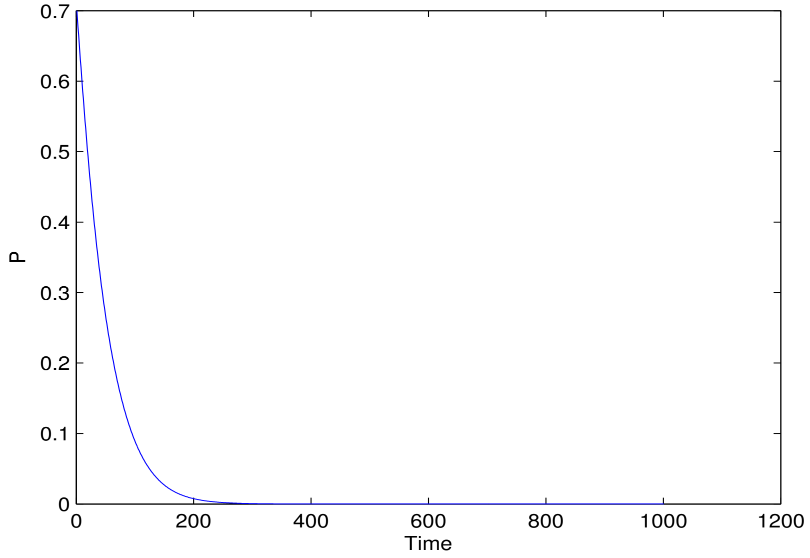

FIGURE 16. Diagram for the predator population with both the controls

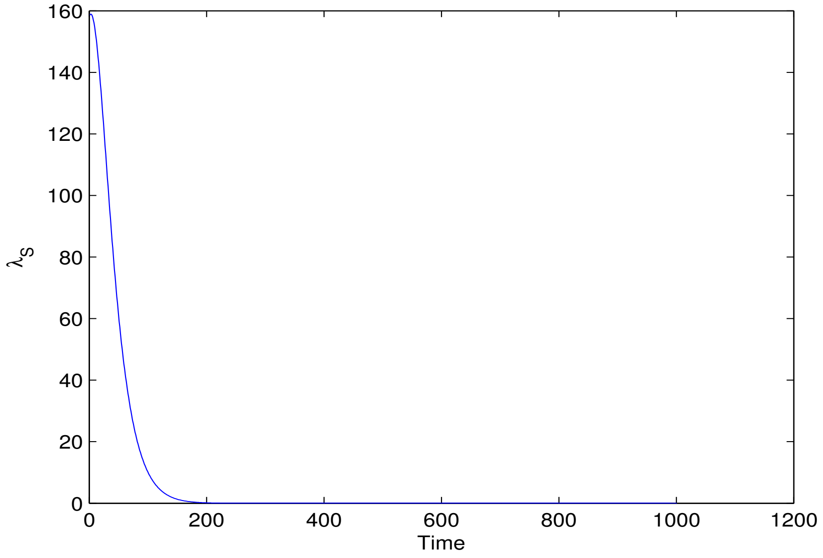

FIGURE 17. Figure for the adjoint variable λ_1.

The impact of the Allee effect plays a crucial role in locating the existence of periodic orbits. The relationships between mate mate-finding Allee effect and other strategies that lead to the Allee effect are the mass release of sterile individuals, release of generalist natural enemies,

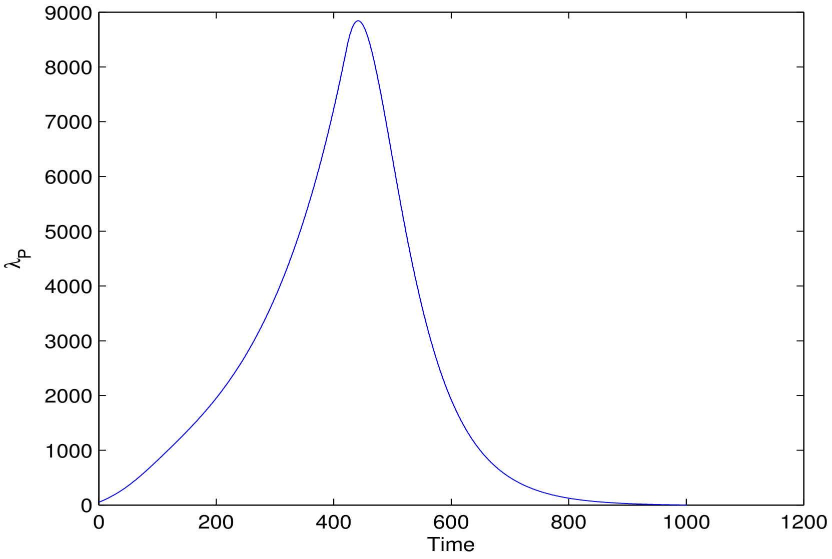

FIGURE 18. Figure for the adjoint variable λ_2.

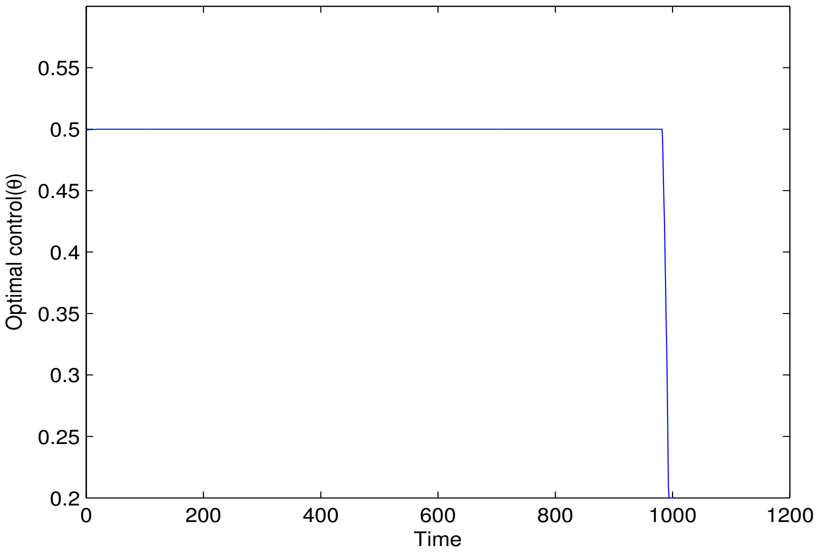

FIGURE 19. Variation of the θ with respect to time

and exploitation with constant yield [**?**]. It is difficult to find the extinction threshold since the demographic Allee effect is hard to calculate from demographic data.

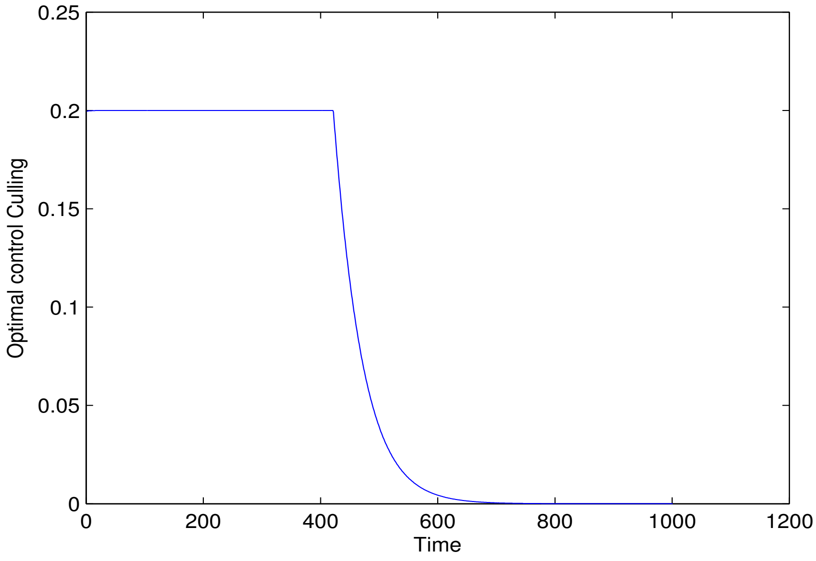

FIGURE 20. Variation of the h with respect to time

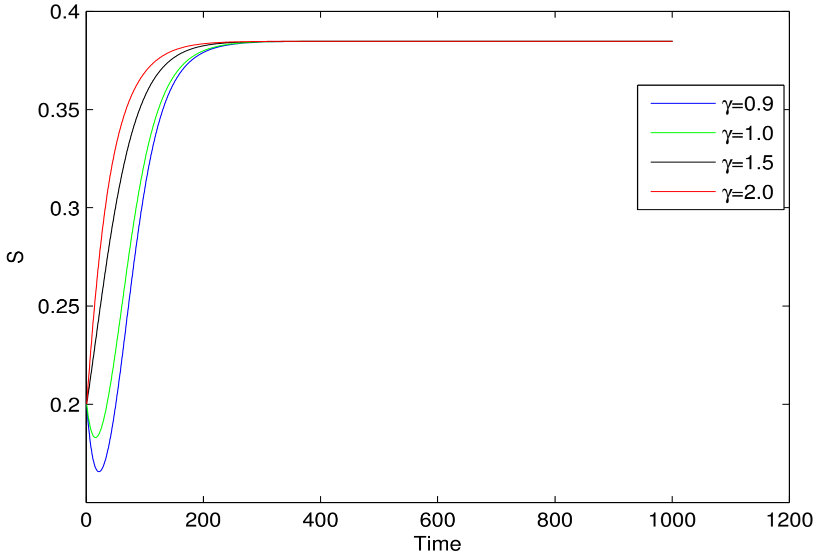

FIGURE 21. Variation of the prey population with different values of γ with respect to time

7. CONCLUSION

Five cats that were introduced in 1949 to the Kerguelen islands have multiplied to thousands and are thought to kill around three million seabirds annually [55]. The elimination of the

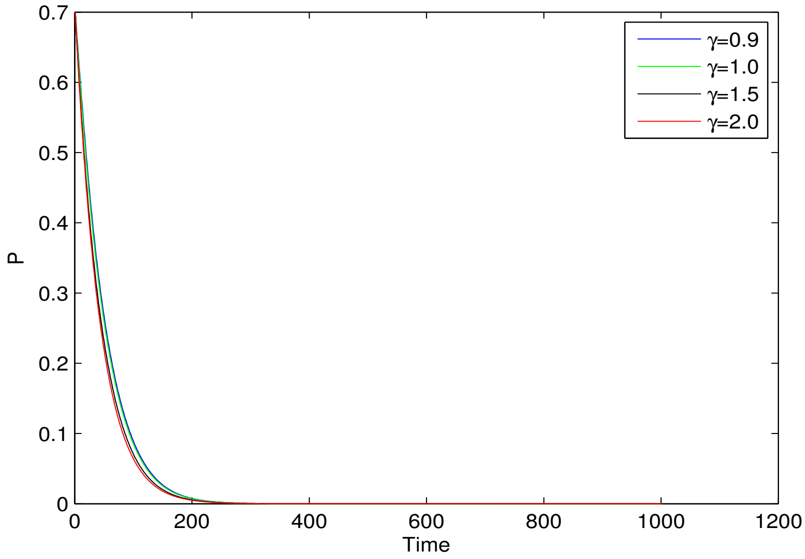

FIGURE 22. Variation of the prey population with different values of γ with respect to time

population of invasive cats on islands has been the objective of several control initiatives in an effort to preserve the number of local birds [56].

In this paper, we have researched an optimal control issue that mixes both mechanical and biological control together. In relation to invading mammals and the damage they produce, the combination of culling and the Allee effect in an integrated management approach is a frequently discussed subject. Our main findings suggest that the implementation of the optimal control policy not only diminishes the predator biomass but also increases the biomass of the prey. The authors claim that this is not a previously discussed application of a problem of optimal control theory to the preservation of endangered species and hence can be considered as a novel approach. On the other side, the system (3) makes perfect ecological sense since, if the prey population is sufficiently lower than the predator population, the extinction of prey species implies the extinction of predator species as well.

The Allee effect and culling effect along with optimal control theory have been implemented in our work in order to minimize the population of the predator. The findings in this article should undoubtedly aid ecologists and, as a result, could improve theoretical ecology. In the future, we will observe the effects of time delay and diffusion in the food chain model [57] and eco–epidemiological model [58] with the defense mechanism for prey species and Allee effect.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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