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DYNAMICS OF A LESLIE-GOWER FOOD CHAIN MODEL WITH FEAR, AND ANTI-PREDATOR CAPABILITY

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Abstract: The dynamics of food chains and ecosystems depend heavily on fear and anti-predator activity. grasping how prey-predator interactions affect not just individual species but also the larger community structure and ecosystem functioning requires a grasp of these fundamental ideas. Therefore, a novel food chain model made up of the Lotka-Volterra interaction and a modified Leslie-Gower model has been presented for investigation in the presence of fear and anti-predator notions. The model's local dynamics were examined. Requirements for persistence were identified. The global dynamics were examined using the Lyapunov function. Investigations of local bifurcations had been conducted. Lastly, to validate the results and comprehend the impact of fear and anti-predators, a numerical simulation was employed.

Keywords: alternative food; anti-predator; fear; Leslie-Gower model.

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1. INTRODUCTION

One of the main goals of ecology and mathematical biology is to comprehend the dynamic relationship between prey and predator. A key element in the prey-predator relationship is the predator's consumption rate, frequently characterized by functional response [1-3]. Of the four

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basic interactions (prey-predator, competition, interference, and mutualism), the prey-predator interaction is the most common and well-known for generating oscillatory dynamics [4]. Food chains are the fundamental components that organize the networks of interactions in natural ecosystems. Food chains have a tremendously rich dynamic, according to long-standing modeling work on the subject [5-7]. First, a simple model of interacting species was independently developed by Lotka and Volterra and is now known by their joint names [8-9]. They showed that prey-predator systems oscillate indefinitely for any initial condition if the prey growth rate is constant and the predator functional response is linear. However, relationships in the actual world are complex and call for more subtle strategies. The Allee effect [10-12], harvesting [13-15], collective defense [16-18], the terror effect [19-21], prey refuge [22-24], and so on are examples of such complications. Researchers have intensively investigated the food chain systems, including some factors, focusing on persistence and permanence, stability analysis, periodic solutions, and global dynamics. See [25-27] and the references therein.

To make the standard Lotka-Volterra system more realistic, Holling suggested three different kinds of functional responses for different species to mimic predatory behavior, based on a review of several recent findings [4]. In autonomous predator-prey systems with these functional responses, the existence and asymptotic stability of equilibrium points and limit cycles are comparatively typical biological phenomena. Over the past few decades, three types of food chain research and their uses have expanded and evolved. This in turn brought to light a variety of biological problems that could be observed in interactions between species. Since these systems explain a larger number of potential scenarios in reality, they are thought to be more realistic and thorough than binary systems. The functional responses among the populations of the three species are used in almost all of the food chain models examined in the ecological literature. But in this case, a different functional reaction might be more suitable. In [28], the authors presented a mathematical model that combined the functional responses of Lotka-Volterra and Sokol-Howell. They noted that stable points and bistable behaviors are among the system's many dynamic features.

A well-known mathematical framework for describing the dynamics of prey-predator interactions in ecological systems is the Leslie-Gower model. With more realistic assumptions on the carrying capacities of both predator and prey populations, it is an expansion of the Lotka-Volterra model. Leslie and Gower introduced the model in [29-30], and it bears their names. Later, [31-33] proposes and studies a modified Leslie-Gower prey-predator model. Because it provides an additional food supply, the predator in these situations functions as a generalist predator. The decrease in the number of predators due to the absence of their preferred prey is known as the Leslie-Gower phrase. Certain predator species may be able to adapt to a different food supply in this situation, but because their preferred meal is scarce, their population growth may be constrained. Specialist predators are considered in the Leslie-Gower prey-predator model [34]. It implies that the supply of their preferred meals per person declines in tandem with a decline in the predator population. On the other hand, if the substitute meal is beneficial, the modified Leslie-Gower model removes anomalies and enhances interaction predictions. Further studies to understand the dynamics of the Leslie-Gower prey-predator model have been recently carried out [35-37] and the references therein.

However, by considering elements like the fear effect and antipredator behavior, additional complexity can be added. Research has shown that fear-induced indirect effects significantly affect the dynamics of prey-predator relationships and the ecological system as a whole. A mathematical principle of fear was developed by Wang et al. [19]. It has been demonstrated that predator species cause psychological stress in their prey, which causes them to change their typical behavior when they are looking for food. This tension is thought to be caused by the fear of being caught and killed by predators. The prey species may benefit in the short run from this, but there may be long-term drawbacks. Their perceived threat from predators influences not just their dietary preferences but also their survival chances and rate of reproduction in comparison to normal adult populations. These assertions are supported by recent field tests and theoretical studies. Contradictory findings have been found in several studies, including the possibility that the indirect consequences of fear outweigh the direct consequences of predatory action, see [38–40] for the 2D system, and [41-42] for the three species systems.

Many prey species alter their behavior in the presence of predators and display a variety of antipredator behaviors, including foraging activity, habitat adaptations, alertness, and specific physiological changes, to lower the danger of predation [43-46]. Reduced production and foraging may be the outcome of sustained high-level anti-predatory behavior, which has benefits and drawbacks for the prey. The main benefit of anti-predator behavior is that it minimizes predation; however, the drawback is that hunger stunts growth because there is less predation, which affects prey reproduction. On the other hand, most evaluations of three-species food chain systems do not account for all the advantages and disadvantages of anti-predator behavior.

In this paper, a new three-species food chain model with producers, consumers, and predators is proposed for study. The interactions between producers and consumers follow Lotka-Volterra type interactions, while the interactions between consumers and predators follow a modified Leslie-Gower prey-predator model using Holling type-II functional response. Furthermore, fear and anti-predator factors are thought to be at play in the suggested food chain. Furthermore, the next section has taken into account the model formulation with its positivity and boundedness. The system's local dynamic behavior is covered in Section 3. Section 4 discusses the persistence of the system. Section 5 has offered the analysis of global stability, whereas Section 6 has presented the analysis of local bifurcation. In Section 7, some numerical simulations are performed to confirm the conclusions obtained. Section 8 provides the study's conclusion.

2. THE MODEL FORMULATION

A novel food chain model comprising producers, consumers, and predators was developed in this section. When the consumer is absent, the producer expands logistically; otherwise, the consumer feeds on the producer under the Lotka-Volterra function. Their fear of predation significantly impacts the feeding process of consumers; moreover, its number declines exponentially in the absence of both producers and predation. The predator is thought to hunt the consumer under the second kind's Holling-type function. Ultimately, the predator has a different source of food in addition to the consumer, which means that its growth will be logistical. Additionally, the consumer can defend himself ferociously, which can occasionally result in the predator dying. The following system of nonlinear equations can be used to quantitatively characterize the dynamics of the suggested food chain based on the fundamental assumptions of the proposed model mentioned above.

$$\begin{aligned}\frac{dX}{dT} &= rX\left(1-\frac{X}{k}\right)-\frac{a_0XY}{1+nZ} = Xf_1(X,Y,Z),\\ \frac{dY}{dT} &= \frac{a_1XY}{1+nZ}-\frac{a_2YZ}{m_0+Y}-d_0Y = Yf_2(X,Y,Z),\\ \frac{dZ}{dT} &= sZ\left(1-\frac{Z}{bY+c}\right)-\frac{qYZ}{1+m_1Z} = Zf_3(X,Y,Z),\end{aligned} \tag{1}$$

where $X(0) \geq 0$, $Y(0) \geq 0$, and $Z(0) \geq 0$. Table 1 shows the variables and descriptions of the positive parameters.

Table 1: Description of model parameters and variables:

Symbols	Description
$\boldsymbol{X(T)}$	The producer density at time $\boldsymbol{T}$.
$\boldsymbol{Y(T)}$	The consumer density at time $\boldsymbol{T}$.
$\boldsymbol{Z(T)}$	The predator density at time $\boldsymbol{T}$.
$\boldsymbol{r, s}$	The producer and predator intrinsic growth rates, respectively.
$\boldsymbol{k}$	The producer carrying capacity.
$\boldsymbol{a_0}$	The consumer attack rate against producers.
$\boldsymbol{a_1 = e_0 a_0}$	The producer conversion rate with $\boldsymbol{e_0} \in (\boldsymbol{0}, \boldsymbol{1})$.
$\boldsymbol{a_2}$	The predator attack rate.
$\boldsymbol{n}$	Consumer level of fear of predators.
$\boldsymbol{m_0}$	The half-saturation constant.
$\boldsymbol{d_0}$	The consumer natural death rate.
$\boldsymbol{q}$	The consumer antipredator level.
$\boldsymbol{m_1}$	Predator avoidance efficiency of the anti-predator consumer's capability.
$\boldsymbol{b}$	The level of consumer preference by the predator when feeding.
$\boldsymbol{c}$	The alternative food for predators.

The non-dimensional parameters and variables listed below will be used to minimize the system's complexity (1).

$$x = \frac{X}{k}, y = \frac{a_0 Y}{r}, z = nZ, t = rT, \rho_1 = \frac{a_1 k}{r}, \rho_2 = \frac{a_2}{rnm_0},\ \rho_3 = \frac{r}{a_0 m_0},$$
$$\rho_4 = \frac{d_0}{r}, \rho_5 = \frac{s}{r}, \rho_6 = \frac{1}{nc}, \rho_7 = \frac{br}{a_0 c}, \rho_8 = \frac{q}{a_0}, \rho_9 = \frac{m_1}{n}.$$

The non-dimensional system is described as follows:

$$\begin{aligned}
\frac{dx}{dt} &= x\left(1 - x - \frac{y}{1+z}\right) = x f_1(x,y,z),\\
\frac{dy}{dt} &= y\left(\frac{\rho_1 x}{1+z} - \frac{\rho_2 z}{1+\rho_3 y} - \rho_4\right) = y f_2(x,y,z),\\
\frac{dz}{dt} &= z\left(\rho_5 - \frac{\rho_5 \rho_6 z}{1+\rho_7 y} - \frac{\rho_8 y}{1+\rho_9 z}\right) = z f_3(x,y,z).
\end{aligned} \tag{2}$$

For system (2), the interaction functions f_1,f_2, and f_3 are specified on $\mathbb{R}_+^3 = \{(x,y,z): x(0) \geq 0, y(0) \geq 0, z(0) \geq 0\}$. They are continuous with continuous partial derivatives. Hence, these functions meet Lipschitz's requirements. Under the initial assumptions $x(0) \geq 0, y(0) \geq 0$, and $z(0) \geq 0$, the solution is unique, consistent with the fundamental theorem of existence and uniqueness.

Theorem 1. For any $t \geq 0$, all system (2) solutions with positive initial conditions are positive.

Proof. Define $\Gamma = \{(x,y,z) \in \mathbb{R}_+^3: x > 0, y > 0, z > 0\}$. Using the initial conditions $x(0) > 0, y(0) > 0$, and $z(0) > 0$ on the equations of system (2) provides the following result:

$$x(t) = x(0)e^{\int_0^t\left[1 - x(s) - \frac{y(s)}{1+z(s)}\right]ds}$$

$$y(t) = y(0)e^{\int_0^t\left[\frac{\rho_1 x(s)}{1+z(s)} - \frac{\rho_2 z(s)}{1+\rho_3 y(s)} - \rho_4\right]ds}$$

$$z(t) = z(0)e^{\int_0^t\left[\rho_5 - \frac{\rho_5 \rho_6 z(s)}{1+\rho_7 y(s)} - \frac{\rho_8 y(s)}{1+\rho_9 z(s)}\right]ds}$$

The exponential function indicates that all solutions in Γ with positive initial conditions remain in the first octant. Hence, the proof is complete. ■

The system's boundedness indicates that it is biologically well-defined in theoretical ecology. Indeed, when the solutions are bounded, neither of the interacting species will grow dramatically or exponentially over time; however, the population of each species is constrained due to a shortage of resources.

Theorem 2. In the region,

$$\Pi = \left\{(x, y, z) \in \mathbb{R}_+^3 : 0 < x(t) \leq 1, 0 < x(t) + y(t) \leq \frac{2}{\sigma}, 0 < z(t) \leq \frac{\sigma + 2\rho_7}{\sigma\rho_6}\right\}.$$

All solutions of the system (2) are uniformly bounded where σ is defined in the proof

Proof. Consider the solution $\big(x(t), y(t), z(t)\big)$ of the system (2). Then the first equation in system (2) indicates that $\frac{dx}{dt} \leq x - x^2$. By using lemma (2.2) [47], this inequality's solution is provided by $x(t) \leq \frac{x_0}{e^{-t}(1-x_0)+x_0}$, where x_0 is the initial value with $x_0 = x(0)$. As t approaches ∞, the solution $x(t)$ ensures that $x \leq 1$.

Consider the function $N(t) = x(t) + y(t)$, then it is obtained that

$$\frac{dN}{dt} \leq 2 - \sigma(x + y),$$

where $\sigma = \min\{1, \rho_4\}$. Therefore, using the lemma (2.1) [47], it is obtained as $t \to \infty$ that, $N = x + y \leq \frac{2}{\sigma}$

Finally, using the upper bound $\frac{2}{\sigma}$ in the third equation of the system (2) indicates that:

$$\frac{dz}{dt} \leq \rho_5 z - \frac{\sigma\rho_5\rho_6}{\sigma + 2\rho_7} z^2.$$

The solution of the last inequality is provided by $z(t) \leq \left[\frac{\sigma\rho_6}{\sigma+2\rho_7} + \left(\frac{1}{z_0} - \frac{\sigma\rho_6}{\sigma+2\rho_7}\right) e^{-\rho_5 t}\right]^{-1}$, where $z_0 = z(0)$.Thus $z \leq \frac{\sigma+2\rho_7}{\sigma\rho_6}$ as $t \to \infty$.

Hence, system (2) solutions in the region Π are uniformly bounded with the positive initial point. Hence, the proof is done. ■

3. Existence and Local Stability Analysis of Equilibria

In the following, the existence of the non-negative equilibrium points (EPs) is investigated, and then their stability conditions are determined. The non-negative EPs are described as follows:

- There is always a trivial point, represented by $e_0 = (0,0,0)$.
- There is always a first axial point, represented by $e_1 = (1,0,0)$.
- There is always a second axial point denoted by $e_2 = \left(0,0,\frac{1}{\rho_6}\right)$.
- There is always a first planar point denoted by $e_3 = \left(1,0,\frac{1}{\rho_6}\right) = (1,0,\tilde{z})$.
- If the following condition holds, then there is a second planar point $e_4 = (\hat{x},\hat{y},0)$, where $\hat{x} = \frac{\rho_4}{\rho_1}$ and $\hat{y} = 1 - \frac{\rho_4}{\rho_1}$.

$$\rho_4 < \rho_1. \tag{3}$$

- Finding the positive roots of System (2)'s equations after setting them to zero will yield the positive equilibrium point, indicated by $e_5 = (\bar{x},\bar{y},\bar{z})$. Straightforward computations provided us with $\bar{x} = \frac{1-\bar{y}+\bar{z}}{1+\bar{z}}$ while the point $(\bar{y},\bar{z})$ is a positive intersection point of the isoclines:

$$\left.\begin{array}{l} h_1(y,z) = \frac{\rho_1\rho_3(1-y+z)y}{1+z} - \rho_3\rho_4 yz - \rho_2 z^2 - \rho_3\rho_4 y + \frac{\rho_1(1-y+z)}{1+z} - (\rho_2+\rho_4)z - \rho_4 = 0 \\ h_2(y,z) = -\rho_5\rho_6\rho_9 z^2 + \rho_5\rho_7\rho_9 yz - \rho_7\rho_8 y^2 - \rho_5(\rho_6-\rho_9)z + (\rho_5\rho_7-\rho_8)y + \rho_5 = 0 \end{array}\right\}. \tag{4}$$

It is easy to verify that as $z \to 0$ the above two isoclines become

$$h_1(y,0) = -\rho_1\rho_3 y^2 + (\rho_1\rho_3 - \rho_3\rho_4 - \rho_1)y + \rho_1 - \rho_4 = 0 \quad .$$

$$h_2(y,0) = -\rho_7\rho_8 y^2 + (\rho_5\rho_7 - \rho_8)y + \rho_5 = 0.$$

Direct computation shows that the first isocline has a unique positive root for y denoted by y_1 with the fulfillment of condition (3). While the second isocline has a unique positive root for y denoted by y_2, where:

$$\left.\begin{array}{l} y_1 = 1 - \frac{\rho_4}{\rho_1} \\ y_2 = \frac{\rho_5}{\rho_8} \end{array}\right\}.$$

Consequently, the isoclines (4) have a unique intersection positive point denoted by $(\bar{y},\bar{z})$ if the condition (3) with the following sufficient conditions hold:

$$\left.\begin{array}{l} y_1 < y_2 \\ \frac{dz}{dy} = -\frac{(\partial h_1/\partial y)}{(\partial h_1/\partial z)} > 0 \\ \frac{dz}{dy} = -\frac{(\partial h_2/\partial y)}{(\partial h_2/\partial z)} < 0 \end{array}\right\}. \tag{5}$$

Furthermore, the positive equilibrium point exists uniquely if, in addition to the conditions (5), the following condition is met.

$$\bar{y} < 1 + \bar{z}. \tag{6}$$

In the following steps, the linearization technique is applied to examine the system's local stability across the earlier-mentioned equilibrium points. The basic Jacobian matrix of system (2) appears to be evaluated as follows:

$$J(x,y,z) = \begin{pmatrix} x\frac{\partial f_1}{\partial x} + f_1 & x\frac{\partial f_1}{\partial y} & x\frac{\partial f_1}{\partial z} \\ y\frac{\partial f_2}{\partial x} & y\frac{\partial f_2}{\partial y} + f_2 & y\frac{\partial f_2}{\partial z} \\ z\frac{\partial f_3}{\partial x} & z\frac{\partial f_3}{\partial y} & z\frac{\partial f_3}{\partial z} + f_3 \end{pmatrix} = (a_{ij})_{3\times 3}, \tag{7}$$

where:

$$a_{11} = -x + \left[1 - x - \frac{y}{1+z}\right],$$

$$a_{12} = -\frac{x}{1+z},$$

$$a_{13} = \frac{xy}{(1+z)^2},$$

$$a_{21} = \frac{\rho_1 y}{1+z},$$

$$a_{22} = \frac{\rho_2\rho_3 yz}{(1+\rho_3 y)^2} + \left[\frac{\rho_1 x}{1+z} - \frac{\rho_2 z}{1+\rho_3 y} - \rho_4\right],$$

$$a_{23} = -\left(\frac{\rho_1 x}{(1+z)^2} + \frac{\rho_2}{1+\rho_3 y}\right) y,$$

$$a_{31} = 0,$$

$$a_{32} = \left(\frac{\rho_5\rho_6\rho_7 z}{(1+\rho_7 y)^2} - \frac{\rho_8}{1+\rho_9 z}\right) z,$$

$$a_{33} = -\frac{\rho_5\rho_6 z}{1+\rho_7 y} + \frac{\rho_8\rho_9 yz}{(1+\rho_9 z)^2} + \left[\rho_5 - \frac{\rho_5\rho_6 z}{1+\rho_7 y} - \frac{\rho_8 y}{1+\rho_9 z}\right].$$

Now, at e_0, matrix (7) becomes:

$$J(e_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\rho_4 & 0 \\ 0 & 0 & \rho_5 \end{pmatrix}. \tag{8}$$

Thus, $J(e_0)$ has the following eigenvalues: $\lambda_{01} = 1$, $\lambda_{02} = -\rho_4$, and $\lambda_{03} = \rho_5$. Therefore, e_0 is a saddle point.

At e_1, matrix (7) becomes:

$$J(e_1) = \begin{pmatrix} -1 & -1 & 0 \\ 0 & \rho_1 - \rho_4 & 0 \\ 0 & 0 & \rho_5 \end{pmatrix} \tag{9}$$

Therefore, e_1 is a saddle point, as $J(e_1)$ has the eigenvalues: $\lambda_{11} = -1$, $\lambda_{12} = \rho_1 - \rho_4$, and $\lambda_{13} = \rho_5$.

At e_2, matrix (7) becomes:

$$J(e_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\rho_2}{\rho_6} - \rho_4 & 0 \\ 0 & \left(\rho_5\rho_7 - \frac{\rho_6\rho_8}{\rho_6+\rho_9}\right)\frac{1}{\rho_6} & -\rho_5 \end{pmatrix} \tag{10}$$

Therefore, e_2 is a saddle point, as $J(e_2)$ has the eigenvalues: $\lambda_{21} = 1$, $\lambda_{22} = -\frac{\rho_2}{\rho_6} - \rho_4$, and $\lambda_{23} = -\rho_5$.

Moreover, at e_3, matrix (7) transfers to:

$$J(e_3) = \begin{pmatrix} -1 & -\frac{\rho_6}{\rho_6+1} & 0 \\ 0 & \frac{\rho_1\rho_6}{\rho_6+1} - \frac{\rho_2}{\rho_6} - \rho_4 & 0 \\ 0 & \left(\rho_5\rho_7 - \frac{\rho_6\rho_8}{\rho_6+\rho_9}\right)\frac{1}{\rho_6} & -\rho_5 \end{pmatrix} \tag{11}$$

Thus, $J(e_3)$ has the eigenvalues: $\lambda_{31} = -1$, $\lambda_{32} = \frac{\rho_1\rho_6}{\rho_6+1} - \frac{\rho_2}{\rho_6} - \rho_4$, and $\lambda_{33} = -\rho_5$. Therefore, e_3 is locally asymptotically stable if the condition (12) holds, becomes a non-hyperbolic point in case of equality in equation (12), and a saddle point otherwise:

$$\frac{\rho_1\rho_6}{\rho_6+1} < \frac{\rho_2}{\rho_6} + \rho_4. \tag{12}$$

Additionally, at e_4, the matrix (7) becomes:

$$J(e_4) = \begin{pmatrix} -\hat{x} & -\hat{x} & \hat{x}\hat{y} \\ \rho_1\hat{y} & 0 & -\left(\rho_1\hat{x} + \frac{\rho_2}{1+\rho_3\hat{y}}\right)\hat{y} \\ 0 & 0 & \rho_5 - \rho_8\hat{y} \end{pmatrix} = \left(\hat{a}_{ij}\right). \tag{13}$$

Thus, the eigenvalues of $J(e_4)$ are $\lambda_{41,42} = \frac{-\hat{x}}{2} \pm \frac{\sqrt{\hat{x}^2-4\rho_1\hat{x}\hat{y}}}{2}$ and $\lambda_{43} = \rho_5 - \rho_8\hat{y}$. Therefore, as λ_{41} and λ_{42} have negative real parts, e_4 is locally asymptotically stable if the condition (14) holds, becomes a non-hyperbolic point when the inequality sign becomes an equality sign in condition (14), while it is a saddle point otherwise:

$$\rho_5 < \rho_8\hat{y}. \tag{14}$$

Finally, the matrix (7) at e_5 transfers to:

$$J(e_5) = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} & \bar{a}_{13} \\ \bar{a}_{21} & \bar{a}_{22} & \bar{a}_{23} \\ 0 & \bar{a}_{32} & \bar{a}_{33} \end{pmatrix}. \tag{15}$$

where:

$$\bar{a}_{11} = -\bar{x},\ \bar{a}_{12} = -\frac{\bar{x}}{1+\bar{z}},\ \bar{a}_{13} = \frac{\bar{x}\bar{y}}{(1+\bar{z})^2},\ \bar{a}_{21} = \frac{\rho_1\bar{y}}{1+\bar{z}},\ \bar{a}_{22} = \frac{\rho_2\rho_3\bar{y}\bar{z}}{(1+\rho_3\bar{y})^2},$$

$$\bar{a}_{23} = -\left(\frac{\rho_1\bar{x}}{(1+\bar{z})^2} + \frac{\rho_2}{1+\rho_3\bar{y}}\right)\bar{y},\ \bar{a}_{31} = 0,\ \bar{a}_{32} = \left(\frac{\rho_5\rho_6\rho_7\bar{z}}{(1+\rho_7\bar{y})^2} - \frac{\rho_8}{1+\rho_9\bar{z}}\right)\bar{z},$$

$$\bar{a}_{33} = -\frac{\rho_5\rho_6\bar{z}}{1+\rho_7\bar{y}} + \frac{\rho_8\rho_9\bar{y}\bar{z}}{(1+\rho_9\bar{z})^2}.$$

As a result, the characteristic equation of $J(e_5)$ can be expressed as follows:

$${\lambda_3}^3 + A_1{\lambda_3}^2 + A_2\lambda_3 + A_3 = 0, \tag{16}$$

where:

$$A_1 = -(\bar{a}_{11} + \bar{a}_{22} + \bar{a}_{33}),$$

$$A_2 = \bar{a}_{11}\bar{a}_{22} + \bar{a}_{11}\bar{a}_{33} + \bar{a}_{22}\bar{a}_{33} - \bar{a}_{12}\bar{a}_{21} - \bar{a}_{23}\bar{a}_{32},$$

$$A_3 = -\bar{a}_{33}(\bar{a}_{11}\bar{a}_{22} - \bar{a}_{12}\bar{a}_{21}) + \bar{a}_{32}(\bar{a}_{11}\bar{a}_{23} - \bar{a}_{21}\bar{a}_{13}),$$

with:

$$\begin{aligned} A_1A_2 - A_3 = -(\bar{a}_{11} + \bar{a}_{22})[\bar{a}_{11}\bar{a}_{22} - \bar{a}_{12}\bar{a}_{21}] - (\bar{a}_{22} + \bar{a}_{33})[\bar{a}_{22}\bar{a}_{33} - \bar{a}_{23}\bar{a}_{32}] \\ -\bar{a}_{11}\bar{a}_{33}(\bar{a}_{11} + 2\bar{a}_{22} + \bar{a}_{33}) + \bar{a}_{13}\bar{a}_{21}\bar{a}_{32}. \end{aligned}$$

According to the Routh-Hurwitz criterion, $e_5 = (\bar{x}, \bar{y}, \bar{z})$ is locally asymptotically stable provided $A_1 > 0, A_3 > 0$ and $A_1A_2 > A_3$, which is true if and only if the following conditions are satisfied:

$$\frac{\rho_8}{1+\rho_9\bar{z}} < \frac{\rho_5\rho_6\rho_7\bar{z}}{(1+\rho_7\bar{y})^2} \tag{17}$$

$$\frac{\rho_2\rho_3\bar{y}\bar{z}}{(1+\rho_3\bar{y})^2} < \bar{x} \tag{18}$$

$$\frac{\rho_8\rho_9\bar{y}}{(1+\rho_9\bar{z})^2} + \frac{\rho_2\rho_3\bar{y}}{(1+\rho_3\bar{y})^2} < \frac{\rho_5\rho_6}{1+\rho_7\bar{y}} \tag{19}$$

$$\frac{\rho_2\rho_3\bar{z}}{(1+\rho_3\bar{y})^2} < \frac{\rho_1}{(1+\bar{z})^2} \tag{20}$$

$$\frac{\rho_1\bar{y}}{(1+\bar{z})^3} < \frac{\rho_1\bar{x}}{(1+\bar{z})^2} + \frac{\rho_2}{1+\rho_3\bar{y}} \tag{21}$$

$$\frac{\rho_2\rho_3}{(1+\rho_3\bar{y})^2}\left[\frac{\rho_5\rho_6\bar{z}}{1+\rho_7\bar{y}} - \frac{\rho_8\rho_9\bar{y}\bar{z}}{(1+\rho_9\bar{z})^2}\right] < \left(\frac{\rho_1\bar{x}}{(1+\bar{z})^2} + \frac{\rho_2}{1+\rho_3\bar{y}}\right)\left(\frac{\rho_5\rho_6\rho_7\bar{z}}{(1+\rho_7\bar{y})^2} - \frac{\rho_8}{1+\rho_9\bar{z}}\right) \tag{22}$$

As a result, the following theorem is established.

Theorem 3. The point e_5 is locally asymptotically stable if the conditions (17–22) are fulfilled.

4. PERSISTENCE

The survival of all species is the meaning of persistence in biological systems. Hence, in the sense of mathematics, system (2) will be persistent if it has no omega limit set that is part of its boundary domain for all positive initial points.

The subsystems located in the positive quadrant of the $xz-$ and $xy-$planes of a system (2) can be represented by the following formulas:

$$\begin{aligned} \frac{dx}{dt} &= x(1-x) = F_1(x,z), \\ \frac{dz}{dt} &= z(\rho_5 - \rho_5\rho_6 z) = F_2(x,z). \end{aligned} \tag{23}$$

$$\begin{aligned} \frac{dx}{dt} &= x(1-x-y) = G_1(x,y), \\ \frac{dy}{dt} &= y(\rho_1 x - \rho_4) = G_2(x,y). \end{aligned} \tag{24}$$

The subsystems (23) and (24) have positive equilibrium points that correspond to the projection of $e_3 = (1,0,\frac{1}{\rho_6})$ and $e_4 = (\hat{x},\hat{y},0)$ on the corresponding planes of the system (2) respectively. To prove whether periodic dynamics exist near the interior positive points of subsystems (23) and (24), the Dulac function approach [48] is used.

Let $g_1(x,z) = \frac{1}{xz}$, and $g_2(x,y) = \frac{1}{xy}$ be continuously differentiable functions that are defined for all $(x,z),(x,y) \in \mathbb{R}_+^2$ and are located in the interior of the positive quadrant of the xz-plane and xy-plane, with $g_1(x,z), g_2(x,y) > 0$. Additionally, the basic calculation produces the following:

$$\Delta_1(x,z) = \frac{\partial}{\partial x}(g_1.F_1) + \frac{\partial}{\partial z}(g_1.F_2) = -\frac{1}{z} - \frac{\rho_5\rho_6}{x}$$

$$\Delta_2(x,y) = \frac{\partial}{\partial x}(g_2.G_1) + \frac{\partial}{\partial y}(g_2.G_2) = -\frac{1}{y}$$

Then $\Delta_1(x,z) < 0$ and $\Delta_2(x,y) < 0$ for any value of $(x,z),(x,y) \in \mathbb{R}_+^2$. As a result, there are no periodic dynamics in the positive quadrants of the xz-plane and xy-plane.

Theorem 4. System (2) is uniformly persistent if the conditions below are met.

$$\frac{\rho_2}{\rho_6} + \rho_4 < \frac{\rho_1\rho_6}{1+\rho_6} \tag{25}$$

$$\rho_8\hat{y} < \rho_5 \tag{26}$$

Proof. Define $F(x,y,z) = x^{\alpha_1}y^{\alpha_2}z^{\alpha_3}$, where $\alpha_1, \alpha_2, \alpha_3$ are arbitrary positive constants, and $F(x,y,z) > 0$ for all $(x,y,z) \in \mathbb{R}_+^3$ with $F(x,y,z) \to 0$ whenever x, y or z goes to zero. Now, let

$$\varphi(x,y,z) = \frac{F'(x,y,z)}{F(x,y,z)} = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3.$$

The functions $f_1, f_2,$ and f_3 are defined in system (2).

For the persistence of system (2), the average Lyapunov technique demands establishing that $\varphi(x,y,z) > 0$ at all boundary equilibrium points. Therefore:

$$\varphi(x,y,z) = \alpha_1\left[1 - x - \frac{y}{1+z}\right] + \alpha_2\left[\frac{\rho_1 x}{1+z} - \frac{\rho_2 z}{1+\rho_3 y} - \rho_4\right] + \alpha_3\left[\rho_5 - \frac{\rho_5\rho_6 z}{1+\rho_7 y} - \frac{\rho_8 y}{1+\rho_9 z}\right].$$

Then:

$$\varphi(e_0) = \alpha_1 - \alpha_2\rho_4 + \alpha_3\rho_5.$$

Clearly, by allowing the arbitrary positive constants α_1 and α_3 to be sufficiently greater than the positive constant α_2, $\varphi(e_0) > 0$ is obtained.

$$\varphi(e_1) = \alpha_2[\rho_1 - \rho_4] + \alpha_3[\rho_5].$$

Hence $\varphi(e_1) > 0$, whether $(\rho_1 - \rho_4)$ is positive or negative, by selecting the arbitrary positive constants α_2 and α_3 suitably.

$$\varphi(e_2) = \alpha_1[1] + \alpha_2\left[-\frac{\rho_2}{\rho_6} - \rho_4\right].$$

So, $\varphi(e_2) > 0$ by selecting the arbitrary positive constants α_1 and α_2 suitably.

$$\varphi(e_3) = \alpha_2\left[\frac{\rho_1\rho_6}{1+\rho_6} - \frac{\rho_2}{\rho_6} - \rho_4\right].$$

As a result, condition (25) indicates that $\varphi(e_3) > 0$.

Furthermore, at the point e_4, straightforward computation reveals that:

$$\varphi(e_4) = \alpha_3[\rho_5 - \rho_8\hat{y}].$$

Under condition (26), $\varphi(e_4) > 0$. Hence, system (2) is uniformly persistent, and the proof is complete. ■

5. GLOBAL STABILITY ANALYSIS

This section uses Lyapunov functions to investigate the possibility of global stability inside the bounded region Π of the system's (2) locally asymptotically stable equilibrium points, as shown in the theorems below.

Theorem 5. The first planar point $e_3 = \left(1,0,\frac{1}{\rho_6}\right) = (1,0,\tilde{z})$ is globally asymptotically stable, provided the following condition is met.

$$1 + \rho_5\rho_7 z_{max} + \frac{\rho_8}{\rho_6} < \frac{\rho_4}{\rho_1}, \tag{27}$$

where $z_{max} = \frac{\sigma+2\rho_7}{\sigma\rho_6}$ that is given in Theorem 2.

Proof. Consider the following a real-valued function:

$$G_1(x,y,z) = (x - 1 - \ln x) + \frac{1}{\rho_1}y + \left(z - \tilde{z} - \tilde{z}\ln\left(\frac{z}{\tilde{z}}\right)\right).$$

It is a positive definite function since $G_1(e_3) = 0$ and $G_1(x,y,z) > 0$, for all $\{(x,y,z) \in \mathbb{R}_+^3 : x > 0, y \geq 0, z > 0\}$, and $(x,y,z) \neq (1,0,\tilde{z})$. Furthermore, some direct computation yields:

$$\frac{dG_1}{dt} = (x-1)\left(1 - x - \frac{y}{1+z}\right) + \frac{xy}{1+z} - \frac{1}{\rho_1}\frac{\rho_2 yz}{1+\rho_3 y} - \frac{\rho_4}{\rho_1}y + (z-\tilde{z})\left(-\frac{\rho_5\rho_6(z-\tilde{z})}{1+\rho_7 y} - \frac{\rho_5\rho_6\rho_7\tilde{z}y}{1+\rho_7 y} - \frac{\rho_8 y}{1+\rho_9 z}\right).$$

After simple simplification, it is obtained that:

$$\frac{dG_1}{dt} = -(x-1)^2 + \frac{y}{1+z} - \frac{1}{\rho_1}\frac{\rho_2 yz}{1+\rho_3 y} - \frac{\rho_4}{\rho_1}y - \frac{\rho_5\rho_6(z-\tilde{z})^2}{1+\rho_7 y} + \frac{\rho_5\rho_6\rho_7\tilde{z}yz}{1+\rho_7 y} - \frac{\rho_5\rho_6\rho_7\tilde{z}^2 y}{1+\rho_7 y} - \frac{\rho_8 yz}{1+\rho_9 z} + \frac{\rho_8\tilde{z}y}{1+\rho_9 z}.$$

Therefore:

$$\frac{dG_1}{dt} = -(x-1)^2 - \left[\frac{\rho_4}{\rho_1} - 1 - \rho_5\rho_7 z_{max} - \frac{\rho_8}{\rho_6}\right]y - \frac{\rho_5\rho_6(z-\tilde{z})^2}{1+\rho_7 y}.$$

Thus, $\frac{dG_1}{dt}$ is negative definite provided that condition (27) holds. Hence the proof is done. ■

Theorem 6. The second planar point e_4 is globally asymptotically stable, provided the following conditions hold.

$$\frac{\rho_2\hat{y}}{\rho_1} < \frac{\rho_8}{1+\rho_9 z_{max}}. \tag{28}$$

$$\hat{y} + \hat{x}\hat{y}^2 + \rho_5 < \frac{\hat{x}\hat{y}}{1+z_{max}}. \tag{29}$$

$$\frac{1}{2}(1-\hat{y})^2 < 1. \tag{30}$$

$$\frac{3}{2} < \frac{\rho_4}{\rho_1}. \tag{31}$$

Proof. Define $G_2(x,y,z) = \left(x - \hat{x} - \hat{x}\ln\left(\frac{x}{\hat{x}}\right)\right) + \frac{1}{2\rho_1}(y-\hat{y})^2 + z$, which is a real-valued function. It is a positive definite function since $G_2(e_4) = 0$ and $G_2(x,y,z) > 0$, for all $\{(x,y,z) \in \mathbb{R}^3_+ : x > 0, y \geq 0, z \geq 0\}$, and $(x,y,z) \neq (\hat{x},\hat{y},0)$. Furthermore, some direct computation yields:

$$\begin{aligned}\frac{dG_2}{dt} = {} & -(x-\hat{x})^2 - (1-\hat{y})\frac{(x-\hat{x})(y-\hat{y})}{1+z} - \left[\frac{\rho_4}{\rho_1} - \frac{x}{1+z}\right](y-\hat{y})^2 \\ & - \left[\frac{\hat{x}\hat{y}}{1+z} - \frac{x\hat{y}}{1+z} - \frac{\hat{x}\hat{y}^2}{1+z} - \rho_5\right]z - \frac{\hat{x}\hat{y}yz}{1+z} - \frac{\rho_2 y^2 z}{\rho_1(1+\rho_3 y)} - \frac{\rho_5\rho_6 z^2}{1+\rho_7 y} \\ & - \left[\frac{\rho_8}{1+\rho_9 z} - \frac{\rho_2\hat{y}}{\rho_1(1+\rho_3 y)}\right]yz.\end{aligned}$$

Using the given conditions leads to:

$$\begin{aligned}\frac{dG_2}{dt} < {} & -(x-\hat{x})^2 - \frac{(1-\hat{y})}{1+z}(x-\hat{x})(y-\hat{y}) - \left[\frac{\rho_4}{\rho_1} - \frac{x}{1+z}\right](y-\hat{y})^2 \\ & - \left[\frac{\hat{x}\hat{y}}{1+z} - \frac{x\hat{y}}{1+z} - \frac{\hat{x}\hat{y}^2}{1+z} - \rho_5\right]z.\end{aligned}$$

By using further simplification, it is obtained that:

$$\frac{dG_2}{dt} < -\left[1 - \frac{1}{2}\left(\frac{1-\hat{y}}{1+z}\right)^2\right](x-\hat{x})^2 - \left[\frac{\rho_4}{\rho_1} - \frac{x}{1+z} - \frac{1}{2}\right](y-\hat{y})^2$$
$$-\left[\frac{\hat{x}\hat{y}}{1+z} - \frac{x\hat{y}}{1+z} - \frac{\hat{x}\hat{y}^2}{1+z} - \rho_5\right]z.$$

Hence, $\frac{dG_2}{dt}$ is a negative-definite if the conditions (28-31) hold, thus proof is complete. ∎

Theorem 7. The positive point e_5 is globally asymptotically stable, provided the following conditions hold.

$$\frac{1}{2}(1-\bar{y})^2 + \frac{1}{2}\left(\frac{\bar{y}}{\bar{A}}\right)^2 < 1, \tag{32}$$

$$\frac{\rho_9\bar{y}}{\bar{D}} + 1 < \frac{\rho_5\rho_6}{1+\rho_7 y_{max}}, \tag{33}$$

$$1 + \frac{1}{2}\left(\frac{\bar{x}\bar{y}}{\bar{A}} + \frac{\rho_2 y_{max}}{\rho_1} - \frac{\rho_5\rho_6\rho_7\bar{z}}{\bar{C}} + \rho_8\right)^2 + \frac{1}{2} < \frac{\rho_4}{\rho_1} + \frac{\rho_2\bar{z}}{\rho_1(1+\rho_3 y_{max})\bar{B}}. \tag{34}$$

Proof. Define $G_3(x,y,z) = \left(x - \bar{x} - \bar{x}\ln\left(\frac{x}{\bar{x}}\right)\right) + \frac{1}{2\rho_1}(y-\bar{y})^2 + \left(z - \bar{z} - \bar{z}\ln\left(\frac{z}{\bar{z}}\right)\right)$, which is a real-valued function. It is a positive definite function since $G_3(e_5) = 0$ and $G_3(x,y,z) > 0$, for all $\{(x,y,z) \in \mathbb{R}_+^3 : x > 0, y \geq 0, z > 0\}$, and $(x,y,z) \neq (\bar{x},\bar{y},\bar{z})$. By using some direct computation yields:

$$\frac{dG_2}{dt} = -(x-\bar{x})^2 - \frac{(1-\bar{y})}{A}(x-\bar{x})(y-\bar{y}) + \frac{\bar{y}}{A\bar{A}}(x-\bar{x})(z-\bar{z})$$
$$-\left[\frac{\bar{x}\bar{y}}{A\bar{A}} + \frac{\rho_2 y}{\rho_1 B} - \frac{\rho_5\rho_6\rho_7\bar{z}}{C\bar{C}} + \frac{\rho_8}{D}\right](y-\bar{y})(z-\bar{z})$$
$$-\left[\frac{\rho_4}{\rho_1} + \frac{\rho_2\bar{z}}{\rho_1 B\bar{B}} - \frac{x}{A}\right](y-\bar{y})^2 - \left[\frac{\rho_5\rho_6}{C} - \frac{\rho_9\bar{y}}{D\bar{D}}\right](z-\bar{z})^2,$$

where $A = (1+z)$, $\bar{A} = (1+\bar{z})$, $B = (1+\rho_3 y)$, $\bar{B} = (1+\rho_3\bar{y})$, $C = (1+\rho_7 y)$, $\bar{C} = (1+\rho_7\bar{y})$, $D = (1+\rho_9 z)$, and $\bar{D} = (1+\rho_9\bar{z})$.

By using further simplification, it is obtained that:

$$\frac{dG_2}{dt} = -\left[1 - \frac{1}{2}\left(\frac{1-\bar{y}}{A}\right)^2 - \frac{1}{2}\left(\frac{\bar{y}}{A\bar{A}}\right)^2\right](x-\bar{x})^2 - \left[\frac{\rho_5\rho_6}{C} - \frac{\rho_9\bar{y}}{D\bar{D}} - 1\right](z-\bar{z})^2$$
$$-\left[\frac{\rho_4}{\rho_1} + \frac{\rho_2\bar{z}}{\rho_1 B\bar{B}} - \frac{x}{A} - \frac{1}{2}\left(\frac{\bar{x}\bar{y}}{A\bar{A}} + \frac{\rho_2 y}{\rho_1 B} - \frac{\rho_5\rho_6\rho_7\bar{z}}{C\bar{C}} + \frac{\rho_8}{D}\right)^2 - \frac{1}{2}\right](y-\bar{y})^2.$$

Hence, $\frac{dG_3}{dt}$ is negative definite provided that conditions (32-34) hold. Hence the proof is done. ■

6. LOCAL BIFURCATION ANALYSIS

This section uses the Sotomayor theorem [48] for local bifurcation to study how changing parameters affect the system's (2) qualitative dynamics near non-hyperbolic points. Rewrite system (2) with the vector form:

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X},\rho), \mathbf{X} = (x,y,z)^T, \mathbf{F} = \left(xf_1(\mathbf{X},\mu), yf_2(\mathbf{X},\mu), zf_3(\mathbf{X},\mu)\right)^T,$$

where the system (2) specifies $f_i(\mathbf{X},\rho), \forall i = 1,2,3$. The potential bifurcation parameter $\rho \in \mathbb{R}_+$ is also specified. Direct computation of the second and third derivatives of vector $\mathbf{F}$ gives the following:

$$D^2\mathbf{F}(\mathbf{X},\rho).(\mathbf{V},\mathbf{V}) = (c_{i1})_{3\times1}, \tag{35}$$

where $\mathbf{V} = (v_1, v_2, v_3)^T$ be any vector and

$$c_{11} = -2v_1^2 - \frac{2}{(1+z)}v_1v_2 + \frac{2y}{(1+z)^2}v_1v_3 + \frac{2x}{(1+z)^2}v_2v_3 - \frac{2xy}{(1+z)^3}v_3^2\ ,$$

$$c_{21} = \frac{2\rho_1}{1+z}v_1v_2 - \frac{2\rho_1 y}{(1+z)^2}v_1v_3 - 2\left(\frac{\rho_1 x}{(1+z)^2} + \frac{\rho_2}{(1+\rho_3 y)^2}\right)v_2v_3 + \frac{2\rho_2\rho_3 z}{(1+\rho_3 y)^3}v_2^2 + \frac{2\rho_1 xy}{(1+z)^3}v_3^2,$$

$$c_{31} = -2\frac{\rho_5\rho_6\rho_7^2z^2}{(1+\rho_7 y)^3}v_2^2 + \left(\frac{4\rho_5\rho_6\rho_7 z}{(1+\rho_7 y)^2} - \frac{2\rho_8}{(1+\rho_9 z)^2}\right)v_2v_3 - 2\left(\frac{\rho_5\rho_6}{1+\rho_7 y} - \frac{\rho_8\rho_9 y}{(1+\rho_9 z)^3}\right)v_3^2.$$

Furthermore,

$$D^3\mathbf{F}(\mathbf{X},\rho).(\mathbf{V},\mathbf{V},\mathbf{V}) = (d_{i1})_{3\times1}, \tag{36}$$

where:

$$d_{11} = \frac{6}{(1+z)^2}v_1v_2v_3 - \frac{6x}{(1+z)^3}v_2{v_3}^2 - \frac{6y}{(1+z)^3}v_1{v_3}^2 + \frac{6xy}{(1+z)^4}{v_3}^3,$$

$$\begin{aligned} d_{21} = &\frac{6\rho_1 y}{(1+z)^3}v_1v_3^2 - \frac{6\rho_1}{(1+z)^2}v_1v_2v_3 - \frac{6\rho_2\rho_3^2 z}{(1+\rho_3 y)^4}v_2^3 + \frac{6\rho_2\rho_3}{(1+\rho_3 y)^3}v_2^2v_3 \\ &+ \frac{6\rho_1 x}{(1+z)^3}v_2v_3^2 - \frac{6\rho_1 xy}{(1+z)^4}v_3^3 \end{aligned},$$

$$d_{31} = 6\frac{\rho_5\rho_6\rho_7^3z^2}{(1+\rho_7 y)^4}v_2^3 - 12\frac{\rho_5\rho_6\rho_7^2 z}{(1+\rho_7 y)^3}v_3v_2^2 + 6\left(\frac{\rho_5\rho_6\rho_7}{(1+\rho_7 y)^2} + \frac{\rho_8\rho_9}{(1+\rho_9 z)^3}\right)v_2v_3^2 - 6\frac{\rho_8\rho_9^2 y}{(1+\rho_9 z)^4}v_3^3.$$

Theorem 8: Near the first planer point e_3, system (2) experiences a transcritical bifurcation when the parameter ρ_4 passes through the positive value $\rho_4^* = \frac{\rho_1\rho_6}{\rho_6+1} - \frac{\rho_2}{\rho_6}$ if the following condition holds:

$$2\rho_1\left(1+\frac{1}{\rho_6}\right)^{-1}\sigma_1 + 2\rho_2\rho_3\frac{1}{\rho_6} \neq 2\left(\rho_1\left(1+\frac{1}{\rho_6}\right)^{-2} + \rho_2\right)\sigma_3, \tag{37}$$

where σ_1 and σ_3 are given in the proof. Otherwise, a pitchfork bifurcation takes place when

$$6\rho_2\rho_3\sigma_3 + 6\rho_1{\sigma_3}^2\left(1+\frac{1}{\rho_6}\right)^{-3} \neq 6\rho_1\sigma_1\sigma_3\left(1+\frac{1}{\rho_6}\right)^{-2} + \frac{6\rho_2\rho_3^2}{\rho_6}. \tag{38}$$

Proof: The matrix (11) at (e_3, ρ_4^*) yields:

$$J_1 = J(e_3,\rho_4^*) = \begin{pmatrix} -1 & -\frac{\rho_6}{\rho_6+1} & 0 \\ 0 & 0 & 0 \\ 0 & \left(\rho_5\rho_7 - \frac{\rho_6\rho_8}{\rho_6+\rho_9}\right)\frac{1}{\rho_6} & -\rho_5 \end{pmatrix}.$$

The eigenvalues of J_1 are as follows: $\lambda_{31} = -1$, $\lambda_{32} = 0$ and $\lambda_{33} = -\rho_5$. Hence a non-hyperbolic point e_3 has been obtained. Let $\mathbf{V}_1 = (v_{11}, v_{21}, v_{31})^T$ and $\mathbf{W}_1 = (w_{11}, w_{21}, w_{31})^T$ be the eigenvectors corresponding $\lambda_{32} = 0$ of J_1 and ${J_1}^T$ respectively. The straightforward computation yields that $\mathbf{V}_1 = (\sigma_1, 1, \sigma_3)^T$, and $\mathbf{W}_1 = (0,1,0)^T$, where $\sigma_1 = -\frac{\rho_6}{\rho_6+1}$ and $\sigma_3 = \frac{\rho_7}{\rho_6} - \frac{\rho_8}{\rho_5(\rho_6+\rho_9)}$. Moreover, equation (35) is used to provide the following:

$$\mathbf{F}_{\rho_4} = (0,-y,0)^T \Longrightarrow \mathbf{F}_{\rho_4}(e_3,\rho_4^*) = (0,0,0)^T \Longrightarrow \mathbf{W}_1^T\mathbf{F}_{\rho_4}(e_3,\rho_4^*) = 0.$$

$$D\mathbf{F}_{\rho_4}.\mathbf{V}_1 = (0,-1,0)^T \Longrightarrow \mathbf{W}_1^T.D\mathbf{F}_{\rho_4}(e_3,\rho_4^*).\mathbf{V}_1 = -1.$$

$$D^2\mathbf{F}(e_3,\rho_4^*).(\mathbf{V}_1,\mathbf{V}_1) = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix},$$

where:

$$m_1 = -2\sigma_1^2 - \frac{2}{\left(1+\frac{1}{\rho_6}\right)}\sigma_1 + \frac{2}{\left(1+\frac{1}{\rho_6}\right)^2}\sigma_3\ ,$$

$$m_2 = 2\rho_1\left(1+\frac{1}{\rho_6}\right)^{-1}\sigma_1 - 2\left(\rho_1\left(1+\frac{1}{\rho_6}\right)^{-2} + \rho_2\right)\sigma_3 + 2\rho_2\rho_3\frac{1}{\rho_6},$$

$$m_3 = -2\rho_5\rho_7^2\frac{1}{\rho_6} + \left(4\rho_5\rho_7 - 2\rho_8\left(1+\rho_9\frac{1}{\rho_6}\right)^{-2}\right)\sigma_3 - 2\rho_5\rho_6\sigma_3^2.$$

Hence, due to condition (37), it is obtained that:

$$\mathbf{W}_1^T[D^2\mathbf{F}(e_3,\rho_4^*).(\mathbf{V}_1,\mathbf{V}_1)] = (0,1,0)^T\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = m_2 \neq 0.$$

Hence, when $\rho_4 = \rho_4^*$, system (2) experiences a transcritical bifurcation at the equilibrium point e_3. Otherwise, when condition (37) is not satisfied, then using (36) yields:

$$D^3\mathbf{F}(e_3,\rho_4^*).(\mathbf{V}_1,\mathbf{V}_1,\mathbf{V}_1) = \begin{pmatrix} m_{11} \\ m_{21} \\ m_{31} \end{pmatrix},$$

where:

$$m_{11} = \frac{6\sigma_1\sigma_3}{(1+\frac{1}{\rho_6})^2} - \frac{6{\sigma_3}^2}{\left(1+\frac{1}{\rho_6}\right)^3},$$

$$m_{21} = -\frac{6\rho_1\sigma_1\sigma_3}{\left(1+\frac{1}{\rho_6}\right)^2} - \frac{6\rho_2\rho_3^2}{\rho_6} + 6\rho_2\rho_3\sigma_3 + \frac{6\rho_1{\sigma_3}^2}{\left(1+\frac{1}{\rho_6}\right)^3},$$

$$m_{31} = \frac{6\rho_5\rho_7^3}{\rho_6} - 12\rho_5\rho_7^2\sigma_3 + 6\left(\rho_5\rho_6\rho_7 + \frac{\rho_8\rho_9}{\left(1+\rho_9\frac{1}{\rho_6}\right)^3}\right){\sigma_3}^2.$$

Hence, due to condition (38), it is obtained that:

$$\mathbf{W}_1^T.D^3\mathbf{F}(e_3,\rho_4^*).(\mathbf{V}_1,\mathbf{V}_1,\mathbf{V}_1) = m_{21} \neq 0.$$

Hence, when $\rho_4 = \rho_4^*$, system (2) experiences a pitchfork bifurcation at the equilibrium point e_3. Thus, the proof is completed. ■

Theorem 9: Near the second planar equilibrium point, system (2) experiences a transcritical bifurcation when the parameter ρ_5 passes through the positive value $\rho_5^* = \rho_8\hat{y}$ if the following condition holds:

$$\rho_8\rho_9\hat{y} \neq \rho_8\delta_2 + \frac{\rho_5\rho_6}{1+\rho_7\hat{y}}, \tag{39}$$

where δ_2 is given in the proof. Otherwise, pitchfork bifurcation occurs provided that:

$$\left(\frac{\rho_5\rho_6\rho_7}{(1+\rho_7\hat{y})^2} + \rho_8\rho_9\right)\delta_2 \neq \rho_8\rho_9^2\hat{y}. \tag{40}$$

Proof: The matrix (13) at (e_4, ρ_5^*) yields:

$$J_2 = J(e_4, \rho_5^*) = \begin{pmatrix} -\hat{x} & -\hat{x} & \hat{x}\hat{y} \\ \rho_1\hat{y} & 0 & -\left(\rho_1\hat{x} + \frac{\rho_2}{1+\rho_3\hat{y}}\right)\hat{y} \\ 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of J_2 are as follows: $\lambda_{41,42} = \frac{-\hat{x}}{2} \pm \frac{\sqrt{\hat{x}^2 - 4\rho_1\hat{x}\hat{y}}}{2}$ and $\lambda_{43} = 0$. This transfers the point e_4 to a non-hyperbolic point. Let $\mathbf{V}_2 = (v_{12}, v_{22}, v_{32})^T$ and $\mathbf{W}_2 = (w_{12}, w_{22}, w_{32})^T$ be the eigenvectors corresponding $\lambda_{43} = 0$ of J_2 and ${J_2}^T$ respectively. The straightforward computation yields that $\mathbf{V}_2 = (\delta_1, \delta_2, 1)^T$, and $\mathbf{W}_2 = (0,0,1)^T$, where $\delta_1 = \hat{x} + \frac{\rho_2}{\rho_1(1+\rho_3\hat{y})} > 0$ and $\delta_2 = -\hat{x} + \hat{y} - \frac{\rho_2}{\rho_1(1+\rho_3\hat{y})}$. Moreover, equation (35) is used to provide the following:

$$\mathbf{F}_{\rho_5} = \left(0,0,z - \frac{\rho_6 z^2}{1+\rho_7 y}\right)^T \Rightarrow \mathbf{F}_{\rho_5}(e_4, \rho_5^*) = (0,0,0)^T \Rightarrow \mathbf{W}_2^T\mathbf{F}_{\rho_5}(e_4, \rho_5^*) = 0.$$

$$\mathbf{W}_2^T\left[D\mathbf{F}_{\rho_5}(e_4, \rho_5^*).\mathbf{V}_2\right] = (0,0,1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1 \neq 0.$$

$$D^2\mathbf{F}(e_4, \rho_5^*).(\mathbf{V}_2, \mathbf{V}_2) = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix},$$

where:

$$n_1 = -2\delta_1^2 - 2\delta_1\delta_2 + 2\hat{y}\delta_1 + 2x\delta_2 - 2\hat{x}\hat{y},$$

$$n_2 = 2\rho_1\delta_1\delta_2 - 2\rho_1\hat{y}\delta_1 - 2\left(\rho_1\hat{x} + \frac{\rho_2}{(1+\rho_3\hat{y})^2}\right)\delta_2 + 2\rho_1\hat{x}\hat{y},$$

$$n_3 = -2\rho_8\delta_2 - 2\left(\frac{\rho_5\rho_6}{1+\rho_7\hat{y}} - \rho_8\rho_9\hat{y}\right).$$

Thus, using condition (39) yields that:

$$\mathbf{W}_2^T[D^2\mathbf{F}(e_4, \rho_5^*).(\mathbf{V}_2, \mathbf{V}_2)] = n_3 \neq 0.$$

Hence, when $\rho_5 = \rho_5^*$, system (2) experiences a transcritical bifurcation at the equilibrium point e_4. When the condition (39) fails to be met, equation (36) yields the following result:

$$D^3\mathbf{F}(e_4, \rho_5^*).(\mathbf{V}_2, \mathbf{V}_2, \mathbf{V}_2) = \begin{pmatrix} n_{11} \\ n_{21} \\ n_{31} \end{pmatrix},$$

where:

$$n_{11} = 6\delta_1\delta_2 - 6\hat{x}\delta_2 - 6\hat{y}\delta_1 + 6\hat{x}\hat{y},$$

$$n_{21} = 6\rho_1\hat{y}\delta_1 - 6\rho_1\delta_1\delta_2 + \frac{6\rho_2\rho_3}{(1+\rho_3\hat{y})^3}\delta_2^2 + 6\rho_1\hat{x}\delta_2 - 6\rho_1\hat{x}\hat{y},$$

$$n_{31} = 6\left(\frac{\rho_5\rho_6\rho_7}{(1+\rho_7\hat{y})^2} + \rho_8\rho_9\right)\delta_2 - 6\rho_8\rho_9^2\hat{y}.$$

Hence, due to condition (40), it is obtained that:

$$\mathbf{W}_2^T[D^3\mathbf{F}(e_4,\rho_5^*).(\mathbf{V}_2,\mathbf{V}_2,\mathbf{V}_2)] = n_{31} \neq 0.$$

Thus, the proof is finished and the pitchfork bifurcation occurs. ■

Theorem 10: Assuming the conditions (17)-(19) hold, as the parameter ρ_6 reaches the value $\rho_6^* = \frac{\rho_8\left(\left((\bar{a}_{22}\bar{y}+\bar{a}_{23}\bar{z})a_{11}-\bar{a}_{21}(\bar{a}_{12}\bar{y}+\bar{a}_{13}\bar{z})\right)\rho_9+\bar{a}_{11}\bar{a}_{23}-\bar{a}_{21}\bar{a}_{13}\right)(1+\rho_7\bar{y})^2}{\rho_5\left(\left((\bar{a}_{22}\bar{y}+\bar{a}_{23}\bar{z})a_{11}-\bar{a}_{21}(\bar{a}_{12}\bar{y}+\bar{a}_{13}\bar{z})\right)\rho_7+\bar{a}_{11}\bar{a}_{22}-\bar{a}_{12}\bar{a}_{21}\right)(1+\rho_9\bar{z})^2}$, system (2) experiences a saddle-node bifurcation around the positive point if the following condition holds:

$$w_{13}s_{11} + w_{23}s_{21} + s_{31} \neq 0, \quad (41)$$

where the definition of each new symbol is represented in the proof.

Proof: The matrix (15) at (e_5,ρ_6^*) yields:

$$J_3 = J(e_5,\rho_6^*) = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} & \bar{a}_{13} \\ \bar{a}_{21} & \bar{a}_{22} & \bar{a}_{23} \\ 0 & \bar{a}_{32}^* & \bar{a}_{33}^* \end{pmatrix},$$

where $\bar{a}_{33}^* = \bar{a}_{33}^*(\rho_6^*)$, and $\bar{a}_{32}^* = \bar{a}_{32}^*(\rho_6^*)$.

Simple computations show that the determinant of J_3, represented by A_3 in equation (16), is zero. Therefore, J_3 will have a zero eigenvalue ($\lambda_3^* = 0$) and two additional non-zero eigenvalues. Thus, the point e_5 becomes a non-hyperbolic point. Let $\mathbf{V}_3 = (v_{13},v_{23},v_{33})^T$ and $\mathbf{W}_3 = (w_{13},w_{23},w_{33})^T$ be the eigenvectors corresponding $\lambda_3^* = 0$ of J_3 and ${J_3}^T$ respectively. Then straightforward computation yields that:

$$\mathbf{V}_3 = \begin{pmatrix} \frac{\bar{a}_{12}\bar{a}_{23}-\bar{a}_{13}\bar{a}_{22}}{\bar{a}_{11}\bar{a}_{22}-\bar{a}_{12}\bar{a}_{21}} \\ -\frac{\bar{a}_{11}\bar{a}_{23}-\bar{a}_{13}\bar{a}_{21}}{\bar{a}_{11}\bar{a}_{22}-\bar{a}_{12}\bar{a}_{21}} \\ 1 \end{pmatrix} = \begin{pmatrix} v_{13} \\ v_{23} \\ 1 \end{pmatrix},\ \mathbf{W}_3 = \begin{pmatrix} \frac{\bar{a}_{21}\bar{a}_{32}^*}{\bar{a}_{11}\bar{a}_{22}-\bar{a}_{12}\bar{a}_{21}} \\ -\frac{\bar{a}_{11}\bar{a}_{32}^*}{\bar{a}_{11}\bar{a}_{22}-\bar{a}_{12}\bar{a}_{21}} \\ 1 \end{pmatrix} = \begin{pmatrix} w_{13} \\ w_{23} \\ 1 \end{pmatrix}.$$

Moreover, equation (35) is used to provide the following:

$$\mathbf{F}_{\rho_6} = \left(0,0,-\frac{\rho_5 z^2}{1+\rho_7 y}\right)^T \Rightarrow \mathbf{F}_{\rho_6}(e_5,\rho_6^*) = \left(0,0,-\frac{\rho_5 \bar{z}^2}{1+\rho_7 \bar{y}}\right)^T$$
$$\Rightarrow \mathbf{W}_3^T \mathbf{F}_{\rho_6}(e_5,\rho_6^*) = -\frac{\rho_5 \bar{z}^2}{1+\rho_7 \bar{y}} \neq 0 .$$

In addition, it is obtained that:

$$D^2\mathbf{F}(e_5,\rho_6^*).(\mathbf{V}_3,\mathbf{V}_3) = \begin{pmatrix} s_{11} \\ s_{21} \\ s_{31} \end{pmatrix},$$

where $s_{11} = c_{11}(e_5,\rho_6^*,\mathbf{V}_3)$, $s_{21} = c_{21}(e_5,\rho_6^*,\mathbf{V}_3)$, and $s_{31} = c_{31}(e_5,\rho_6^*,\mathbf{V}_3)$. Hence, due to condition (41), it is obtained that:

$$\mathbf{W}_3^T[D^2\mathbf{F}(e_5,\rho_6^*).(\mathbf{V}_3,\mathbf{V}_3)] = w_{13}s_{11} + w_{23}s_{21} + s_{31} \neq 0.$$

Hence, when $\rho_6 = \rho_6^*$, system (2) experiences a Saddle-node bifurcation at the equilibrium point e_5. Thus, the proof is completed. ■

7. NUMERICAL SIMULATION

This section will numerically solve the food chain system (2) by picking biologically acceptable values for the parameters $\rho_i, (i = 1,2,3,\dots,9)$ represented by the set (42). The objective is to confirm the previously stated outcomes as well as to understand and explain the impact of changes in system parameters. The numerical solutions will be presented in a variety of ways using MATLAB version R2023b. All the obtained results use the initial points

$$I_1 = (0.75,0.75,0.75), I_2 = (0.1,0.25,0.9), I_3 = (0.9,0.1,0.9), I_4 = (1,0.5,0.2),\ I_5 = (1,1,0.5),$$
$$I_6 = (0.5,0.25,0.75), I_7 = (0.25,0.5,0.1), I_8 = (0.1,0.9,1), I_9 = (0.02,0.02,0.02),$$

and the following parameters set

$$\begin{gathered}\rho_1 = 0.75, \rho_2 = 0.05, \rho_3 = 0.5, \rho_4 = 0.2, \rho_5 = 0.5,\\ \rho_6 = 0.9, \rho_7 = 0.1, \rho_8 = 0.25, \rho_9 = 0.1\end{gathered} \tag{42}$$

Moreover, the red dot stands for the final state of the solution, while the blue dots stand for the above initial points. It is obtained that for the set (42) with the given initial data, system (2) approaches a positive equilibrium point as shown in Fig. 1.

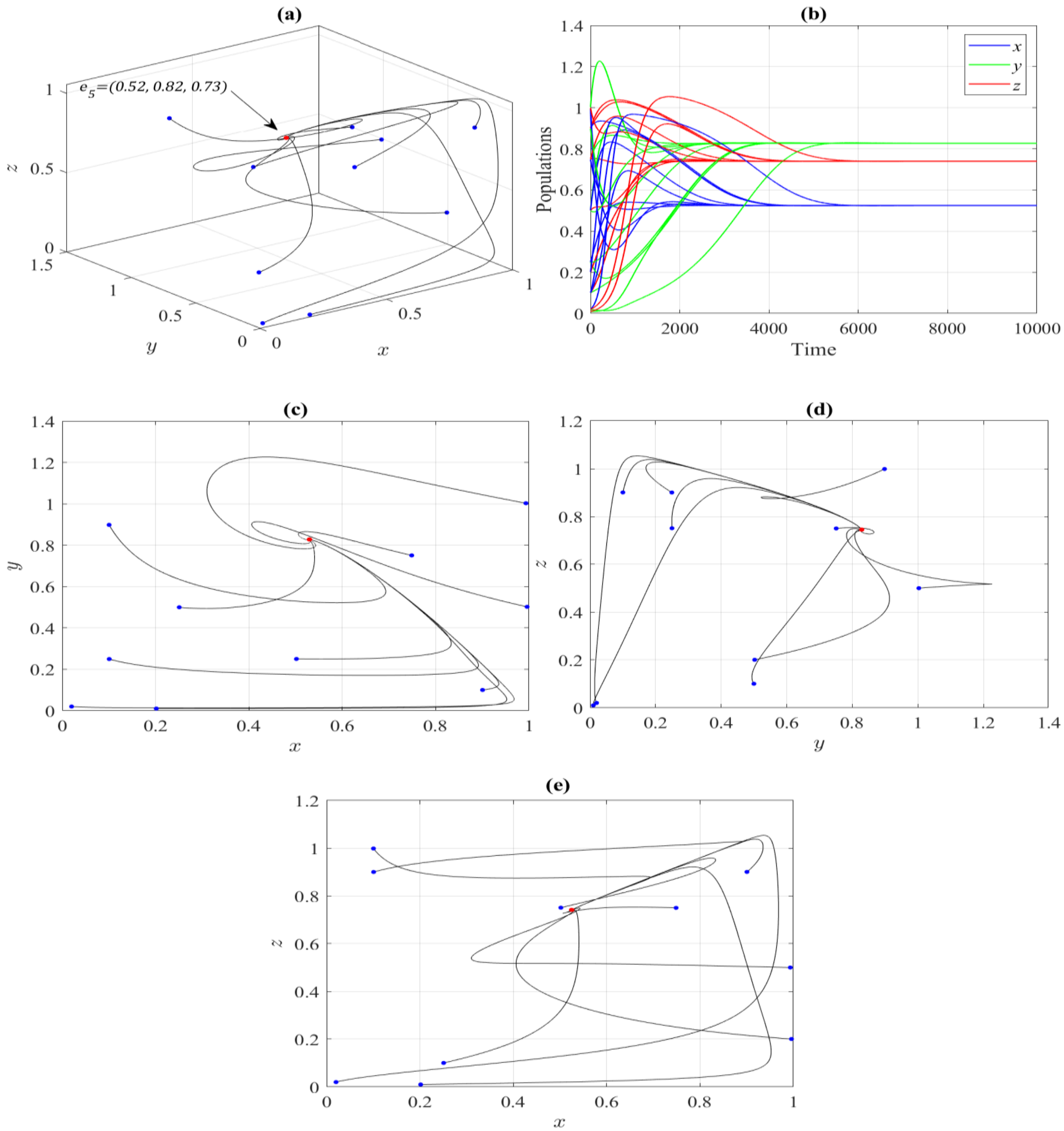

Figure 1. The trajectory of the system (2) using the parameters set (42). (a) Shows the 3D phase portrait of the system (2) from different initial points. (b) The time series. (c) The projection of the phase portrait on the xy-plane. (d) The projection of the phase portrait on the yz-plane. (e) The projection of the phase portrait on the xz-plane.

According to Fig. 1, system (2) approaches asymptotically to the positive equilibrium point $e_5 = (0.52,0.82,0.73)$ from different initial points.

Now, the effect of changing the parameter ρ_1 on the dynamics of the system (2) indicates that when $\rho_1 \leq 0.54$, the system approaches the first planar point e_3 from different initial points, the system approaches the positive point e_5 when $0.54 < \rho_1 \leq 3.5$, which is presented by Fig 1. Moreover, when $\rho_1 > 3.5$ reveals that the system exhibits 3D periodic dynamics see Fig. 2.

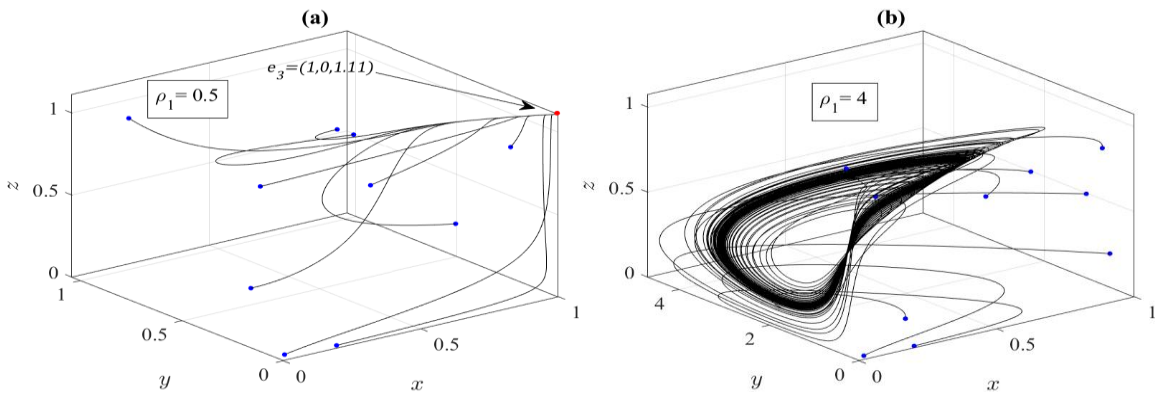

Figure 2. The 3D phase portrait of the system (2) using the parameters set (42) with different values of ρ_1. (a) Phase portrait shows the approaching of the system solution to the first planer point $e_3 = (1,0,1.11)$ when $\rho_1 = 0.5$. (b) 3D periodic dynamics when $\rho_1 = 4$.

The analysis of the impact of varying the parameter ρ_2 on the system's (2) dynamics reveals that it approaches the positive point e_5 when $\rho_2 < 0.15$ (see Fig. 1). Also, it approaches the first planer point e_3 when $\rho_2 \geq 0.15$, see Fig. 3.

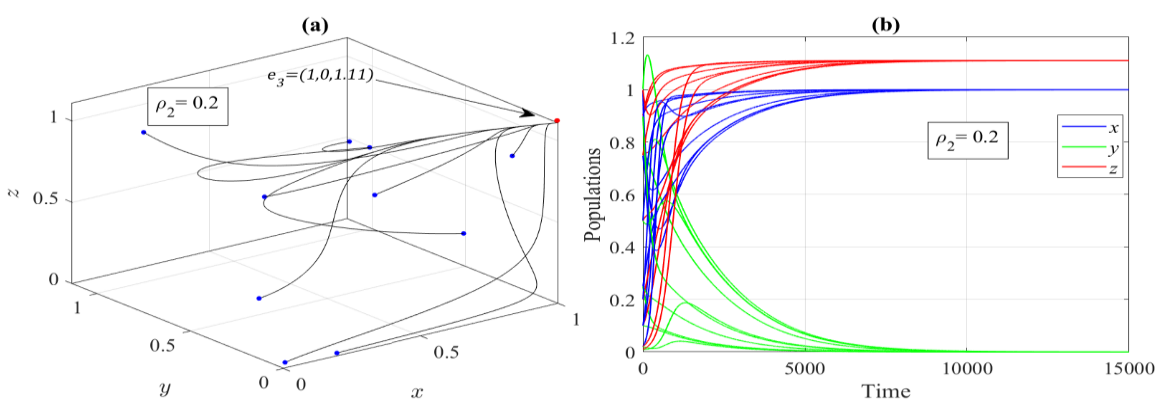

Figure 3. The trajectory of the system (2) using the parameters set (42) with $\rho_2 = 0.2$. (a) 3D phase portrait of the system (2) that approaches $e_3 = (1,0,1.11)$. (b) The time series.

It is observed that system (2) dynamics do not qualitatively change when the parameter ρ_3, ρ_7, and ρ_9 values vary. On the contrary, system (2) exhibits 3D periodic dynamics when $\rho_4 < 0.01$, it approaches asymptotically the positive point e_5 when $0.01 \leq \rho_4 < 0.3$ (see Fig. 1). Finally, system (2) approaches the first planar point e_3 when $\rho_4 \geq 0.3$, see Fig 4.

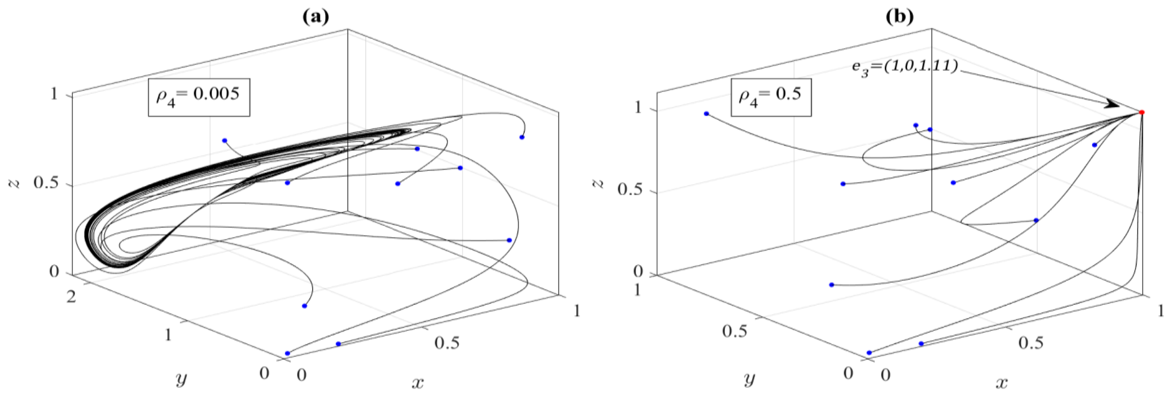

Figure 4. The 3D phase portrait of the system (2) using the parameters set (42). (a) 3D periodic dynamics when $\rho_4 = 0.005$. (b) System (2) approaches $e_3 = (1,0,1.11)$ with $\rho_4 = 0.5$.

The effect of changing the parameter ρ_5 on the dynamics of system (2) indicates that when $\rho_5 < 0.2$ the system approaches e_4 as in Fig. 5. Moreover, the system approaches e_5 when $\rho_5 \geq 0.2$, see Fig. 1.

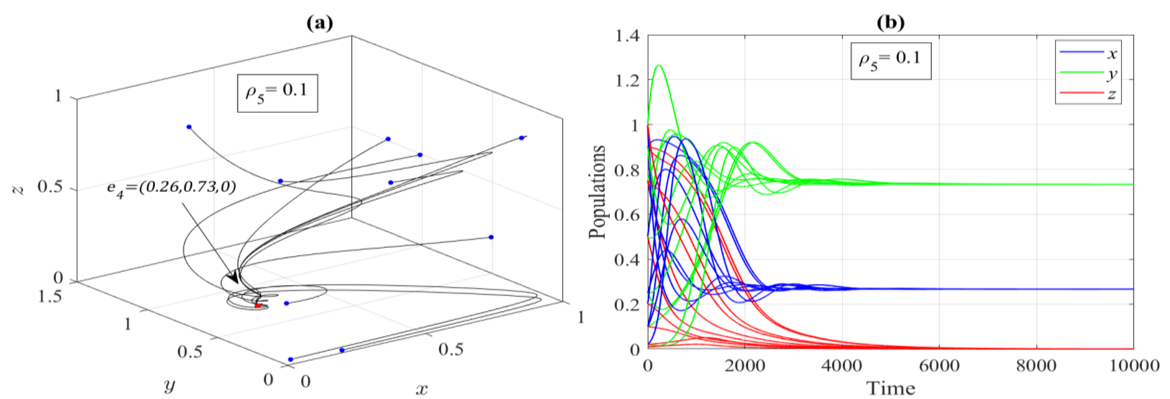

Figure 5. The trajectory of the system (2) using the set of parameters (42). (a) The 3D phase portrait of the system (2) approaches $e_4 = (0.26,0.73,0)$ when $\rho_5 = 0.1$. (b) The time series.

On the other hand, it is obtained that the parameters ρ_6 and ρ_8 will behave oppositely to the parameters ρ_2 and ρ_5 respectively. For more information see Table 2.

Table 2. The food chain system (2)'s dynamical behavior as a function of the parameters for ρ_6 and ρ_8 using the set of parameters (42).

The parameter	The range	The dynamic behaviour
$\boldsymbol{\rho_6}$	$\boldsymbol{\rho_6 \leq 0.6}$	System (2) approaches $\boldsymbol{e_3}$
	$\boldsymbol{\rho_6 > 0.6}$	System (2) approaches $\boldsymbol{e_5}$
$\boldsymbol{\rho_8}$	$\boldsymbol{\rho_8 < 0.68}$	System (2) approaches $\boldsymbol{e_5}$
	$\boldsymbol{\rho_8 \geq 0.68}$	System (2) approaches $\boldsymbol{e_4}$

8. DISCUSSION AND CONCLUSIONS

A Leslie-Gower food chain model consisting of producer-consumer-predators was formulated mathematically. Although consumers have a fear of predators, they are capable of defending themselves against predation. The solution properties of the proposed food chain model are obtained. All the feasible equilibrium points are determined. Their stability analysis is established. The persistence requirements are determined. The bifurcation analysis of the system is studied.

Finally, the system is solved numerically using an estimated set of parameter values and starting from different sets of initial points to verify the obtained analytical results and understand the impact of varying the parameter values. Regarding numerical results, it is obtained that decreasing (increasing) the consumer fear, alternative food for predators, and intrinsic growth rate of producers in comparison with that of predators stabilizes (loses the persistence) the food chain system. On the other hand, decreasing (increasing) the consumer fear, or half saturation constant, or intrinsic growth rate of producers in comparison with predators' attack rate loses the persistence (stabilizes) the food chain system. Similar results are obtained when decreasing (increasing) the half-saturation constant in comparison with the consumer antipredator level or intrinsic growth rate of producers in comparison with the consumer death rate. Finally, decreasing (increasing) the intrinsic growth rate of producers in comparison with the multiplication quotient between producer conversion rate and producer carrying capacity destabilizes but keeps the persistence (losing of persistence) of the food chain system. Similar results are obtained when increasing (or decreasing) the intrinsic growth rate of producers in comparison with the consumer death rate

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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