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STOCHASTIC DYNAMIC BEHAVIOR FOR THE PREY-PREDATOR PROBLEM WITH HOLLING-TYPE II FUNCTIONAL RESPONSE

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Abstract. The purpose of our study is to examine the stochastic stability of one of the important problems associated with ecology, namely the problem of prey and predators. This study was based on the type II functional response. After constructing the stochastic mathematical system that describes the problem, we demonstrated the system's biological acceptability by providing the necessary conditions for the solution to be positive as well as studying the stationary distribution. By doing so, we may have studied the stability of the system. The comparison approach is often employed to determine the prerequisites for the extinction or persistence of prey. Finally, to demonstrate the results' accuracy and realism, we present a numerical analysis of the problem in question.

Keywords: stochastic differential equations; stochastic dynamical systems; stochastic prey-predator model; stationary distribution; the local Lipschitz condition; the extinction and persistence; positive solution.

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1. INTRODUCTION

One of the most important problems associated with ecology is the issue of prey and predators. This topic has occupied the thoughts of many researchers interested in creating an ecological balance by finding a symbiotic relationship between prey and predator in one environment [29].

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A two-dimensional prey-predator model [23, 30, 47], in [14, 41] a multi-Predator model were studied, or in [17, 39, 46] a multi-prey model were studied. Models of diseased predator or diseased prey animals were studied in [13, 15, 25, 40, 50, 51]. Models with functional responses were also presented [8, 28, 30, 36]. The mechanism of pest control is the main point in agriculture. Biological pest control is a popular approach. There are new results related to pest control [18, 45, 44, 49].

Kumar, and Gunasundari [22], studied the communication between two prey and one predator species and are discuss the stability analysis. Using ergodic theory, Prasad, and Kumar [42], investigated the dynamic behavior of the three-species system consisting of prey, predator, and scavenger. Additionally, they have examined the dynamic bifurcation character of the prey-predator stochastic bifurcation events using a scavenger system.

An appropriate Lyapunov functions have been used to verify the existence of stable and unique ergodic distributions [21, 37, 43]. An ecosystem consisting of three species was studied in [7] with the interaction of prey and predator and a third species acting as a predator host for stability. The model is the second-type semantic responses in nonlinear differential equations. In order to model the phenomenon of predation, Holling [11] depend upon experiments, proposed three different types of functional responses. So, we will study the standard model that depends on the following model:

$$(1) \qquad \frac{dx_1}{dt} = rx_1(t)\left(1 - \frac{x_1(t)}{K}\right) - b\frac{x_1(t)x_2(t)}{\beta + x_1(t)}, \quad \frac{dx_2}{dt} = c\frac{x_1(t)x_2(t)}{\beta + x_1(t)} - d_1 x_2$$

wherever *r, K, b, c, d* > 0.$x_1(t)$ $(x_2(t))$ it epitomizes the density of prey (rep. predator) types. For more details about system 1, see B. Liu, Z. Tengb, L. Chen [26]. We outline some ideas regarding the necessary prerequisites in Section 2 so that the work requirements can be fulfilled. The development of the stochastic system for our primary problem is the focus of Section 3. The analysis of the behavior of the stochastic system, which contains the positive solution, and the research of the stationary distribution will be covered in Section 4. Section 5 presents the numerical results. In Section 6, we briefly discuss the main findings of this article.

2. PRELIMINARIES

Unless otherwise stated, the quartet $\Omega \equiv (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ will be indicated to the complete probability space.

We will provide some basic notations and definitions necessary to complete the search requirements. Let $\mathbb{R}_+^n$ be the subset of $\mathbb{R}^n$ with positive coordinate and an Euclidean metric.

Furthermore, for $z:(0,\infty)\longrightarrow\mathbb{R}$, define

$$\langle z(t)\rangle^{*}=\lim_{t\longrightarrow\infty}\sup\frac{1}{t}\int_0^t z(s)\,ds,\text{ and }\quad \langle z(t)\rangle_{*}=\lim_{t\longrightarrow\infty}\inf\frac{1}{t}\int_0^t z(s)\,ds$$

Definition 2.1. [24, 32] If $\lim_{t\to\infty}X_t=0$, then the X_t is called go extinct, a.s.

Lemma 2.1. [37] If $M:\mathbb{R}^{+}\times\Omega\longrightarrow\mathbb{R}$ be a function such that $M(\cdot,\omega):\mathbb{R}^{+}\longrightarrow\mathbb{R}$ is continuous for all ω , $M_t(\cdot)\equiv M(t,\cdot):\Omega\longrightarrow\mathbb{R}$ is local martingale and $M(0,\omega)=0$ for every ω. Then

(i) $\mathbb{P}\left\{\lim_{t\to\infty}\frac{M_t}{\langle M,M\rangle_t}=0\right\}=1$, whenever $\mathbb{P}\left\{\lim_{t\to\infty}\langle M,M\rangle_t=\infty\right\}=1$.

(ii) $\mathbb{P}\left\{\lim_{t\to\infty}\frac{M_t}{t}=0\right\}=1$, whenever $\mathbb{P}\left\{\limsup_{t\to\infty}\frac{\langle M,M\rangle_t}{t}=\infty\right\}=1$.

Lemma 2.2. [2, 48] Let $x:\Omega\times\mathbb{R}^{+}\longrightarrow(0,\infty)$ and $G:\Omega\times\mathbb{R}^{+}\longrightarrow\mathbb{R}$ be two functions of class C^1 and $\lim_{t\longrightarrow\infty}\frac{G_t}{t}=0$. Then for every $t\in\mathbb{R}^{+}$ we have

1. For some $\lambda,T\in\mathbb{R}$ and $,\lambda_0>0$ with

$$\ln X_t\leq\lambda t-\lambda_0\int_0^t X_s\,ds+G(t)\ \text{ a.s, for every } t\in[T,+\infty)$$

yield $\langle X_t\rangle^{*}\leq\frac{\lambda}{\lambda_0}$ a.s, whenever $\lambda\in[0,+\infty)$, $\lim_{t\longrightarrow\infty}X_t=0$ a.s, whenever $\lambda\in(-\infty,0)$.

(ii) For some $\lambda,\lambda_0,T\in(0,+\infty)$, with

$$\ln X_t\geq\lambda t-\lambda_0\int_0^t X_s\,ds+G(t)\ \text{ a.s., for every } t\in[T,+\infty)$$

yield $\langle X_t\rangle_{*}\geq\frac{\lambda}{\lambda_0}$ a.s. Now, if $X:\mathbb{R}^{+}\times\Omega\longrightarrow\mathbb{R}^{l}$ is a homogeneous Markov stochastic process such that

$$dX(t)=b(X)dt+\sum_{r=1}^{k}\sigma_r(X)\,dB_r(t). \tag{2}$$

Definition 2.2. [1,3]: the system (2) is said to be admit a stationary distribution $\pi(\cdot)$ when the distribution of $X(t)$ converges to $\pi=\pi_\gamma$, where γ is an initial distribution satisfy $\lim_{t\longrightarrow\infty}\mathbb{P}_\gamma\{X(t)\in M\}=\pi(M)$, for every measurable set M.

Hypothesis 2.1. [20]: Let $A(x)$ be the diffusion matrix [10]. The following axioms are admitted by a bonded open set $U\subset\mathbb{R}^{l}$ with a regular boundary ∂U.

(H1) The least eigenvalue of $A(x)$ is enclosed by zero in U.

(H2) whenever $x\in\mathbb{R}^{l}\setminus U$, then for every compact set K in $\mathbb{R}^{l}$ we have $\sup_{x\in K}\mathbb{R}^{l}\,\tau'<\infty$.

Proposition 2.1. [3]*:* Let $f(\cdot)$ be an integrable functional w.r.t μ. If Hypothesis *2.1* satisfies, then $X(t)$ admits a stationary distribution $\mu(\cdot)$. Furthermore, whenever $g(\cdot)$ is an integrable with respect to μ, then we have

$$\mathbb{P}_x\left\{\int_{\mathbb{R}^l} g(x)\mu(dx)\right\} = 1.$$

3. STOCHASTIC MODEL

Indeed, environmental white noise is an essential constituent of the ecosystem as it affects population dynamics [16]. Nonetheless in the deterministic case, not every parameters are affected by the environment. In order to recognize the random effects we formulate the population dynamics as system of SDE's, see [1,21, 27,31, 32, 33, 34, 35, 48]. For this purpose we will use the technique of [53]. For the system 1, specified $t \in (0, +\infty)$ and time instantaneous $t = j\Delta t$, present

$$Z^j = (X^j, Y^j) = (X_{j\Delta t}(\omega), Y_{j\Delta t}(\omega))^T, j = 0, 1, \dots$$

with $X^0 = (X_0(\omega), Y_0(\omega))^T \in \mathbb{R}_+^2$ as the initial value. If $\{R_i^j(m)\}_{m=0}^{\infty}$ is a stochastic process with a normal distribution satisfies

$$E[R_i^{\Delta t}(m)] = 0, \; E[R_i^{\Delta t}(m)]^2 = \sigma_i^2 \Delta t, \; E[R_i^{\Delta t}(m)]^4 = o(\Delta t),$$

where $i = 1, 2$, $m \in \mathbb{Z}^+$, and σ_i^2 represent the intensities of random disruption. Then

$$\begin{aligned} X^{m+1} &= X^m + X^m R_1^{\Delta t}(m) + X^m \left(r - \frac{rX^m}{K} - \frac{bY^m}{\beta + X^m}\right)\Delta t \\ Y^{m+1} &= Y^m + Y^m R_2^{\Delta t}(m) + Y^m \left(\frac{cX^m}{\beta + Y^m} - d_1\right)\Delta t \end{aligned} \tag{3}$$

If $t \longrightarrow 0$, then Z^m weak converges toward the solution of the system:

$$dX_t = X_t\left(r - \frac{r\bar{X}_t}{K} - \frac{bY_t}{\beta + \bar{X}_t}\right)dt + \sigma_1 X_t dB_1(t), \quad dY_t = Y_t\left(\frac{cX_t}{\beta + X_t} - d_1\right)dt + \sigma_2 Y_t dB_2(t) \tag{4}$$

wherever $B_1(t)$ and $B_2(t)$, represent the customary independent Wiener processes (see Theorem 7.1 and Lemma 8.2 in [5]). It is worth noting that the System 4 is more general than System 1 in [19].

4. LONG-TERM BEHAVIOR OF THE STOCHASTIC MODEL

Here we study the behavior of the system (4) and give essential dynamical properties of our problem.

Theorem 4.1. (The Positive Solution) For $(X_0(\omega), Y_0(\omega)) \in \mathbb{R}_+^2$, there is a unique positive solution $(X_t(\omega), Y_t(\omega))$ *of* (4) *on* $[0, +\infty)$, *and* $\mathbb{P}\{\omega: (X_t(\omega), Y_t(\omega)) \in \mathbb{R}_+^2\} = 1$.

Proof: Use the transformation $X_t = e^u$, $Y_t = e^v$ in (4) we get:

(5) $$du = \left(r - \frac{re^u}{K} - \frac{be^v}{\beta + e^u} - \frac{\sigma_1^2}{2}\right)dt + \sigma_1 dB_1(t), \quad dv = \left(\frac{ce^u}{\beta + e^u} - d_1 - \frac{\sigma_2^2}{2}\right)dt + \sigma_2 dB_2(t)$$

where $u_0(\omega) = \ln X_0(\omega)$, $v_0(\omega) = \ln Y_0(\omega)$. Wherever τ_e is the explosion time, the system (4) admits the local solution on $[0, \tau_e)$. This follows because the factors of system (4) gratify the local Lipschitz form. Using Itô's formula, (4) admits a unique local solution $(X_t(\omega), Y_t(\omega)) \in \mathbb{R}_+^2$ for $(X_0(\omega), Y_0(\omega)) \in \mathbb{R}_+^2$. This solution is global. For, let $k_0 \longrightarrow +\infty$ such that $(X_0(\omega), Y_0(\omega)) \in D_{k_0} = [1/k_0, k_0] \times [1/k_0, k_0]$. Describe the stopping time via

$$\tau_k = \inf\left\{t \in [0, \tau_e) : \max\{X_t, Y_t\} \geq k \text{ or } \min\{X_t, Y_t\} \leq \frac{1}{k}\right\}, \; k \in \mathbb{Z} \text{ with } \; k > k_0.$$

For the empty set $\emptyset$, put $\inf \emptyset = \infty$. Then $\tau_\infty \leq \tau_e$ a.s. when $\tau_\infty = \lim_{k\to\infty} \tau_k$. Suppose that $\tau_\infty = \infty$. or else, there exist a $\mathrm{T} > 0$ and $0 < \varepsilon < 1$ with $\mathbb{P}\{\tau_\infty \leq \mathrm{T}\} > \varepsilon$. So , there exists a $k_1 > k_0$ which satisfies $\mathbb{P}\{\tau_k \leq \mathrm{T}\} \geq \varepsilon$ for $k \geq k_1$. At present, for $(X_t(\omega), Y_t(\omega)) \in \mathbb{R}_+^2$, define

$$V(X_t, Y_t) \coloneqq (X_t + 1 - \ln X_t) + (Y_t + 1 - \ln Y_t).$$

We conclude from Itô's formula that

$$\begin{aligned} dV &= dX_t - \frac{1}{X_t}dX_t + dY_t - \frac{1}{Y_t}dY_t \\ &= \left(rX_t - \frac{rX_t^2}{K} - \frac{bX_tY_t}{\beta + X_t} - r + \frac{rX_t}{K} + \frac{bY_t}{\beta + X_t} + \frac{cX_tY_t}{\beta + X_t} - d_1Y_t - \frac{cX_t}{\beta + X_t} + d_1 + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2}\right)dt \\ &\quad + \sigma_1(X_t - 1)dB_1(t) + \sigma_2(Y_t - 1)dB_2(t) \\ &\leq \left[\left(r + \frac{r}{K}\right)X_t + (b + c)Y_t + d_1 - r + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2}\right]dt \\ &\quad + \sigma_1(X_t - 1)dB_1(t) + \sigma_2(Y_t - 1)dB_2(t) \end{aligned}$$

According to Lemma 4.1 in [4], we have

$$\left(r + \frac{r}{K}\right)X_t + (b + c)Y_t \leq 2\left[\left(r + \frac{r}{K}\right)(X_t - \ln X_t + 1) + (b + c)(Y_t - \ln Y_t + 1)\right]$$

Let $C_3 = \max\{C_1, C_2\}$, where $C_1 = d_1 - r + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2}$, $C_2 = \max\{r + \frac{r}{K}, b + c\}$. Consequently,

$$dV \leq C_3(V + 1)dt + \sigma_1(X_t - 1)dB_1(t) + \sigma_2(Y_t - 1)dB_2(t).$$

Evaluate the integrations

$$\int_0^{\tau_k \wedge \mathrm{T}} dV \leq \int_0^{\tau_k \wedge \mathrm{T}} C_3(V + 1)\, ds + \int_0^{\tau_k \wedge \mathrm{T}} \sigma_1(X_t - 1)\, dB_1(s) + \int_0^{\tau_k \wedge \mathrm{T}} \sigma_2(Y_t - 1)\, dB_2(s),$$

then taking the expectation and using Grownwall's inequality we get

$$EV(X_{\tau_k\wedge \mathrm{T}},Y_{\tau_k\wedge \mathrm{T}}) \le [V(X_0,Y_0)+C_3\mathrm{T}]e^{C_3\mathrm{T}} = M_0,$$

So we get

$$V(X_{\tau_k\wedge \mathrm{T}},Y_{\tau_k\wedge \mathrm{T}}) \ge (\tfrac{1}{k} - 1 - \ln\tfrac{1}{k}) \wedge (k - 1 - \ln k).$$

Then one can be derived that

$$M_0 \ge E[1_{\Omega_t}V(X_{\tau_k\wedge \mathrm{T}},Y_{\tau_k\wedge \mathrm{T}})] \ge \varepsilon\left[\tfrac{1}{k} - 1 - \ln\tfrac{1}{k}\right] \wedge [k-1-\ln k],$$

wherever 1_{Ω_t} is a characteristic function of Ω_k. But that is a contradiction to the hypothesis.

Theorem 4.2. Assume $1+\frac{1}{\beta}\bar{X}_t - c\bar{X}_t \ge 0,\ 1<\frac{c}{d_1}$. If

$$\omega < \min\left\{\left(\frac{r}{K} - b\bar{Y}_t - \sigma_1^2 - \frac{l_1 b}{2} + \frac{b\bar{X}_t}{2}\right)\bar{X}_t^{\,2}, \left(l_1 b\bar{X}_t - l_1^2\sigma_1^2 - \frac{l_1 b}{2} + \frac{b\bar{X}_t}{2}\right)\bar{Y}_t^{\,2}\right\},$$

where $\omega = \frac{\sigma_1^2}{2}\bar{X}_t + \frac{l_1\sigma_2^2}{2}\bar{Y}_t + \sigma_1^2\bar{X}_t^{\,2} + l_1^2\sigma_2^2\bar{Y}_t^{\,2}$ and $l_1 = \frac{b(\beta+c\bar{X}_t)}{c(\beta+\bar{X}_t)}$,

then the system (4) admits an ergodic stationary distribution.

Proof. If $d_1 < c$, then the equilibrium of (1) is $\bar{E} = (\bar{X}_t,\bar{Y}_t) = \left(\frac{d_1\beta}{(c-d_1)}, \frac{c\beta r[K(c-d_1)-d_1\beta]}{Kb(c-d_1)}\right)$.

Define

$$V = X_t - \bar{X}_t - \bar{X}_t\ln\frac{X_t}{\bar{X}_t} + l_1\left(Y_t - \bar{Y}_t - \bar{Y}_t\ln\frac{Y_t}{\bar{Y}_t}\right) + \frac{1}{2}[(X_t-\bar{X}_t)+l_1(Y_t-\bar{Y}_t)]^2$$

$$= V_1 + V_2$$

where

$$V_1 = X_t - \bar{X}_t - \bar{X}_t\ln\frac{X_t}{\bar{X}_t} + l_1\left(Y_t - \bar{Y}_t - \bar{Y}_t\ln\frac{Y_t}{\bar{Y}_t}\right),\quad V_2 = \frac{1}{2}[(X_t-\bar{X}_t)+l_1(Y_t-\bar{Y}_t)]^2 .$$

We conclude from Itô's formula that

$$dV_1 = LV_1dt + \sigma_1(X_t-\bar{X}_t)dB_1(t) + l_1(Y_t-\bar{Y}_t)dB_2(t),$$

where

$$LV_1 = (X_t-\bar{X}_t)\left(r - \frac{rX_t}{K} - \frac{bY_t}{\beta+X_t}\right) + l_1(Y_t-\bar{Y}_t)\left(\frac{cX_t}{\beta+X_t} - d_1\right) + \frac{\sigma_1^2}{2}\bar{X}_t + \frac{l_1\sigma_2^2}{2}\bar{Y}_t$$

$$= (X_t-\bar{X}_t)\left(-\frac{r}{K}(X_t-\bar{X}_t) - \frac{b}{\beta+\bar{x}}(Y_t-\bar{Y}_t)\right) + \frac{l_1c\left(\frac{1}{\beta}\right)(X_t-\bar{X}_t)(Y_t-\bar{Y}_t)}{\left[1+\frac{1}{\beta}X_t\right]\left[1+c\left(\frac{1}{\beta}\right)\bar{X}_t\right]} + \frac{\sigma_1^2}{2}\bar{X}_t + \frac{l_1\sigma_2^2}{2}\bar{Y}_t$$

$$\le -\frac{r}{K}(X_t-\bar{X}_t)^2 - \frac{b}{\beta+\bar{X}_t}(X_t-\bar{X}_t)(Y_t-\bar{Y}_t) + \frac{l_1c\left(\frac{1}{\beta}\right)(X_t-\bar{X}_t)(Y_t-\bar{Y}_t)}{1+c\left(\frac{1}{\beta}\right)\bar{X}_t} + \frac{\sigma_1^2}{2}\bar{X}_t + \frac{l_1\sigma_2^2}{2}\bar{Y}_t$$

$$\leq -\frac{r}{K}(X_t - \bar{X}_t)^2 + \frac{\sigma_1^2}{2}\bar{X}_t + \frac{l_1\sigma_2^2}{2}\bar{Y}_t$$

We conclude from Itô's formula that

$$V_2 = \frac{1}{2}[(X_t - \bar{X}_t) + l_1(Y_t - \bar{Y}_t)]^2$$

$$dV_2 = LV_2 dt + (\sigma_1 X_t + l_1\sigma_2 Y_t)(X_t - \bar{X}_t)dB_1(t) + l_1(\sigma_1 X_t + l_1\sigma_2 Y_t)(Y_t - \bar{Y}_t)dB_2(t),$$

and

$$LV_2 \leq \left(\left(\frac{1}{\beta}\right)\bar{Y}_t + \sigma_1^2 + \frac{1}{2}l_1\left(\frac{1}{\beta}\right)\bar{Y}_t - \frac{1}{2}\frac{b}{\beta+\bar{X}_t}\bar{X}_t\right)(X_t - \bar{X}_t)^2$$

$$-(l_1\frac{b}{\beta+\bar{X}_t}\bar{X}_t - l_1^2\sigma_2^2 - \frac{1}{2}l_1\left(\frac{1}{\beta}\right)\bar{Y}_t + \frac{1}{2}\frac{b}{\beta+\bar{X}_t}\bar{X}_t\ \ (Y_t - \bar{Y}_t)^2 + \sigma_1^2\bar{X}_t{}^2 + l_1^2\sigma_2^2\bar{Y}_t{}^2$$

Consequently, $\frac{X_t{}^2}{2} \leq (X_t - \bar{X}_t) + \bar{X}_t{}^2$ and $\frac{Y_t{}^2}{2} \leq (Y_t - \bar{Y}_t) + Y_t{}^2$. Therefore,

$$LV \leq -\left(\frac{r}{K} - \left(\frac{1}{\beta}\right)\bar{Y}_t - \sigma_1^2 + \frac{1}{2}l_1\left(\frac{1}{\beta}\right)\bar{Y}_t + \frac{1}{2}\frac{b}{\beta+\bar{X}_t}\bar{X}_t\right)(X_t - \bar{X}_t)^2$$

$$-(l_1\frac{b}{\beta+\bar{X}_t}\bar{X}_t - l_1^2\sigma_2^2 - \frac{1}{2}l_1\left(\frac{1}{\beta}\right)\bar{Y}_t + \frac{1}{2}\frac{b}{\beta+\bar{X}_t}\bar{X}_t)\ \ (Y_t - \bar{Y}_t)^2$$

$$+\frac{\sigma_1^2}{2}\bar{X}_t + \frac{l_1^2\sigma_2^2}{2}\bar{Y}_t + \sigma_1^2\bar{X}_t{}^2 + l_1^2\sigma_2^2\bar{Y}_t{}^2\ .$$

When

$$\omega < \min\left\{\left(\frac{r}{K} - \left(\frac{1}{\beta}\right)\bar{Y}_t - \sigma_1^2 + \frac{1}{2}l_1\left(\frac{1}{\beta}\right)\bar{Y}_t + \frac{1}{2}\frac{b}{\beta+\bar{x}}\bar{X}_t\right)\bar{X}_t{}^2, (l_1\frac{b}{\beta+\bar{X}_t}\bar{X}_t - l_1^2\sigma_2^2 - \frac{1}{2}l_1\left(\frac{1}{\beta}\right)\bar{Y}_t + \frac{1}{2}\frac{b}{\beta+\bar{X}_t}\bar{X}_t)\ \ \bar{Y}_t{}^2\right\},$$

the ellipsoid

$$-\left(\frac{r}{K} - \left(\frac{1}{\beta}\right)\bar{Y}_t - \sigma_1^2 + \frac{1}{2}l_1\left(\frac{1}{\beta}\right)\bar{Y}_t + \frac{1}{2}\frac{b}{\beta+\bar{X}_t}\bar{X}_t\right)(X_t - \bar{X}_t)^2$$

$$-\left(l_1\frac{b}{\beta+\bar{X}_t}\bar{X}_t - l_1^2\sigma_2^2 - \frac{1}{2}l_1\left(\frac{1}{\beta}\right)\bar{Y}_t + \frac{1}{2}\frac{b}{\beta+\bar{X}_t}\bar{X}_t\right)\ (Y_t - \bar{Y}_t)^2 + \omega = 0$$

lies entirely in $\mathbb{R}_+^2$. Let $\mathfrak{N}$ be an open set containing the ellipsoid with $\overline{\mathfrak{N}} \subseteq E_2 \setminus \mathfrak{N}$, so there exists a $\bar{K} > 0$ with $LV \leq -\bar{K}$ whenever $(X_t, Y_t) \in E_2 \setminus \mathfrak{N}$. That is, axiom (H2) in Hypothesis 2.5 is satisfy. Furthermore, for every $(X_t, Y_t) \in \overline{\mathfrak{N}}$ and $\upsilon \in \mathbb{R}^2$, there is $\lambda > 0$ such that

$$\sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j = \sigma_1 X_t{}^2\xi_1^2 + \sigma_2 Y_t{}^2\xi_2^2 \geq \lambda\|\upsilon\|^2.$$

Note that λ it can be chosen as $\lambda \coloneqq \min\{\sigma_1 X_t{}^2, \sigma_2 Y_t{}^2, (X_t, Y_t) \in \mathfrak{N}\}$. Therefore axiom (H1) in Hypothesis 2.1 is hold. Hence, by Proposition 2.1, we get the result.

Theorem 4.3. The population X_t in (4), satisfied:

(6) $$\lim_{t\longrightarrow+\infty} \sup \frac{1}{t} \ln X_t \leq 0 \text{ almost surely,}$$

for any initial value $X_0 \in \mathbb{R}_+$.

Proof: Since $dX_t = X_t\left(r - \frac{r\bar{X}_t}{K} - \frac{bY_t}{\beta+\bar{X}_t}\right)dt + \sigma_1 X_t dB_1(t)$, Itô's formula yield:

$$d\ln X_t = \left(r - \frac{rX_t}{K} - \frac{bY_t}{\beta+X_t} - \frac{\sigma_1^2}{2}\right)dt + \sigma_1 dB_1(t) \leq \left(r - \frac{rX_t}{K} - \frac{\sigma_1^2}{2}\right)dt + \sigma_1 dB_1(t)$$

Now,

$$d\ln z = \left(r - \frac{r}{K}z - \frac{\sigma_1^2}{2}\right)dt + \sigma_1 dB_1(t),\ z_0 = x(0).$$

Set $V_1 = e^t \ln z$. We conclude from Itô's formula that

$$dV_1 = LV_1\, dt + e^t\sigma_1 dB_1(t), \text{ where } LV_1 = e^t\left(\ln z + r - \frac{r}{K}z - \frac{\sigma_1^2}{2}\right).$$

Take the integral from zero toward t, we can get that

$$e^t \ln z\,(t) - \ln z_0 = \int_0^t e^s\left[\ln z(s) + r - \frac{r}{K}z(s) - \frac{\sigma_1^2}{2}\right]ds + \int_0^t e^s\sigma_1\, dB_1(s).$$

Denote $M_1(t) = \int_0^t e^s\sigma_1\, dB_1(s)$, then quadratic variation is $\langle M_1(t), M_1(t)\rangle = \int_0^t e^{2s}\sigma_1^2\, ds$.

By the exponential martingale inequality [9], for $T_0, c_1,\ c_2 > 0$, yields

(7) $$\mathbb{P}\left\{\sup_{0\leq t\leq T_0}\left[M_1(t) - \frac{c_1}{2}\langle M_1(t), M_1(t)\rangle\right] > c_2\right\} \leq e^{-c_1c_2}$$

Using the analogous technique as Zhu *et al.* [48], put $T_0 = \lambda_0 v,\ c_1 = \varepsilon e^{-\lambda_0 v}$, $c_2 = \frac{\theta e^{\lambda_0 v}\ln\lambda_0}{\varepsilon}$, where $\lambda_0 \in \mathbb{Z}^+,\ v > 0,\ \varepsilon \in (0,1)$ and $\theta > 1$. So,

$$\mathbb{P}\left\{\sup_{0\leq t\leq \lambda_0 v}\left[M_1(t) - \frac{\varepsilon e^{-\lambda_0 v}}{2}\langle M_1(t), M_1(t)\rangle\right] > \frac{\theta e^{\lambda_0 v}\ln\lambda_0}{\varepsilon}\right\} \leq \lambda_0^{-\theta}$$

Since $\sum_{\lambda_0=1}^{\infty}\lambda_0^{-\theta} < \infty$ .Applying Borel–Cantalli Lemma, there is $\Omega_i \subset \Omega$ with the property if $\varpi \in \Omega_i$, then there is a $\lambda_i = \lambda_i(\varpi)$, so for every $\lambda_0 > \lambda_i$, we derive

$$M_1(t) \leq \frac{\varepsilon e^{-\lambda_0 v}}{2}\langle M_1(t), M_1(t)\rangle + \frac{\theta e^{\lambda_0 v}\ln\lambda_0}{\varepsilon},\ 0 \leq t \leq \lambda_0 v.$$

Pick $\Omega_0 = \bigcap_{i=1}^2\Omega_i$. Hence $\mathbb{P}(\Omega_0) = 1$. For any $\varpi \in \Omega_0$, define $\lambda_0(\varpi) = \max\{\lambda_i(\varpi): i = 1,2,\dots,n\}$. So, $\sum_{i=1}^n M_1(t) \leq \frac{\varepsilon e^{-\lambda_0 v}}{2}\langle M_1(t), M_1(t)\rangle + \frac{\theta e^{\lambda_0 v}\ln\lambda_0}{\varepsilon}$, $0 \leq t \leq \lambda_0$. holds. Consequently, for $0 \leq t \leq \lambda_0 v$, it holds that

$$e^t \ln z\,(t) - \ln z_0 \leq \int_0^t e^s\left[\ln z(s) + r - \frac{r}{K}z(s) + \frac{\sigma_1^2}{2}(\varepsilon e^{s-\lambda_0 v} - 1)\right]ds + \frac{\theta e^{\lambda_0 v}\ln\lambda_0}{\varepsilon}$$

Thus, there is M_1 verify

$$\ln z(t) + r - \frac{r}{K} z(t) + \frac{\sigma_1^2}{2}\left(\varepsilon e^{t-\lambda_0 v} - 1\right) \le M_1, \text{ over } [0, \lambda_0 v]..$$

If $(\lambda_0 - 1)v \le t \le \lambda_0 v$, $\lambda_0 = \lambda_0(\varpi)$, then

$$e^t \ln z(t) - \ln z_0 \le M_1(e^t - 1) + \frac{\theta e^{\lambda_0 v} \ln \lambda_0}{\varepsilon}$$

So $\limsup_{t\to\infty} \frac{\ln z(t)}{\ln t} \le \frac{\theta e^v}{\varepsilon}$. Consequently, by letting $v \downarrow 0$, $\varepsilon \uparrow 1$ and $\theta \uparrow 1$, we get

$$\limsup_{t\to\infty} \frac{\ln z(t)}{\ln t} \le 1 \quad \text{a.s.}$$

Corollary 4.1. Consider the sumptions of Theorem 4.3. Then $\limsup_{t\to\infty} \frac{1}{t} \ln X_t \le 0$ a.s.

Proof By Theorem 4.3,

$$\limsup_{t\to\infty} \frac{1}{t} \ln X_t = \limsup_{t\to\infty} \frac{1}{\ln t} \ln X_t \limsup_{t\to\infty} \frac{1}{t} \ln t \le 0.$$

Theorem 4.4. For X_t in (4), we have

(1) if $r - \frac{\sigma_1^2}{2} < 0$, then $\lim_{t\to+\infty} X_t = 0$.

(2) if $r - \frac{\sigma_1^2}{2} > 0$, then $\langle X_t \rangle^* > 0$.

Proof: (1) By

$$dX_t = X_t\left(r - \frac{rX_t}{K} - \frac{bY_t}{\beta + X_t}\right)dt + \sigma_1 X_t dB_1(t) \le X_t\left(r - \frac{rX_t}{K}\right)dt + \sigma_1 X_t dB_1(t)$$

we construction a comparison stochastic system:

$$dX_t = X_t\left(r - \frac{r}{K}X_t\right)dt + \sigma_1 X_t dB_1(t), \text{ with initial condition } X_0.$$

We conclude from Itô's formula that

$$d \ln X_t = \left(r - \frac{r}{K}X_t - \frac{\sigma_1^2}{2}\right)dt + \sigma_1 dB_1(t).$$

By performing the integration from zero toward t for the above equation, yields

$$\ln X_t - \ln X_0 = \int_0^t \left[r - \frac{r}{K}X_s - \frac{\sigma_1^2}{2}\right] ds + \int_0^t \sigma_1\, dB_1(s)$$

$$= \int_0^t \left[r - \frac{r}{K}X_s - \frac{\sigma_1^2}{2}\right] ds + \int_0^t \sigma_1\, dB_1(s),$$

So $\ln X_t = \ln X_0 + \int_0^t \left[r - \frac{r}{K}X_s - \frac{\sigma_1^2}{2}\right] ds + \int_0^t \sigma_1\, dB_1(s)$ .Thus

$$X_t = X_0 \exp\left\{\int_0^t \left[r - \frac{r}{K}X_s - \frac{\sigma_1^2}{2}\right] ds + M_1(t)\right\},$$

Where $M_1(t)=\int_0^t \sigma_1\, dB_1(s)$. Hence $\lim_{t\longrightarrow+\infty}\sup\frac{M_1(t)}{t}=0$. Consequently,

$\limsup_{t\to\infty}\frac{1}{t}\ln X_t\le r-\frac{\sigma_1^2}{2}<0$ a.s.. So $\limsup_{t\to\infty}\frac{1}{t}\ln X_t<0$, hence $\lim_{t\longrightarrow+\infty}X_t=0$.

(2) It is sufficient to show that there exists a $u>0$ such that any solution of (4) fulfills $\langle X_t\rangle^*\ge u>0$. Assume contrary that the consequence is incorrect. Choose ε_1 be arbitrary small so that $-d_1-\frac{\sigma_2^2}{2}+\frac{c}{\beta}\varepsilon_1<0$, $r-\frac{\sigma_1^2}{2}-\frac{r}{K}\varepsilon_1>0$. So for all $\varepsilon_1>0$, there exists the solution $(\bar{X}_t,\bar{Y}_t)$ with $\mathbb{P}\{\langle\bar{X}_t\rangle^*<\varepsilon_1\}>0$. Consequently, $d\ln\bar{Y}_t\le\left(\frac{c}{\beta}\bar{X}_t-d_1-\frac{\sigma_2^2}{2}\right)dt+\sigma_2 dB_2(t)$. By performing the integration from zero toward t for the above equation and then divide by t, yields

$\frac{1}{t}(\ln\bar{Y}_t-\ln\bar{Y}_0)\le\frac{1}{t}\int_0^t\left(-d_1-\frac{\sigma_2^2}{2}\right)ds+\frac{1}{t}\int_0^t\frac{c}{\beta}\bar{X}_s\, ds+\frac{1}{t}\int_0^t\sigma_2\, dB_2(s)=-d_1-\frac{\sigma_2^2}{2}+\frac{c}{\beta}\frac{1}{t}\int_0^t\bar{X}_s\, ds+\frac{M_2(t)}{t}$,

so

$$\frac{1}{t}(\ln\bar{Y}_t-\ln\bar{Y}_0)\le-d_1-\frac{\sigma_2^2}{2}+\frac{c}{\beta}\frac{1}{t}\int_0^t\bar{X}_s\, ds+\frac{M_2(t)}{t}\ , \tag{8}$$

where $M_2(t)=\int_0^t\sigma_2\, dB_2(s)$. So, $\limsup_{t\longrightarrow+\infty}\frac{M_2(t)}{t}=0$.

Hence,

$\lim_{t\longrightarrow+\infty}\sup\frac{1}{t}\ln\bar{Y}_t\le-d_1-\frac{\sigma_2^2}{2}+\frac{c}{\beta}\varepsilon_1<0$. So $\lim_{t\longrightarrow+\infty}\bar{Y}_t=0$.

Furthermore,

$$d\ln\bar{X}_t=\left[r-\frac{r}{K}\bar{X}_t-\frac{b}{\beta+\bar{X}_t}\bar{Y}_t-\frac{\sigma_1^2}{2}\right]dt+\sigma_2 dB_2(t)$$

Consequently,

$\frac{1}{t}[\ln\bar{X}_t-\ln\bar{X}_0]=\frac{1}{t}\int_0^t\left(r-\frac{\sigma_1^2}{2}\right)ds-\frac{1}{t}\int_0^t\frac{r}{K}\bar{X}_s\, ds+\frac{1}{t}\int_0^t\frac{b}{\beta+\bar{X}_s}\bar{Y}_s\, ds+\frac{M_1(t)}{t}$

$=r-\frac{\sigma_1^2}{2}-\frac{1}{t}\int_0^t\frac{r}{K}\bar{X}_s\, ds-\frac{1}{t}\int_0^t\frac{b}{\beta+\bar{X}_s}\bar{Y}_s\, ds+\frac{M_1(t)}{t}$.

Therefore, $\limsup_{t\longrightarrow+\infty}\frac{M_1(t)}{t}=0$ is verified.

Hence,

$$\limsup_{t\longrightarrow+\infty}\frac{1}{t}\ln\bar{X}_t=r-\frac{\sigma_2^2}{2}+\frac{r}{K}\varepsilon_1\ >\ 0.$$

This lead to a contradiction with Theorem 4.1. Therefore, $\langle X_t\rangle^* > 0$.

Theorem 4.5: Consider the model 3. *If* $\frac{c}{\beta}K(r-\frac{\sigma_1^2}{2}) < r(d_1+\frac{\sigma_2^2}{2})$, then the population Y_t will lean towards extinct a.s.

Proof: If $r-\frac{\sigma_1^2}{2} \le 0$, then $\langle X_t\rangle^* < 0$. According to the same method as inequality 7, we get

$$\frac{1}{t}[\ln Y_t - \ln Y_0] \le -d_1 - \frac{\sigma_2^2}{2} + \frac{c}{\beta}\frac{1}{t}\int_0^t X_s\, ds + \frac{M_2(t)}{t} .$$

Consequently, $\limsup_{t\to+\infty}\frac{1}{t}\ln Y_t \le -d_1 - \frac{\sigma_2^2}{2} < 0$. So $\lim_{t\to\infty} Y_t = 0$. Furthermore, if $r-\frac{\sigma_1^2}{2} > 0$, then for every $\varepsilon_2 > 0$, there exists $\tau \in (0,+\infty)$ so that $\frac{M_2(t)}{t} \le \varepsilon_2$. Then

$$\ln X_t - \ln X_0 \le \int_0^t \left(r-\frac{\sigma_1^2}{2}\right) ds - \frac{r}{K}\int_0^t X_s\, ds + \frac{M_1(t)}{t} \le \left(r-\frac{\sigma_1^2}{2}+\varepsilon_2\right)t - \frac{r}{K}\int_0^t X_s\, ds$$

So $\langle X_t\rangle^* \le \frac{K(r-\sigma_1^2/2+\varepsilon_2)}{r}$ by Lemma 2.3. Let $\varepsilon_2 \to 0$, then $\langle X_t\rangle^* \le \frac{K(r-\sigma_1^2/2)}{r}$. Therefore

$$\lim_{t\to+\infty}\sup\frac{1}{t}\ln Y_t \le -d_1 - \frac{\sigma_2^2}{2} + \frac{c}{\beta}\langle X_t\rangle^*,$$

so

$$\lim_{t\to+\infty}\sup\frac{1}{t}\ln Y_t \le -d_1 - \frac{\sigma_2^2}{2} + \frac{c}{\beta}\frac{K(r-\sigma_1^2/2)}{r} = \frac{cK(r-\sigma_1^2/2) - r\beta(-d_1-\sigma_2^2/2)}{r\beta} \tag{9}$$

Then $\limsup_{t\to+\infty}\frac{1}{t}\ln Y_t < 0$. As a result $\lim_{t\to\infty} Y_t = 0$.

5. NUMERICAL ANALYSIS FOR THE PROBLEM

In order to make our conclusions more realistic and verify the results obtained and find out how much they correspond to reality we conduct a numerical simulation. This is done using Milstein's higher order Model [12]. The analogous estimation equations are

$$\begin{aligned} X_{k+1} &= X_k + X_k\left(r - \frac{r}{K}X_k - \frac{bX_k}{\beta+X_k}\right)\Delta t + \sigma_1 X_k\sqrt{\Delta t}\xi_k + \frac{\sigma_1^2 X_k}{2}(\xi_k^2-1)\Delta t \\ Y_{k+1} &= Y_k + Y_k\left(\frac{cX_k}{\beta+X_k} - d_1\right)\Delta t + \sigma_2 Y_k\sqrt{\Delta t}\varsigma_k + \frac{\sigma_2^2 Y_k}{2}(\varsigma_k^2-1)\Delta t \end{aligned} \tag{10}$$

Consider the system (4), take $(X_0, Y_0) = (0.9, 0.8)$ as the initial value and take:

(11) $\quad r = 0.4, K = 1.3,\ c = 2.6, \quad b = 0.25,,\ d_1 = 0.2,\ \beta = 8$

Due to $d_1 < c$, the model 2 be existent the critical point $\bar{E} = (\bar{X}, \bar{Y})$, where $\bar{X} \approx 0.6667$, $\bar{Y} \approx 0.7795$. So as to demonstrate the outcome of white noise on populations X_t and Y_t, take $\sigma_1 = 0.05$, $\sigma_2 = 0.05$, as shown in Figure 1.

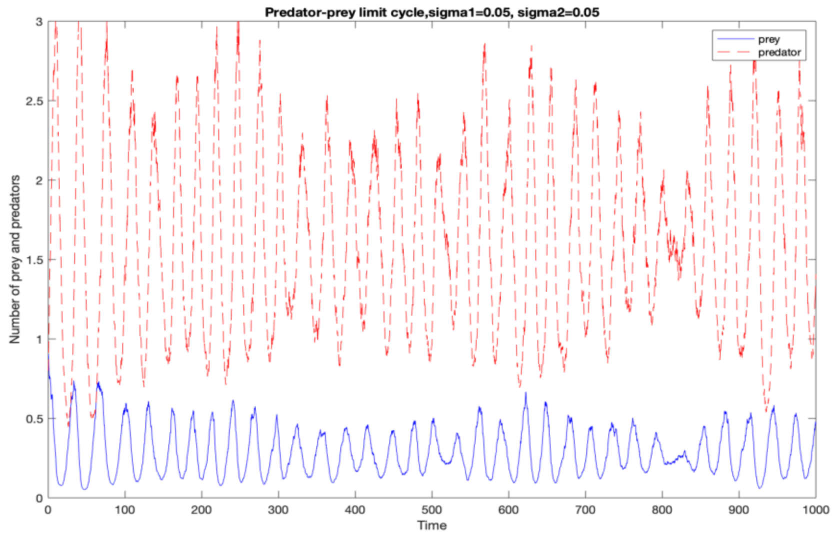

Figure 1. The stochastic system 4 with $\sigma_1 = \sigma_2 = 0.05$, where $(X_0, Y_0) = (0.9, 0.8)$ and further parameters are used as 11.

Furthermore, set $\sigma_1 = \sigma_2 = 0.1$ and the rest of the values are as given in 11. Thus

$$1 + \frac{1}{\beta}\bar{X} - \left(\frac{c}{\beta}\right)\bar{X} = 1 + \left(\frac{1}{8}\right)(0.6667) - \left(\frac{2.6}{8}\right)(0.6667)$$

$$= 1 + 0.0833375 - 0.2166775 \approx 0.8666 > 0$$

$$\beta + \bar{X} - c\bar{X} \approx 0.2167 \geq 0 \text{ , } \left(\frac{r}{K} - b\bar{Y} - \sigma_1^2 - \frac{l_1 b}{2} + \frac{b\bar{X}}{2}\right)\bar{X}^2 \approx 0.0364,$$

$$\left(l_1 b\bar{X} - l_1^2\sigma_2^2 - \frac{l_1 b}{2} + \frac{b\bar{X}}{2}\right)\bar{Y}^2 \approx 0.0675, \text{ and } \omega = \frac{\sigma_1^2}{2}\bar{X} + \frac{l_1\sigma_2^2}{2}\bar{Y} + \sigma_1^2\bar{X}^2 + l_1^2\sigma_2^2\bar{Y}^2 \approx 0.0161.$$

Consequently the condition of Theorem 4.3 is fulfilled. Thus there is an ergodic stationary distribution in the model 4. When

$$\sigma_1 = 0.1, \; r - \frac{\sigma_1^2}{2} = 0.395 > 0.$$

Thus by Theorem 4.4 (2) $\langle X_t \rangle^* > 0$. If the condition remains fixed, then $\langle Y_t \rangle^* > 0$. Set $\sigma_1 = 0.1$, $\sigma_2 = 0.7$ and all other parameters still invariant as shown in Figure 2. By calculating, $r - \frac{\sigma_1^2}{2} \approx 0.395 > 0$ and $\frac{c}{\beta}K\left(r - \frac{\sigma_1^2}{2}\right) - r\left(d_1 + \frac{\sigma_2^2}{2}\right) = -0.0111 < 0$, fulfills the hypothesis of Theorems 4.4(2) and 4.5. Consequently, $\langle X_t \rangle^* > 0$ and $\lim_{t\to\infty} Y_t = 0$.

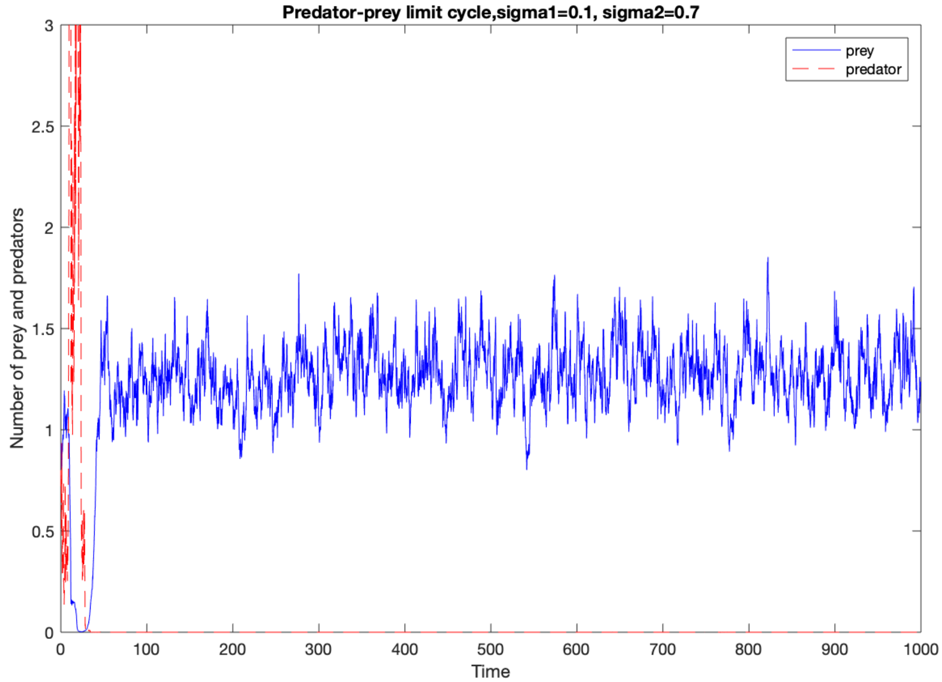

Figure 2. The stochastic system (4) with $\sigma_1 = 0.1 = \sigma_2 = 0.7$, where $(X_0, Y_0) = (0.9, 0.8)$ and other parameters are taken as 11.

By increasing the value of σ_1 such that $\sigma_1 = 0.9$, we get $r - \frac{\sigma_1^2}{2} = -0.005 < 0$. Therefore the hypothesis of Theorem 4.4(1) fulfills. In other words, the population X_t will lean towards extinct. So, if $\sigma_2 = 0.2$ and other parameters still in accordance with 11, then

$$\frac{c}{\beta}K(r - \frac{\sigma_1^2}{2}) - r(d_1 + \frac{\sigma_2^2}{2}) \approx -0.9011 < 0$$

where the population X_t is extinct. Therefore, Y_t will go to extinct.

6. CONCLUSIONS

We can outline the results of this paper as follows:

1. The features and long-term behaviour of the system (4) will change as σ_1 and σ_2 increase.
2. When $\sigma_1 = \sigma_2 = 0$ in the Model 4, white noise has no effect. But the effect is more obvious when the values σ_1 and σ_2 become greater than zero.
3. If $r - \frac{\sigma_1^2}{2} > 0$, then $\langle X_t \rangle^* > 0$. If σ_2 is large enough, then Y_t will be inclined to extinction a.s.
4. The X_t and Y_t will be inclined to extinction a.s. when σ_1 is large enough.

5. From the proof of Theorem 4.2, for an appropriate Lyapunov function V we get $LV < 0$, and so by Theorem 3.6 in [10] (see also [19]), the reference solution of the system (4) is globally asymptotically stable. Consequently, the reference solution of the system (4) is dissipative [52].

6. From the proof of Theorem 4.4(1), we get $\limsup_{t\to\infty}\frac{1}{t}\ln X_t < 0$. Thus, the trivial solution of the first equation of the system (4) is almost sure exponential stable in the sense of Definition 3.1 in [38, pp.119]. Similarly, from the proof of Theorem 4.5 we get $\limsup_{t\to+\infty}\frac{1}{t}\ln Y_t < 0$, so the second equation of the system (4) is almost sure exponential stable.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interest.

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