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A MATHEMATICAL MODEL OF HIV/AIDS WITH AWARENESS AND ITS APPLICATION TO OPTIMAL CONTROL STRATEGIES: A CASE STUDY IN THAILAND

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Abstract. HIV infection can progress to AIDS, the most severe stage of the disease, characterized by a critically weakened immune system. In some regions, AIDS is a leading cause of mortality. This study develops a mathematical model of HIV/AIDS aimed at reducing new infections, using Thailand as a case study. The model includes epidemic and endemic analyses, along with stability evaluations for both equilibrium points. Additionally, an optimal control analysis is conducted to identify strategic interventions for minimizing all stages of HIV infection.

Keywords: mathematical modeling; stability analysis; optimal control theory; HIV-AIDS.
2020 AMS Subject Classification: 92B05.

1. Introduction

HIV, or Human Immunodeficiency Virus, is a virus that compromises the immune system by targeting and destroying CD4 cells, a type of white blood cell essential for immune defense. As HIV progresses, it weakens the immune system, reducing the body's ability to combat infections and certain cancers. If left untreated, HIV can advance to AIDS (Acquired Immunodeficiency Syndrome), the most severe stage of the infection.

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HIV/AIDS is among the world's most deadly infectious diseases, with a particularly devastating impact in Sub-Saharan Africa. Over recent decades, it has significantly affected health outcomes and life expectancy in this region [22]. According to the Global Burden of Disease study, nearly one million people die annually from HIV/AIDS, accounting for approximately 1.5 % of global deaths. The mortality rates vary widely across regions, with some countries in southern Sub-Saharan Africa experiencing rates exceeding 100 deaths per 100,000 people.

Since the onset of the HIV epidemic, approximately 88.4 million people have been infected with the virus, and about 42.3 million have died from HIV-related causes [32]. By the end of 2023, an estimated 0.6 % of adults aged 15–49 worldwide were living with HIV, including 20.5 million women and 18.1 million men. Additionally, 1.4 million children under the age of 15 were living with HIV, with around 630,000 deaths attributed to HIV-related causes.

Numerous mathematical models of HIV/AIDS have been developed, each differing in focus and approach. In 2014, Fatma Bozkurt et al. [9] presented a study on the HIV epidemic, dividing the population into three groups: HIV-negative individuals, HIV-positive individuals unaware of their infection, and HIV-positive individuals aware of their status. The study found that the disease becomes more endemic due to immigration, while the prevalence decreases when infected individuals become aware of their status through screening and contact tracing, subsequently refraining from sexual activity. Conversely, in the absence of contact tracing, the prevalence increases. The study concluded that the most effective strategy to reduce infection rates and prevalence is to educate individuals about HIV and raise awareness of the consequences of unsafe sexual practices and other risky behaviors.

In 2020, S. Saravanakumar et al. [27] proposed a model examining risk factors in HIV/AIDS transmission dynamics, with a specific focus on the female sexual network in India. The study highlighted the role of female sex workers in the spread of HIV. The model categorized the population into several groups: HIV-susceptible males, susceptible females, susceptible female sex workers, HIV-infected males, infected females, infected female sex workers, AIDS-infected males, and AIDS-infected females, including sex workers. The findings provide insights into the dynamics of HIV prevalence, enabling the development of more effective prevention strategies.

In 2021, Tigabu Kasia Ayele et al. [29] developed a mathematical model of HIV/AIDS with optimal control, focusing on a case study in Ethiopia. The study categorized the population into six groups: individuals aware of HIV/AIDS who are not yet exposed, individuals unaware of HIV/AIDS who are not yet exposed, individuals unknowingly infected, individuals aware of their infection but not yet progressed to AIDS, individuals undergoing treatment but not fully recovered, and individuals infected with HIV displaying AIDS symptoms. The findings revealed that a combination of optimal control strategies significantly reduces the number of unaware susceptible individuals, undiagnosed infections, diagnosed infections, and diagnosed cases with AIDS symptoms.

In 2022, Roberto Arias, Kevin De Angeles, and colleagues [26] proposed a mathematical model of the HIV/AIDS epidemic. The study divided the population into four groups: susceptible individuals, HIV-infected individuals, AIDS-infected individuals, and a ”removed” category representing those isolated, cured, or permanently immune. However, it is important to note that HIV cannot be cured. Their experiments indicated that one HIV-infected individual could potentially transmit the virus to the entire susceptible population, leading to widespread infection.

In the same year, Cristian Camilo Espitia Morillo and colleagues [5] analyzed a mathematical model of HIV/AIDS that incorporated sexual preferences under antiretroviral therapy. Their findings suggested that reducing the rate of homosexual partnerships could significantly decrease transmission rates and help achieve a disease-free equilibrium.

In the same year, Cristian C. Espitia et al. [6] developed a model of HIV/AIDS that considered sexual preferences under antiretroviral therapy, using a case study from San Juan de Pasto, Colombia. The model analyzed the impact of bisexual behavior in a global community and is governed by nonlinear equations representing various groups: susceptible homosexual men, untreated infected homosexual men, susceptible women, untreated infected women, susceptible heterosexual men, untreated infected heterosexual men, individuals receiving antiretroviral treatment, and individuals living with AIDS. The study concluded that the most effective way to reduce transmission and achieve a disease-free equilibrium is primarily by decreasing the number of homosexual partners. While increasing the departure rate of infected individuals reduces infections among heterosexual men and women, it is not sufficient to fully prevent or control the rate of contagion.

In 2024, Syeda Alishwa Zanib and colleagues [33] proposed a model of HIV/AIDS that incorporates both the fisher-folk community and the general community. The study categorized individuals in the general population into several groups: those who are purely susceptible, individuals exposed to the virus, those aware of HIV/AIDS, patients undergoing treatment after showing symptoms and becoming infectious, and individuals who progress to AIDS in the absence of adequate therapy. Similarly, the fisher-folk community was classified using the same categories. The study highlighted gaps in the representation of exposed individuals within both communities, offering valuable insights into the dynamics of HIV/AIDS and its societal impact.

In the same year, M. O. Ogunmodimu and colleagues [21] proposed a mathematical model addressing HIV/AIDS prevention in the context of the ”undetectable equals untransmittable” principle. Their research focused on HIV/AIDS transmission in Africa, using Cape Verde as a case study, and incorporated antiretroviral therapy (ART) as a key component. The study analyzed the qualitative properties of the model, including the boundedness and positivity of the solutions, alongside other essential mathematical proofs. The findings emphasized the importance of educating HIV-endemic communities about the disease and its fatality. Furthermore, the study recommended that governments and health organizations ensure access to ART treatment at significantly subsidized costs for those in need.

In June 2024, Idris Ahmed and colleagues [12] proposed a mathematical model incorporating HIV/AIDS treatment strategies, including antiretroviral therapy. The study examined the dynamics of HIV/AIDS transmission and employed numerical methods to analyze the behavior of each compartment within the model. The results highlighted the importance of implementing non-pharmaceutical interventions as effective control strategies.

The studies we reviewed represent only a small selection of the available literature. However, few have explored the transitional state between HIV and AIDS or considered a class of individuals who are aware of their infection and take measures to prevent transmitting the disease to others. In the remainder of this paper, we first introduce our model, which incorporates control measures, followed by an analysis of the model, including the solution boundaries and the basic reproduction number. We then present the disease-free and endemic equilibrium points along with their stability conditions. Subsequently, we apply the optimal control model and provide numerical simulations. Finally, we discuss the simulation results and conclude with a summary and discussion of our findings.

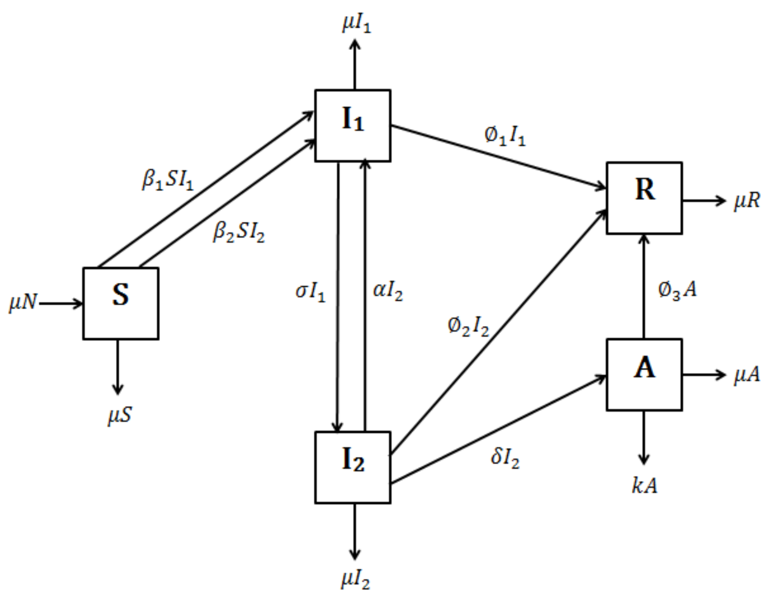


FIGURE 1. HIV/AIDS model

In this section, we present our model along with an optimal control strategy. The population is divided into five categories: susceptible individuals (S), asymptomatic HIV-infected individuals (I_1), symptomatic HIV-infected individuals (I_2), AIDS patients (A), and a semi-recovered group who are aware of their condition and take precautions to prevent transmission (R).

We assume that susceptible individuals can become infected through contact with asymptomatic and symptomatic HIV patients. Upon infection, individuals initially enter the asymptomatic category (I_1) and can reduce their risk of transmission through medical care. If left untreated, asymptomatic individuals may progress to the symptomatic category (I_2). Symptomatic patients, if provided with appropriate care, can revert to the asymptomatic state (I_1). Without treatment or adherence to care guidelines, symptomatic patients may advance to the AIDS stage (A).

AIDS patients (A) represent individuals with severe symptoms who face a high risk of mortality. However, with effective healthcare, they can transition into the semi-recovered category (R), where they can live a normal life, are aware of their condition, and do not transmit the disease to others. Similarly, symptomatic patients receiving proper care may also move directly to the semi-recovered group. The progression of the disease is described by the following system of equations:

$$\frac{dS}{dt} = \mu N - \beta_1 S I_1 - \beta_2 S I_2 - \mu S \tag{1}$$

$$\frac{dI_1}{dt} = \beta_1 S I_1 + \beta_2 S I_2 - \sigma I_1 - \mu I_1 - \phi_1 I_1 + \alpha I_2 \tag{2}$$

$$\frac{dI_2}{dt} = \sigma I_1 - \delta I_2 - \mu I_2 - \phi_2 I_2 - \alpha I_2 \tag{3}$$

$$\frac{dA}{dt} = \delta I_2 - \kappa A - \mu A - \phi_3 A \tag{4}$$

$$\frac{dR}{dt} = \phi_1 I_1 + \phi_2 I_2 + \phi_3 A - \mu R \tag{5}$$

where

- N is the total population.
- μ is the natural birth and natural death rate.
- β_1 is the transmission rate from asymptomatic patients to susceptible population.
- β_2 is the transmission rate from symptomatic patients to susceptible population.
- σ progressing rate from asymptomatic HIV patients becoming symptomatic patients.
- α is the progression rate of treatments for symptomatic patients becoming asymptomatic patients.
- δ is progression rate of symptomatic HIV group becoming AIDS group.
- κ is the AIDS-related death rate.
- ϕ_1 is the ART rate for HIV infections.
- ϕ_2 is the additional medication rate for symptomatic patients.
- ϕ_3 is the additional medication rate for AIDS group.

Written in a vector from, the above equations become

$$\frac{dX}{dt} = F(X)$$

with $X = (S, I_1, I_2, A, R)^T$.

It is important to note that while approximately 60% of the costs for HIV patients are attributed to antiretroviral therapy (ART), an additional 40% is associated with related expenses, such as doctor visits and treatments for other infection-related medical conditions.

2. Epidemic Analysis

Next we will find a disease-free equilibrium by setting I_1, I_2, A and R in the equations to zero and solve for S. We have $S = N$. Thus now we have the DFE:

$$\varepsilon_0 = (N, 0, 0, 0, 0)$$

Since $\frac{dN}{dt} = -\kappa A < 0$, the solutions of the system are bounded, and the function $N(t)$ is monotonically decreasing.

To calculate the basic reproduction number, R_0, for this model, we employ the method of van den Driessche and Watmough. The associated next-generation matrices are defined as follows:

$$\begin{aligned} \mathscr{F} &= \begin{bmatrix} \beta_1 S I_1 + \beta_2 S I_2 \\ 0 \\ 0 \end{bmatrix}, \\ \mathscr{V} &= \begin{bmatrix} \sigma I_1 + \mu I_1 + \phi_1 I_1 - \alpha I_2 \\ -\sigma I_1 + \delta I_2 + \mu I_2 + \phi_2 I_2 + \alpha I_2 \\ -\delta I_2 + \kappa A + \mu A + \phi_3 A \end{bmatrix}. \end{aligned}$$

Then

$$F = \begin{bmatrix} \frac{\partial \mathscr{F}_{I1}}{\partial I_1} & \frac{\partial \mathscr{F}_{I1}}{\partial I_2} & \frac{\partial \mathscr{F}_{I1}}{\partial A} \\ \frac{\partial \mathscr{F}_{I2}}{\partial I_1} & \frac{\partial \mathscr{F}_{I2}}{\partial I_2} & \frac{\partial \mathscr{F}_{I2}}{\partial A} \\ \frac{\partial \mathscr{F}_{A}}{\partial I_1} & \frac{\partial \mathscr{F}_{A}}{\partial I_2} & \frac{\partial \mathscr{F}_{A}}{\partial A} \end{bmatrix} = \begin{bmatrix} \beta_1 S & \beta_2 S & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{6}$$

and

$$V = \begin{bmatrix} \frac{\partial \mathscr{V}_{I1}}{\partial I_1} & \frac{\partial \mathscr{V}_{I1}}{\partial I_2} & \frac{\partial \mathscr{V}_{I1}}{\partial A} \\ \frac{\partial \mathscr{V}_{I2}}{\partial I_1} & \frac{\partial \mathscr{V}_{I2}}{\partial I_2} & \frac{\partial \mathscr{V}_{I2}}{\partial A} \\ \frac{\partial \mathscr{V}_{A}}{\partial I_1} & \frac{\partial \mathscr{V}_{A}}{\partial I_2} & \frac{\partial \mathscr{V}_{A}}{\partial A} \end{bmatrix} = \begin{bmatrix} \sigma + \mu + \phi_1 & -\alpha & 0 \\ -\sigma & \delta + \mu + \phi_2 + \alpha & 0 \\ 0 & -\delta & \kappa + \mu + \phi_3 \end{bmatrix}, \tag{7}$$

At the DFE point, we have

$$F(\varepsilon_0) = \begin{bmatrix} \beta_1 N & \beta_2 N & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let

$$\begin{aligned} M^* &= \beta_1 N, \\ N^* &= \beta_2 N. \end{aligned}$$

Then we write

$$F(\varepsilon_0) = \begin{bmatrix} M^* & N^* & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then we have,

$$\begin{aligned} \det V &= \big((\sigma+\mu+\phi_1)(\delta+\mu+\phi_2+\alpha)(\kappa+\mu+\phi_3)+0+0\big)-\sigma\alpha(\kappa+\mu+\phi_3) \\ &= (\sigma+\mu+\phi_1)(\delta+\mu+\phi_2+\alpha)(\kappa+\mu+\phi_3)-\sigma\alpha(\kappa+\mu+\phi_3) \end{aligned}$$

Let

$$K = (\sigma+\mu+\phi_1)(\delta+\mu+\phi_2+\alpha)(\kappa+\mu+\phi_3)-\sigma\alpha(\kappa+\mu+\phi_3)$$

Thus,

$$\operatorname{adj} V = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} (\delta+\mu+\phi_2+\alpha)(\kappa+\mu+\phi_3) & \sigma(\kappa+\mu+\phi_3) & \sigma\delta \\ \alpha(\kappa+\mu+\phi_3) & (\sigma+\mu+\phi_1)(\kappa+\mu+\phi_3) & \delta(\kappa+\mu+\phi_3) \\ 0 & 0 & (\sigma+\mu+\phi_1)(\delta+\mu+\phi_2+\alpha)-\sigma\alpha \end{bmatrix}^T$$

$$= \begin{bmatrix} (\delta+\mu+\phi_2+\alpha)(\kappa+\mu+\phi_3) & \alpha(\kappa+\mu+\phi_3) & 0 \\ \sigma(\kappa+\mu+\phi_3) & (\sigma+\mu+\phi_1)(\kappa+\mu+\phi_3) & 0 \\ \sigma\delta & \delta(\kappa+\mu+\phi_3) & (\sigma+\mu+\phi_1)(\delta+\mu+\phi_2+\alpha)-\sigma\alpha \end{bmatrix}$$

and hence ,

$$V^{-1} = \frac{1}{\det V}\cdot \text{adj}V$$

$$= \frac{1}{K}\begin{bmatrix} (\delta+\mu+\phi_2+\alpha)(\kappa+\mu+\phi_3) & \alpha(\kappa+\mu+\phi_3) & 0 \\ \sigma(\kappa+\mu+\phi_3) & (\sigma+\mu+\phi_1)(\kappa+\mu+\phi_3) & 0 \\ \sigma\delta & \delta(\kappa+\mu+\phi_3) & (\sigma+\mu+\phi_1)(\delta+\mu+\phi_2+\alpha)-\sigma\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(\delta+\mu+\phi_2+\alpha)(\kappa+\mu+\phi_3)}{K} & \frac{\alpha(\kappa+\mu+\phi_3)}{K} & 0 \\ \frac{\sigma(\kappa+\mu+\phi_3)}{K} & \frac{(\sigma+\mu+\phi_1)(\kappa+\mu+\phi_3)}{K} & 0 \\ \frac{\sigma\delta}{K} & \frac{\delta(\kappa+\mu+\phi_3)}{K} & \frac{(\sigma+\mu+\phi_1)(\delta+\mu+\phi_2+\alpha)-\sigma\alpha}{K} \end{bmatrix}.$$

Thus

$$FV^{-1} = \begin{bmatrix} M^* & N^* & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{(\delta+\mu+\phi_2+\alpha)(\kappa+\mu+\phi_3)}{K} & \frac{\alpha(\kappa+\mu+\phi_3)}{K} & 0 \\ \frac{\sigma(\kappa+\mu+\phi_3)}{K} & \frac{(\sigma+\mu+\phi_1)(\kappa+\mu+\phi_3)}{K} & 0 \\ \frac{\sigma\delta}{K} & \frac{\delta(\kappa+\mu+\phi_3)}{K} & \frac{(\sigma+\mu+\phi_1)(\delta+\mu+\phi_2+\alpha)-\sigma\alpha}{K} \end{bmatrix}$$

$$= \begin{bmatrix} M^*\cdot\frac{(\delta+\mu+\phi_2+\alpha)(\kappa+\mu+\phi_3)}{K}+N^*\cdot\frac{\sigma(\kappa+\mu+\phi_3)}{K} & M^*\cdot\frac{\alpha(\kappa+\mu+\phi_3)}{K}+N^*\cdot\frac{(\sigma+\mu+\phi_1)(\kappa+\mu+\phi_3)}{K} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let

$$
\begin{aligned}
C^* &= M^* \cdot \frac{(\delta+\mu+\phi_2+\alpha)(\kappa+\mu+\phi_3)}{K} + N^* \cdot \frac{\sigma(\kappa+\mu+\phi_3)}{K} \\
D^* &= M^* \cdot \frac{\alpha(\kappa+\mu+\phi_3)}{K} + N^* \cdot \frac{(\sigma+\mu+\phi_1)(\kappa+\mu+\phi_3)}{K}
\end{aligned}
$$

The basic reproductive number is then determined as the spectral radius of FV^{-1}, which yields.

$$
\begin{aligned}
\det(\lambda I - FV^{-1}) &= \begin{vmatrix} C^*-\lambda & D^* & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} \\
&= (C^*-\lambda)(-\lambda)(-\lambda) = 0
\end{aligned}
$$

thus $\lambda = C^*, 0$ which are real numbers. Hence,

$$
\begin{aligned}
R_0 &= M^* \cdot \frac{(\delta+\mu+\phi_2+\alpha)(\kappa+\mu+\phi_3)}{K} + N^* \cdot \frac{\sigma(\kappa+\mu+\phi_3)}{K} \\
&= \frac{\beta_1 N(\delta+\mu+\phi_2+\alpha)(\kappa+\mu+\phi_3) + \beta_2 N\sigma(\kappa+\mu+\phi_3)}{K} \\
&= \frac{\beta_1 N(\delta+\mu+\phi_2+\alpha)(\kappa+\mu+\phi_3) + \beta_2 N\sigma(\kappa+\mu+\phi_3)}{(\sigma+\mu+\phi_1)(\delta+\mu+\phi_2+\alpha)(\kappa+\mu+\phi_3) - \sigma\alpha(\kappa+\mu+\phi_3)} \\
&= \frac{\beta_1 N(\delta+\mu+\phi_2+\alpha)(\kappa+\mu+\phi_3) + \beta_2 N\sigma(\kappa+\mu+\phi_3)}{(\kappa+\mu+\phi_3)\big((\mu+\phi_1)(\delta+\mu+\phi_2+\alpha) + \sigma(\delta+\mu+\phi_2)\big)}
\end{aligned}
$$

Based on the work in [31], we immediately obtain the result below :

Theorem The disease-fee equilibrium of the model is locally asymptotically stable if $R_0 < 1$, and unstable if $R_0 > 1$.

To analyze the global asymptotic stability of the disease-free equilibrium (DFE), a common approach involves constructing an appropriate Lyapunov function. However, we found it more straightforward to utilize the following result proposed by Castillo-Chavez et al.

Lemma Consider a model system written in the form

$$
\begin{aligned}
\frac{dX_1}{dt} &= F(X_1, X_2), \\
\frac{dX_2}{dt} &= G(X_1, X_2), \qquad G(X_1, 0) = 0
\end{aligned}
$$

where $X_1 \in \mathbb{R}^m$ denotes (its components) the number of uninfected individuals and $X_2 \in \mathbb{R}^n$ denotes (its components) the number of infected individuals including latent, infectious, etc;

$X_0 = (X_1^*)$ denotes the disease-free equilibrium of the system.

Also assume the conditions $(H1)$ and $(H2)$ below:

$(H1)$ For $\frac{dX_1}{dt} = F(X_1^*, 0)$ is globally asymptotically stable;

$(H2)$ $G(X_1, X_2) = AX_2 - \hat{G}(X_1, X_2)$, $\hat{G}(X_1, X_2) \geq 0$ for $(X_1, X_2) \in \Omega$, where the Jacobian $A = \frac{\partial G}{\partial X_2}(X_1^*, 0)$ is an M-matrix (the off diagonal elements of A are non-negative) and Ω is the region where the model makes biological sense.

Then the DFE $X_0 = (X_1^*, 0)$ is globally asymptotically stable.

Theorem The disease-free equilibrium of the model is globally asymptotic stable.

Proof. We only need to show that the condition $(H1)$ and $(H2)$ hold.

In our ODE system, $X_1 = S$, $X_2 = (I_1, I_2, A)$, and $X_1^* = N$. We note that the system is linear and its solution can be easily found as:

$$\frac{dX_1}{dt} = F(X_1, X_2) = \left[\ \mu N - \beta_1 S I_1 - \beta_2 S I_2 - \mu S\ \right].$$

We have

$$\frac{dX_1}{dt} = F(X_1, 0) = \left[\ \mu N - \mu S\ \right]$$

is linear and its solution can be easily found as follows:

For S:

$$\begin{aligned}
e^{\mu t}\frac{dS}{dt} + e^{\mu t}\mu S &= e^{\mu t}\mu N \\
\frac{d}{dt}(e^{\mu t}S) &= e^{\mu t}\mu N\ , \\
\int \frac{d}{dt}(e^{\mu t}S)dt &= \int e^{\mu t}\mu N dt\ , \\
&= Ne^{\mu t} - \int e^{\mu t}N'(t)dt; \text{By Parts}, \\
e^{\mu t}S &= Ne^{\mu t} - \int e^{\mu t}N'(t)dt\ , \\
S(t) &= N - \frac{\int e^{\mu t}N'(t)dt}{e^{\mu t}}\ .
\end{aligned}$$

Since the integral over certain intervals approximates the integrand, which is a real number, it follows that $S(t) \to N$ as $t \to \infty$. Therefore, $X_1^* = N$ is globally asymptotically stable.

Next consider that

$$\frac{dX_2}{dt} = G(X_1, X_2) \quad = \quad \begin{bmatrix} \beta_1 S I_1 + \beta_2 S I_2 - \sigma I_1 - \mu I_1 - \phi_1 I_1 + \alpha I_2 \\ \sigma I_1 - \delta I_2 - \mu I_2 - \phi_2 I_2 - \alpha I_2 \\ \delta I_2 - \kappa A - \mu A - \phi_3 A \end{bmatrix},$$

and thus

$$A \quad = \quad \begin{bmatrix} \beta_1 N - \sigma - \mu - \phi_1 & \beta_2 N + \alpha & 0 \\ \sigma & -\delta - \mu - \phi_2 - \alpha & 0 \\ 0 & \delta & -\kappa - \mu - \phi_3 \end{bmatrix}.$$

Hence

$$\begin{aligned}
\hat{G}(X_1, X_2) &= AX_2 - G(X_1, X_2) \\
&= \begin{bmatrix} \beta_1 N - \sigma - \mu - \phi_1 & \beta_2 N + \alpha & 0 \\ \sigma & -\delta - \mu - \phi_2 - \alpha & 0 \\ 0 & \delta & -\kappa - \mu - \phi_3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ A \end{bmatrix} - \\
&\quad \begin{bmatrix} \beta_1 S I_1 + \beta_2 S I_2 - \sigma I_1 - \mu I_1 - \phi_1 I_1 + \alpha I_2 \\ \sigma I_1 - \delta I_2 - \mu I_2 - \phi_2 I_2 - \alpha I_2 \\ \delta I_2 - \kappa A - \mu A - \phi_3 A \end{bmatrix} \\
&= \begin{bmatrix} (\beta_1 N - \sigma - \mu - \phi_1) I_1 + (\beta_2 N + \alpha) I_2 \\ \sigma I_1 - (\delta - \mu - \phi_2 - \alpha) I_2 \\ \delta I_2 - (\kappa - \mu - \phi_3) A \end{bmatrix} - \\
&\quad \begin{bmatrix} \beta_1 S I_1 + \beta_2 S I_2 - \sigma I_1 - \mu I_1 - \phi_1 I_1 + \alpha I_2 \\ \sigma I_1 - \delta I_2 - \mu I_2 - \phi_2 I_2 - \alpha I_2 \\ \delta I_2 - \kappa A - \mu A - \phi_3 A \end{bmatrix} \\
&= \begin{bmatrix} \beta_1 I_1 (N - S) + \beta_2 I_2 (N - S) \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

Now we can write the matrix in the form

$$\therefore \hat{G}(X_1, X_2) \quad = \quad [\beta_1 I_1 (N - S) + \beta_2 I_2 (N - S), 0, 0]^T.$$

Since $0 \leq S \leq N$, it is obvious that $\hat{G}(X_1, X_2) \geq 0$. The stability of the disease-free equilibrium (DFE) determines the short-term dynamics of the epidemic, while the long-term behavior of the disease is governed by the stability of the endemic equilibrium. In this section, we analyze the endemic properties of our model.

3. Endemic Equilibrium

When the disease is presence in the population, I_1^* and $I_2^* \neq 0$, there may be several critical points where I_1^* and $I_2^* \neq 0$, which are the endemic equilibrium points (EEP) of the model. These points will be denoted as $\varepsilon_0^* = (S^*, I_1^*, I_2^*, A^*, R^*)$ which are determined from the model as follows:

$$
\begin{aligned}
\frac{dS^*}{dt} &= \mu N - \beta_1 S^* I_1^* - \beta_2 S^* I_2^* - \mu S^* , \\
\frac{dI_1^*}{dt} &= \beta_1 S^* I_1^* + \beta_2 S^* I_2^* - \sigma I_1^* - \mu I_1^* - \phi_1 I_1^* + \alpha I_2^* , \\
\frac{dI_2^*}{dt} &= \sigma I_1^* - \delta I_2^* - \mu I_2^* - \phi_2 I_2^* - \alpha I_2^* , \\
\frac{dA^*}{dt} &= \delta I_2^* - \kappa A^* - \mu A^* - \phi_3 A^* , \\
\frac{dR^*}{dt} &= \phi_1 I_1^* + \phi_2 I_2^* + \phi_3 A^* - \mu R^* .
\end{aligned}
$$

Its components must satisfy,

$$
\begin{aligned}
S^* &= \frac{\mu N}{\beta_1 I_1^* + \beta_2 I_2^* + \mu} , \\
I_1^* &= = \frac{I_2^*(\delta + \mu + \phi_2 + \alpha)}{\sigma} , \\
I_2^* &= \frac{\sigma I_1^*}{\delta + \mu + \phi_2 + \alpha} , \\
A^* &= \frac{\delta I_2^*}{\kappa + \mu + \phi_3} , \\
R^* &= \frac{\phi_1 I_1^* + \phi_2 I_2^* + \phi_3 A^*}{\mu} .
\end{aligned}
$$

We first show the following theorem.

Theorem The positive endemic equilibrium exists and is unique if and only if $R_0 > 1$.

Proof. Note that

$$N = S^* + I_1^* + I_2^* + A^* + R^* \;.$$

Substitute of S^*, R^* in N, we have

$$N = \frac{\mu N}{\beta_1 I_1^* + \beta_2 I_2^* + \mu} + I_1^* + I_2^* + A^* + \frac{\phi_1 I_1^* + \phi_2 I_2^* + \phi_3 A^*}{\mu} \;.$$

Let

$$\begin{aligned} a_1 &= \delta + \mu + \phi_2 + \alpha \;, \\ a_2 &= \kappa + \mu + \phi_3 \;, \\ a_3 &= \sigma + \mu + \phi_1 \;. \end{aligned}$$

Substitute of A^* in N, we have

$$N = \frac{\mu N}{\beta_1 I_1^* + \beta_2 I_2^* + \mu} + I_1^* + I_2^* + \frac{\delta I_2^*}{a_2} + \frac{\phi_1 I_1^* + \phi_2 I_2^* + \frac{\phi_3 \delta I_2^*}{a_2}}{\mu}$$

$$N = \frac{\mu N}{\beta_1 I_1^* + \beta_2 I_2^* + \mu} + I_1^* + I_2^* + \frac{\delta I_2^*}{a_2} + \frac{\phi_1 I_1^* a_2 + \phi_2 I_2^* a_2 + \phi_3 \delta I_2^*}{\mu a_2}$$

$$\begin{aligned} N &= \mu^2 N a_2 + I_1^* \mu a_2 (\beta_1 I_1^* + \beta_2 I_2^* + \mu) + I_2^* \mu a_2 (\beta_1 I_1^* + \beta_2 I_2^* + \mu) \\ &\quad + \delta I_2^* \mu (\beta_1 I_1^* + \beta_2 I_2^* + \mu) + (\phi_1 I_1^* a_2 + \phi_2 I_2^* a_2 + \phi_3 \delta I_2^*)(\beta_1 I_1^* \\ &\quad + \beta_2 I_2^* + \mu) \div \mu a_2 (\beta_1 I_1^* + \beta_2 I_2^* + \mu) \\ \mu a_2 (\beta_1 I_1^* + \beta_2 I_2^* + \mu) N &= \mu^2 N a_2 + I_1^* \mu a_2 (\beta_1 I_1^* + \beta_2 I_2^* + \mu) + I_2^* \mu a_2 (\beta_1 I_1^* + \beta_2 I_2^* + \mu) \\ &\quad + \delta I_2^* \mu (\beta_1 I_1^* + \beta_2 I_2^* + \mu) + (\phi_1 I_1^* a_2 + \phi_2 I_2^* a_2 + \phi_3 \delta I_2^*)(\beta_1 I_1^* \\ &\quad + \beta_2 I_2^* + \mu) \\ 0 &= \mu a_2 (\beta_1 I_1^* + \beta_2 I_2^* + \mu) N - \mu^2 N a_2 - I_1^* \mu a_2 (\beta_1 I_1^* + \beta_2 I_2^* + \mu) \\ &\quad - I_2^* \mu a_2 (\beta_1 I_1^* + \beta_2 I_2^* + \mu) - \delta I_2^* \mu (\beta_1 I_1^* + \beta_2 I_2^* + \mu) \\ &\quad - (\phi_1 I_1^* a_2 + \phi_2 I_2^* a_2 + \phi_3 \delta I_2^*)(\beta_1 I_1^* + \beta_2 I_2^* + \mu) \\ 0 &= \beta_1 a_2 \mu N I_1^* - \beta_1 a_2 \mu I_1^{*2} - \beta_1 a_2 \mu I_1 * I_2^* - \beta_1 \mu \delta I_1^* I_2^* - \beta_1 a_2 \phi_1 I_1^{*2} \end{aligned}$$

$$-\beta_1 a_2\phi_2 I_1^* I_2^* - \beta_1\delta\phi_3 I_1^* I_2^* + \beta_2 a_2\mu N I_2^* - \beta_2 a_2\mu I_1^* I_2^*$$
$$-\beta_2 a_2\mu I_2^{*2} - \beta_2\mu\delta I_2^{*2} - \beta_2 a_2\phi_1 I_1 * I_2^* - \beta_2 a_2\phi_2 I_2^{*2} - \beta_2\delta\phi_3 I_2^{*2}$$
$$-a_2\mu^2 I_1^* - a_2\mu^2 I_2^* - \delta\mu^2 I_2^* - a_2\mu\phi_1 I_1^* - a_2\mu\phi_2 I_2^* - \mu\delta\phi_3 I_2^*$$

Note that

$$I_1^* = \frac{a_1 I_2^*}{\sigma} .$$

Substitute of I_1^* in I_2^*,we have

$$\begin{aligned}
0 &= \big((\beta_1 a_1 a_2\mu N I_2^* - \beta_1 a_1 a_2\mu I_2^{*2} - \beta_1 a_1\mu\delta I_2^{*2} - \beta_1 a_1 a_2\phi_2 I_2^{*2} \\
&\quad -\beta_1 a_1\delta\phi_3 I_2^{*2} - \beta_2 a_1 a_2\mu I_2^{*2} - \beta_2 a_1 a_2\phi_1 I_2^{*2} - a_1 a_2\mu^2 I_2^* \\
&\quad -a_1 a_2\mu\phi_1 I_2^*) \div (\sigma)\big) + \big((-\beta_1 a_1^2 a_2\mu I_2^{*2} - \beta_1 a_1^2 a_2\phi_1 I_2^{*2}) \\
&\quad \div\sigma^2\big) + \big(\beta_2 a_2\mu N I_2^* - \beta_2 a_2\mu I_2^{*2} - \beta_2\mu\delta I_2^{*2} - \beta_2 a_2\phi_2 I_2^{*2} \\
&\quad -\beta_2\delta\phi_3 I_2^{*2} - a_2\mu^2 I_2^* - \delta\mu^2 I_2^* - a_2\mu\phi_2 I_2^* - \mu\delta\phi_3 I_2^*\big) \\
\frac{\beta_1 a_1^2 a_2\mu I_2^{*2} + \beta_1 a_1^2 a_2\phi_1 I_2^{*2}}{\sigma^2} &= \big(\beta_1 a_1 a_2\mu N I_2^* - \beta_1 a_1 a_2\mu I_2^{*2} - \beta_1 a_1\mu\delta I_2^{*2} - \beta_1 a_1 a_2\phi_2 I_2^{*2} \\
&\quad -\beta_1 a_1\delta\phi_3 I_2^{*2} - \beta_2 a_1 a_2\mu I_2^{*2} - \beta_2 a_1 a_2\phi_1 I_2^{*2} - a_1 a_2\mu^2 I_2^* \\
&\quad -a_1 a_2\mu\phi_1 I_2^* + \beta_2 a_2\mu\sigma N I_2^* - \beta_2 a_2\mu\sigma I_2^{*2} - \beta_2\mu\delta\sigma I_2^{*2} \\
&\quad -\beta_2 a_2\sigma\phi_2 I_2^{*2} - \beta_2\delta\sigma\phi_3 I_2^{*2} - a_2\mu^2\sigma I_2^* - \delta\mu^2\sigma I_2^* \\
&\quad -a_2\mu\sigma\phi_2 I_2^* - \mu\delta\sigma\phi_3 I_2^*\big) \div \sigma \\
\beta_1 a_1^2 a_2\mu I_2^{*2} + \beta_1 a_1^2 a_2\phi_1 I_2^{*2} &= \sigma^2\big((\beta_1 a_1 a_2\mu N I_2^* - \beta_1 a_1 a_2\mu I_2^{*2} - \beta_1 a_1\mu\delta I_2^{*2} - \beta_1 a_1 a_2\phi_2 I_2^{*2} \\
&\quad -\beta_1 a_1\delta\phi_3 I_2^{*2} - \beta_2 a_1 a_2\mu I_2^{*2} - \beta_2 a_1 a_2\phi_1 I_2^{*2} - a_1 a_2\mu^2 I_2^* \\
&\quad -a_1 a_2\mu\phi_1 I_2^* + \beta_2 a_2\mu\sigma N I_2^* - \beta_2 a_2\mu\sigma I_2^{*2} - \beta_2\mu\delta\sigma I_2^{*2} \\
&\quad -\beta_2 a_2\sigma\phi_2 I_2^{*2} - \beta_2\delta\sigma\phi_3 I_2^{*2} - a_2\mu^2\sigma I_2^* - \delta\mu^2\sigma I_2^* \\
&\quad -a_2\mu\sigma\phi_2 I_2^* - \mu\delta\sigma\phi_3 I_2^*) \div \sigma\big) \\
\beta_1 a_1^2 a_2\mu I_2^{*2} + \beta_1 a_1^2 a_2\phi_1 I_2^{*2} &= \sigma(\beta_1 a_1 a_2\mu N I_2^* - \beta_1 a_1 a_2\mu I_2^{*2} - \beta_1 a_1\mu\delta I_2^{*2} - \beta_1 a_1 a_2\phi_2 I_2^{*2} \\
&\quad -\beta_1 a_1\delta\phi_3 I_2^{*2} - \beta_2 a_1 a_2\mu I_2^{*2} - \beta_2 a_1 a_2\phi_1 I_2^{*2} - a_1 a_2\mu^2 I_2^*
\end{aligned}$$

$$
\begin{aligned}
& -a_1a_2\mu\phi_1 I_2^* + \beta_2 a_2\mu\sigma N I_2^* - \beta_2 a_2\mu\sigma I_2^{*2} - \beta_2\mu\delta\sigma I_2^{*2} \\
& -\beta_2 a_2\sigma\phi_2 I_2^{*2} - \beta_2\delta\sigma\phi_3 I_2^{*2} - a_2\mu^2\sigma I_2^* - \delta\mu^2\sigma I_2^* \\
& -a_2\mu\sigma\phi_2 I_2^* - \mu\delta\sigma\phi_3 I_2^*)
\end{aligned}
$$

$$
\begin{aligned}
\beta_1 a_1^2 a_2\mu I_2^{*2} + \beta_1 a_1^2 a_2\phi_1 I_2^{*2} \quad = \quad & \beta_1 a_1 a_2\mu\sigma N I_2^* - \beta_1 a_1 a_2\mu\sigma I_2^{*2} - \beta_1 a_1\mu\delta\sigma I_2^{*2} - \beta_1 a_1 a_2\sigma\phi_2 I_2^{*2} \\
& -\beta_1 a_1\delta\sigma\phi_3 I_2^{*2} - \beta_2 a_1 a_2\sigma\mu I_2^{*2} - \beta_2 a_1 a_2\sigma\phi_1 I_2^{*2} - a_1a_2\mu^2\sigma I_2^* \\
& -a_1a_2\mu\phi_1\sigma I_2^* + \beta_2 a_2\mu\sigma^2 N I_2^* - \beta_2 a_2\mu\sigma^2 I_2^{*2} - \beta_2\mu\delta\sigma^2 I_2^{*2} \\
& -\beta_2 a_2\sigma^2\phi_2 I_2^{*2} - \beta_2\delta\sigma^2\phi_3 I_2^{*2} - a_2\mu^2\sigma^2 I_2^* - \mu^2\delta\sigma^2 I_2^* \\
& -a_2\mu\sigma^2\phi_2 I_2^* - \mu\delta\sigma^2\phi_3 I_2^*
\end{aligned}
$$

$$
\begin{aligned}
0 \quad = \quad & \beta_1 a_1 a_2\mu\sigma N I_2^* - \beta_1 a_1 a_2\mu\sigma I_2^{*2} - \beta_1 a_1\mu\delta\sigma I_2^{*2} - \beta_1 a_1 a_2\sigma\phi_2 I_2^{*2} - \beta_1 a_1\delta\sigma\phi_3 I_2^{*2} \\
& -\beta_2 a_1 a_2\sigma\mu I_2^{*2} - \beta_2 a_1 a_2\sigma\phi_1 I_2^{*2} - a_1a_2\mu^2\sigma I_2^* - a_1a_2\mu\phi_1\sigma I_2^* + \beta_2 a_2\mu\sigma^2 N I_2^* \\
& -\beta_2 a_2\mu\sigma^2 I_2^{*2} - \beta_2\mu\delta\sigma^2 I_2^{*2} - \beta_2 a_2\sigma^2\phi_2 I_2^{*2} - \beta_2\delta\sigma^2\phi_3 I_2^{*2} - a_2\mu^2\sigma^2 I_2^* - \mu^2\delta\sigma^2 I_2^* \\
& -a_2\mu\sigma^2\phi_2 I_2^* - \mu\delta\sigma^2\phi_3 I_2^* - \beta_1 a_1^2 a_2\mu I_2^{*2} - \beta_1 a_1^2 a_2\phi_1 I_2^{*2} \\
0 \quad = \quad & -(\beta_1 a_1 a_2\mu\sigma I_2^{*2} + \beta_1 a_1\mu\delta\sigma I_2^{*2} + \beta_1 a_1 a_2\sigma\phi_2 I_2^{*2} + \beta_1 a_1\delta\sigma\phi_3 I_2^{*2} + \beta_2 a_1 a_2\sigma\mu I_2^{*2} \\
& +\beta_2 a_2\mu\sigma^2 I_2^{*2} + \beta_2\mu\delta\sigma^2 I_2^{*2} + \beta_2 a_1 a_2\sigma\phi_1 I_2^{*2} + \beta_2 a_2\sigma^2\phi_2 I_2^{*2} + \beta_2\delta\sigma^2\phi_3 I_2^{*2} \\
& +\beta_1 a_1^2 a_2\mu I_2^{*2} + \beta_1 a_1^2 a_2\phi_1 I_2^{*2}) I_2^{*2} + (\beta_1 a_1 a_2\mu\sigma N + \beta_2 a_2\mu\sigma^2 N - a_1a_2\mu^2\sigma \\
& -a_2\mu^2\sigma^2 - \mu^2\delta\sigma^2 - a_1a_2\mu\phi_1\sigma - a_2\mu\sigma^2\phi_2 - \mu\delta\sigma^2\phi_3) I_2^*
\end{aligned}
$$

with

$$
\begin{aligned}
G_1 \quad = \quad & -(\beta_1 a_1 a_2\mu\sigma + \beta_1 a_1\mu\delta\sigma + \beta_1 a_1 a_2\sigma\phi_2 + \beta_1 a_1\delta\sigma\phi_3 + \beta_2 a_1 a_2\sigma\mu + \beta_2 a_2\mu\sigma^2 \\
& +\beta_2\mu\delta\sigma^2 + \beta_2 a_1 a_2\sigma\phi_1 + \beta_2 a_2\sigma^2\phi_2 + \beta_2\delta\sigma^2\phi_3 + \beta_1 a_1^2 a_2\mu + \beta_1 a_1^2 a_2\phi_1)\ , \\
G_2 \quad = \quad & \beta_1 a_1 a_2\mu\sigma N + \beta_2 a_2\mu\sigma^2 N - a_1a_2\mu^2\sigma - a_2\mu^2\sigma^2 - \mu^2\delta\sigma^2 - a_1a_2\mu\phi_1\sigma - a_2\mu\sigma^2\phi_2 \\
& -\mu\delta\sigma^2\phi_3\ .
\end{aligned}
$$

Hence

$$
G_1 I_2^{*2} + G_2 I_2^* = 0
$$

For $I_2^* \neq 0$, the root of this quadratic equation must satisfy,

$$I_2^* = \frac{-G_2}{G_1}$$

Consider that

$$\begin{aligned}
\frac{-G_2}{G_1} &= \big(-(\beta_1 a_1 a_2 \mu\sigma N + \beta_2 a_2 \mu \sigma^2 N - a_1 a_2 \mu^2 \sigma - a_2 \mu^2 \sigma^2 - \mu^2 \delta \sigma^2 - a_1 a_2 \mu \phi_1 \sigma \\
&\quad - a_2 \mu \sigma^2 \phi_2 - \mu \delta \sigma^2 \phi_3)\big) \div \big(-(\beta_1 a_1 a_2 \mu \sigma + \beta_1 a_1 \mu \delta \sigma + \beta_1 a_1 a_2 \sigma \phi_2 + \beta_1 a_1 \delta \sigma \phi_3 \\
&\quad + \beta_2 a_1 a_2 \sigma \mu + \beta_2 a_2 \mu \sigma^2 + \beta_2 \mu \delta \sigma^2 + \beta_2 a_1 a_2 \sigma \phi_1 + \beta_2 a_2 \sigma^2 \phi_2 + \beta_2 \delta \sigma^2 \phi_3 \\
&\quad + \beta_1 a_1^2 a_2 \mu + \beta_1 a_1^2 a_2 \phi_1)\big)
\end{aligned}$$

$$\begin{aligned}
&= \big(\beta_1 a_1 a_2 \mu\sigma N + \beta_2 a_2 \mu \sigma^2 N - a_1 a_2 \mu^2 \sigma - a_2 \mu^2 \sigma^2 - \mu^2 \delta \sigma^2 - a_1 a_2 \mu \phi_1 \sigma - a_2 \mu \sigma^2 \phi_2 \\
&\quad - \mu \delta \sigma^2 \phi_3\big) \div \big(\beta_1 a_1 a_2 \mu \sigma + \beta_1 a_1 \mu \delta \sigma + \beta_1 a_1 a_2 \sigma \phi_2 + \beta_1 a_1 \delta \sigma \phi_3 + \beta_2 a_1 a_2 \sigma \mu + \beta_2 a_2 \mu \sigma^2 \\
&\quad + \beta_2 \mu \delta \sigma^2 + \beta_2 a_1 a_2 \sigma \phi_1 + \beta_2 a_2 \sigma^2 \phi_2 + \beta_2 \delta \sigma^2 \phi_3 + \beta_1 a_1^2 a_2 \mu + \beta_1 a_1^2 a_2 \phi_1\big) \\
&= \big(\beta_1 a_1 a_2 \mu\sigma N + \beta_2 a_2 \mu \sigma^2 N - (a_1 a_2 \mu^2 \sigma + a_2 \mu^2 \sigma^2 + \mu^2 \delta \sigma^2 + a_1 a_2 \mu \phi_1 \sigma + a_2 \mu \sigma^2 \phi_2 \\
&\quad + \mu \delta \sigma^2 \phi_3)\big) \div \big(\beta_1 a_1 a_2 \mu \sigma + \beta_1 a_1 \mu \delta \sigma + \beta_1 a_1 a_2 \sigma \phi_2 + \beta_1 a_1 \delta \sigma \phi_3 + \beta_2 a_1 a_2 \sigma \mu + \beta_2 a_2 \mu \sigma^2 \\
&\quad + \beta_2 \mu \delta \sigma^2 + \beta_2 a_1 a_2 \sigma \phi_1 + \beta_2 a_2 \sigma^2 \phi_2 + \beta_2 \delta \sigma^2 \phi_3 + \beta_1 a_1^2 a_2 \mu + \beta_1 a_1^2 a_2 \phi_1\big) \\
&= \frac{a_1 a_2 \mu^2 \sigma + a_2 \mu^2 \sigma^2 + \ldots + \mu \delta \sigma^2 \phi_3}{\beta_1 a_1 a_2 \mu \sigma + \beta_1 a_1 \mu \delta \sigma + \ldots + \beta_1 a_1^2 a_2 \phi_1} \left(\frac{\beta_1 a_1 a_2 \mu \sigma N + \beta_2 a_2 \mu \sigma^2 N}{a_1 a_2 \mu^2 \sigma + a_2 \mu^2 \sigma^2 + \ldots + \mu \delta \sigma^2 \phi_3} - 1 \right) \\
&= \frac{a_1 a_2 \mu^2 \sigma + a_2 \mu^2 \sigma^2 + \ldots + \mu \delta \sigma^2 \phi_3}{\beta_1 a_1 a_2 \mu \sigma + \beta_1 a_1 \mu \delta \sigma + \ldots + \beta_1 a_1^2 a_2 \phi_1} \left(\frac{\beta_1 a_1 a_2 N + \beta_2 a_2 \sigma N}{a_1 a_2 \mu + a_2 \mu \sigma + \mu \delta \sigma + \ldots + \delta \sigma \phi_3} - 1 \right) \\
&= \frac{a_1 a_2 \mu^2 \sigma + a_2 \mu^2 \sigma^2 + \ldots + \mu \delta \sigma^2 \phi_3}{\beta_1 a_1 a_2 \mu \sigma + \beta_1 a_1 \mu \delta \sigma + \ldots + \beta_1 a_1^2 a_2 \phi_1} \left(\frac{\beta_1 a_1 a_2 N + \beta_2 a_2 \sigma N}{a_2 \big(a_1(\mu + \phi_1) + \sigma(\delta + \mu + \phi_2)\big) - \delta \sigma \kappa} - 1 \right)
\end{aligned}$$

Since all parameters are positive, thus we have

$$\frac{\beta_1 a_1 a_2 N + \beta_2 a_2 \sigma N}{a_2 \big(a_1(\mu + \phi_1) + \sigma(\delta + \mu + \phi_2)\big) - \delta \sigma \kappa} > \frac{\beta_1 a_1 a_2 N + \beta_2 a_2 \sigma N}{a_2 \big(a_1(\mu + \phi_1) + \sigma(\delta + \mu + \phi_2)\big)} = R_0 \ .$$

Hence if $R_0 > 1$, we have

$$\frac{\beta_1 a_1 a_2 N + \beta_2 a_2 \sigma N}{a_2 \big(a_1(\mu + \phi_1) + \sigma(\delta + \mu + \phi_2)\big) - \delta \sigma \kappa} > 1 \ .$$

Therefor $\frac{-G_2}{G_1} > 0$ that is $I_2^* > 0$.

Consequently, from

$$I_2^* = \frac{\sigma I_1^*}{\delta + \mu + \phi_2 + \alpha}$$

we conclude that $I_1^* > 0$.

4. Local Stability

We proceed to analyze the stability properties of the endemic equilibrium. First we establish the following result regarding the local stability.

Theorem The positive endemic equilibrium ε^* is locally asymptotically stable.

Proof. The Jacobian of the system of our model is given by

$$J = \begin{bmatrix} -\beta_1 I_1 - \beta_2 I_2 - \mu & -\beta_1 S & -\beta_2 S & 0 \\ \beta_1 I_1 + \beta_2 I_2 & \beta_1 S - \sigma - \mu - \phi_1 & \beta_2 S + \alpha & 0 \\ 0 & \sigma & -\delta - \mu - \phi_2 - \alpha & 0 \\ 0 & 0 & \delta & -\kappa - \mu - \phi_3 \end{bmatrix},$$

and at $x = \varepsilon^*$ we have

$$\begin{aligned} J(\varepsilon^*) &= \begin{bmatrix} -\beta_1 I_1^* - \beta_2 I_2^* - \mu & -\beta_1 S^* & -\beta_2 S^* & 0 \\ \beta_1 I_1^* + \beta_2 I_2^* & \beta_1 S^* - \sigma - \mu - \phi_1 & \beta_2 S^* + \alpha & 0 \\ 0 & \sigma & -\delta - \mu - \phi_2 - \alpha & 0 \\ 0 & 0 & \delta & -\kappa - \mu - \phi_3 \end{bmatrix} \\ &= \begin{bmatrix} -P - \mu & -\beta_1 S^* & -\beta_2 S^* & 0 \\ P & \beta_1 S^* - a_3 & \beta_2 S^* + \alpha & 0 \\ 0 & \sigma & -a_1 & 0 \\ 0 & 0 & \delta & -a_2 \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} P &= \beta_1 I_1^* + \beta_2 I_2^* \,, \\ a_1 &= \delta + \mu + \phi_2 + \alpha \,, \\ a_2 &= \kappa + \mu + \phi_3 \,, \\ a_3 &= \sigma + \mu + \phi_1 \,. \end{aligned}$$

The characteristic polynomial of $J(\varepsilon^*)$ is

$$
\begin{aligned}
0=\det[\lambda I-J(\varepsilon^*)] &= \begin{bmatrix} \lambda+P+\mu & \beta_1 S^* & \beta_2 S^* & 0 \\ -P & \lambda-\beta_1 S^*+a_3 & -\beta_2 S^*-\alpha & 0 \\ 0 & -\sigma & \lambda+a_1 & 0 \\ 0 & 0 & -\delta & \lambda+a_2 \end{bmatrix} \\
&= \lambda^4+(a_1+a_2+a_3+P+\mu-\beta_1 S^*)\lambda^3+(a_1a_3+a_1P+a_3P \\
&\quad +\mu a_1+\mu a_3+a_1a_2+a_2a_3+a_2P+\mu a_2-\beta_1 S^* a_1-\sigma\beta_2 S^* \\
&\quad -\sigma\alpha-\mu\beta_1 S^*-\beta_1 S^* a_2)\lambda^2+(Pa_1a_3+\mu a_1a_3+Pa_1a_2+Pa_2a_3 \\
&\quad +\mu a_1a_2+\mu a_2a_3+a_1a_2a_3-\sigma\alpha P-\mu\beta_1 S^* a_1-\mu\sigma\beta_2 S^*-\mu\sigma\alpha \\
&\quad -\beta_1 S^* a_1a_2-\sigma\beta_2 S^* a_2-\sigma\alpha a_2-\mu\beta_1 S^* a_2)\lambda+(Pa_1a_2a_3 \\
&\quad +\mu a_1a_2a_3-\sigma\alpha Pa_2-\mu\beta_1 S^* a_1a_2-\mu\sigma\beta_2 S^* a_2-\mu\sigma\alpha a_2).
\end{aligned}
$$

Hence

$$
0 = b_0\lambda^4+b_1\lambda^3+b_2\lambda^2+b_3\lambda+b_4 \tag{8}
$$

where

$$
\begin{aligned}
b_0 &= 1\ , \\
b_1 &= a_1+a_2+a_3+P+\mu-\beta_1 S^*\ , \\
b_2 &= a_1a_3+a_1P+a_3P+\mu a_1+\mu a_3+a_1a_2+a_2a_3+a_2P+\mu a_2-\beta_1 S^* a_1-\sigma\beta_2 S^* \\
&\quad -\sigma\alpha-\mu\beta_1 S^*-\beta_1 S^* a_2\ , \\
b_3 &= a_1a_3P+\mu a_1a_3+a_1a_2P+a_2a_3P+\mu a_1a_2+\mu a_2a_3+a_1a_2a_3-\sigma\alpha P-\mu\beta_1 S^* a_1 \\
&\quad -\mu\sigma\beta_2 S^*-\mu\sigma\alpha-\beta_1 S^* a_1a_2-\sigma\beta_2 S^* a_2-\sigma\alpha a_2-\mu\beta_1 S^* a_2\ , \\
b_4 &= a_1a_2a_3P+\mu a_1a_2a_3-\sigma\alpha a_2P-\mu\beta_1 S^* a_1a_2-\mu\sigma\beta_2 S^* a_2-\mu\sigma\alpha a_2\ .
\end{aligned}
$$

To ensure that all root of equation (8) have negative real parts, the Routh - Hurwitz stability criterion [28] will be used. We require that $b_1>0$, $b_1b_2-b_3>0$, $b_3(b_1b_2-b_3)-b_1^2b_4>0$, and $b_4>0$.

Now, we note that

$$\begin{aligned} b_1 &= a_1 + a_2 + a_3 + P + \mu - \beta_1 S^* \\ &= (a_3 - \beta_1 S^*) + a_1 + a_2 + P + \mu \ . \end{aligned}$$

and since

$$I_1^* = \frac{I_2^*(\beta_2 S^* + \alpha)}{a_3 - \beta_1 S^*}$$

thus $a_3 - \beta_1 S^* > 0$, we then have $b_1 > 0$. Next, we have

$$\begin{aligned} b_2 &= a_1a_3 + a_1P + a_3P + \mu a_1 + \mu a_3 + a_1a_2 + a_2a_3 + a_2P + \mu a_2 - \beta_1 S^* a_1 - \sigma\beta_2 S^* \\ &\quad -\sigma\alpha \\ &= a_1(a_3 - \beta_1 S^*) + a_2(a_3 - \beta_1 S^*) + \mu(a_3 - \beta_1 S^*) + a_1a_2 + \mu a_1 + \mu a_2 + a_1 P \\ &\quad + a_2 P + a_3 P - \sigma\beta_2 S^* - \sigma\alpha \ . \end{aligned}$$

Let $\frac{dI_1^*}{dt} = 0$ and now we have

$$\begin{aligned} \beta_1 S^* I_1^* + \beta_2 S^* I_2^* - a_3 I_1^* + \alpha I_2^* &= 0 \\ \beta_2 S^* I_2^* &= a_3 I_1^* - \beta_1 S^* I_1^* - \alpha I_2^* \\ \beta_2 S^* &= \frac{a_3 I_1^*}{I_2^*} - \frac{\beta_1 S^* I_1^*}{I_2^*} - \alpha \ . \end{aligned}$$

Thus

$$\sigma\beta_2 S^* = \frac{\sigma(a_3 I_1^* - \beta_1 S^* I_1^*)}{I_2^*} - \sigma\alpha \ .$$

Now we simplify b_2:

$$\begin{aligned} b_2 &= a_1(a_3 - \beta_1 S^*) + a_2(a_3 - \beta_1 S^*) + \mu(a_3 - \beta_1 S^*) + a_1a_2 + \mu a_1 + \mu a_2 \\ &\quad + a_1 P + a_2 P + a_3 P - \sigma\beta_2 S^* - \sigma\alpha \\ &= a_1(a_3 - \beta_1 S^*) + a_2(a_3 - \beta_1 S^*) + \mu(a_3 - \beta_1 S^*) + a_1a_2 + \mu a_1 + \mu a_2 \\ &\quad + a_1 P + a_2 P + (\beta_1 I_1^* + \beta_2 I_2^*)(\sigma + \mu + \phi_1) - \sigma\beta_2 S^* - \sigma\alpha \\ &= a_1(a_3 - \beta_1 S^*) + a_2(a_3 - \beta_1 S^*) + \mu(a_3 - \beta_1 S^*) + a_1a_2 + \mu a_1 + \mu a_2 \\ &\quad + a_1 P + a_2 P + \beta_1 I_1^* \sigma + \beta_1 I_1^* \mu + \beta_1 I_1^* \phi_1 + \beta_2 I_2^* \sigma + \beta_2 I_2^* \mu + \beta_2 I_2^* \phi_1 \end{aligned}$$

$$
\begin{aligned}
& -\left(\frac{\sigma(a_3 I_1^* - \beta_1 S^* I_1^*)}{I_2^*} - \sigma\alpha\right) - \sigma\alpha \\
= \;& a_1(a_3 - \beta_1 S^*) + a_2(a_3 - \beta_1 S^*) + \mu(a_3 - \beta_1 S^*) + a_1 a_2 + \mu a_1 + \mu a_2 \\
& + a_1 P + a_2 P + \beta_1 I_1^* \sigma + \beta_1 I_1^* \mu + \beta_1 I_1^* \phi_1 + \beta_2 I_2^* \sigma + \beta_2 I_2^* \mu + \beta_2 I_2^* \phi_1 \\
& - \frac{\sigma a_3 I_1^*}{I_2^*} + \frac{\sigma \beta_1 S^* I_1^*}{I_2^*} + \sigma\alpha - \sigma\alpha \\
= \;& a_1(a_3 - \beta_1 S^*) + a_2(a_3 - \beta_1 S^*) + \mu(a_3 - \beta_1 S^*) + a_1 a_2 + \mu a_1 + \mu a_2 \\
& + a_1 P + a_2 P + \beta_1 I_1^* \mu + \beta_1 I_1^* \phi_1 + \beta_2 I_2^* \sigma + \beta_2 I_2^* \mu + \beta_2 I_2^* \phi_1 \\
& + \left(\beta_1 I_1^* \sigma - \frac{\sigma a_3 I_1^*}{I_2^*}\right) + \frac{\sigma \beta_1 S^* I_1^*}{I_2^*} \\
= \;& a_1(a_3 - \beta_1 S^*) + a_2(a_3 - \beta_1 S^*) + \mu(a_3 - \beta_1 S^*) + a_1 a_2 + \mu a_1 + \mu a_2 \\
& + a_1 P + a_2 P + \beta_1 I_1^* \mu + \beta_1 I_1^* \phi_1 + \beta_2 I_2^* \sigma + \beta_2 I_2^* \mu + \beta_2 I_2^* \phi_1 + \sigma I_1^* \\
& \left(\beta_1 - \frac{a_3}{I_2^*}\right) + \frac{\sigma \beta_1 S^* I_1^*}{I_2^*} .
\end{aligned}
$$

Hence $b_2 > 0$. Now, we turn to b_3:

$$
\begin{aligned}
b_3 \;=\;& a_1 a_3 P + \mu a_1 a_3 + a_1 a_2 P + a_2 a_3 P + \mu a_1 a_2 + \mu a_2 a_3 + a_1 a_2 a_3 - \sigma\alpha P \\
& - \mu \beta_1 S^* a_1 - \mu \sigma \beta_2 S^* - \mu\sigma\alpha - \beta_1 S^* a_1 a_2 - \sigma \beta_2 S^* a_2 - \sigma\alpha a_2 - \mu \beta_1 S^* a_2 \\
=\;& \mu a_1(a_3 - \beta_1 S^*) + \mu a_2(a_3 - \beta_1 S^*) + a_1 a_2(a_3 - \beta_1 S^*) + a_1 a_2 P + a_1 a_3 P \\
& + a_2 a_3 P + a_1 a_2 \mu - \sigma\alpha P - \mu \sigma \beta_2 S^* - \mu\sigma\alpha - \sigma \beta_2 S^* a_2 - \sigma\alpha a_2 .
\end{aligned}
$$

Since

$$
\sigma \beta_2 S^* = \frac{\sigma(a_3 I_1^* - \beta_1 S^* I_1^*)}{I_2^*} - \sigma\alpha ,
$$

now we can write

$$
\begin{aligned}
b_3 \;=\;& \mu a_1(a_3 - \beta_1 S^*) + \mu a_2(a_3 - \beta_1 S^*) + a_1 a_2(a_3 - \beta_1 S^*) + a_1 a_2 P + a_1 a_3 P \\
& + a_2 a_3 P + a_1 a_2 \mu - \sigma\alpha P - \mu\left(\frac{\sigma(a_3 I_1^* - \beta_1 S^* I_1^*)}{I_2^*} - \sigma\alpha\right) - \mu\sigma\alpha \\
& - a_2\left(\frac{\sigma(a_3 I_1^* - \beta_1 S^* I_1^*)}{I_2^*} - \sigma\alpha\right) - \sigma\alpha a_2 \\
=\;& \mu a_1(a_3 - \beta_1 S^*) + \mu a_2(a_3 - \beta_1 S^*) + a_1 a_2(a_3 - \beta_1 S^*) + a_1 a_2 P + a_1 a_3 P
\end{aligned}
$$

$$
\begin{aligned}
&+a_2a_3P+a_1a_2\mu-\sigma\alpha P-\frac{\mu\sigma a_3I_1^*}{I_2^*}+\frac{\mu\sigma\beta_1S^*I_1^*}{I_2^*}+\mu\sigma\alpha-\mu\sigma\alpha-\frac{a_2\sigma a_3I_1^*}{I_2^*}\\
&+\frac{a_2\sigma\beta_1S^*I_1^*}{I_2^*}+a_2\sigma\alpha-a_2\sigma\alpha\\
= \;&\mu a_1(a_3-\beta_1S^*)+\mu a_2(a_3-\beta_1S^*)+a_1a_2(a_3-\beta_1S^*)+a_1a_2P+a_1a_2\mu\\
&+(\delta+\mu+\phi_2+\alpha)(\sigma+\mu+\phi_1)(\beta_1I_1^*+\beta_2I_2^*)+a_2(\sigma+\mu+\phi_1)(\beta_1I_1^*+\beta_2I_2^*)\\
&-\sigma\alpha(\beta_1I_1^*+\beta_2I_2^*)-\frac{\mu\sigma a_3I_1^*}{I_2^*}+\frac{\mu\sigma\beta_1S^*I_1^*}{I_2^*}-\frac{a_2\sigma a_3I_1^*}{I_2^*}+\frac{a_2\sigma\beta_1S^*I_1^*}{I_2^*}\\
= \;&\mu a_1(a_3-\beta_1S^*)+\mu a_2(a_3-\beta_1S^*)+a_1a_2(a_3-\beta_1S^*)+a_1a_2P+a_1a_2\mu\\
&+\delta\sigma\beta_1I_1^*+\delta\mu\beta_1I_1^*+\delta\phi_1\beta_1I_1^*+\mu\sigma\beta_1I_1^*+\mu^2\beta_1I_1^*+\mu\phi_1\beta_1I_1^*+\phi_2\sigma\beta_1I_1^*\\
&+\phi_2\mu\beta_1I_1^*+\phi_1\phi_2\beta_1I_1^*+\alpha\sigma\beta_1I_1^*+\alpha\mu\beta_1I_1^*+\alpha\phi_1\beta_1I_1^*+\delta\sigma\beta_2I_2^*+\delta\mu\beta_2I_2^*\\
&+\delta\phi_1\beta_2I_2^*+\mu\sigma\beta_2I_2^*+\mu^2\beta_2I_2^*+\mu\phi_1\beta_2I_2^*+\phi_2\sigma\beta_2I_2^*+\phi_2\mu\beta_2I_2^*+\phi_1\phi_2\beta_2I_2^*\\
&+\alpha\sigma\beta_2I_2^*+\alpha\mu\beta_2I_2^*+\alpha\phi_1\beta_2I_2^*+a_2\sigma\beta_1I_1^*+a_2\mu\beta_1I_1^*+a_2\phi_1\beta_1I_1^*+a_2\sigma\beta_2I_2^*\\
&+a_2\mu\beta_2I_2^*+a_2\phi_1\beta_2I_2^*-\sigma\alpha\beta_1I_1^*-\sigma\alpha\beta_2I_2^*-\frac{\mu\sigma a_3I_1^*}{I_2^*}+\frac{\mu\sigma\beta_1S^*I_1^*}{I_2^*}-\frac{a_2\sigma a_3I_1^*}{I_2^*}\\
&+\frac{a_2\sigma\beta_1S^*I_1^*}{I_2^*}\\
= \;&\mu a_1(a_3-\beta_1S^*)+\mu a_2(a_3-\beta_1S^*)+a_1a_2(a_3-\beta_1S^*)+a_1a_2P+a_1a_2\mu+\delta\sigma\beta_1I_1^*\\
&+\delta\mu\beta_1I_1^*+\delta\phi_1\beta_1I_1^*+\mu^2\beta_1I_1^*+\mu\phi_1\beta_1I_1^*+\phi_2\sigma\beta_1I_1^*+\phi_2\mu\beta_1I_1^*+\phi_1\phi_2\beta_1I_1^*\\
&+\alpha\mu\beta_1I_1^*+\alpha\phi_1\beta_1I_1^*+\delta\sigma\beta_2I_2^*+\delta\mu\beta_2I_2^*+\delta\phi_1\beta_2I_2^*+\mu\sigma\beta_2I_2^*+\mu^2\beta_2I_2^*\\
&+\mu\phi_1\beta_2I_2^*+\phi_2\sigma\beta_2I_2^*+\phi_2\mu\beta_2I_2^*+\phi_1\phi_2\beta_2I_2^*+\alpha\mu\beta_2I_2^*+\alpha\phi_1\beta_2I_2^*+a_2\mu\beta_1I_1^*\\
&+a_2\phi_1\beta_1I_1^*+a_2\sigma\beta_2I_2^*+a_2\mu\beta_2I_2^*+a_2\phi_1\beta_2I_2^*+\left(\mu\sigma\beta_1I_1^*-\frac{\mu\sigma a_3I_1^*}{I_2^*}\right)+\frac{\mu\sigma\beta_1S^*I_1^*}{I_2^*}\\
&+\left(a_2\sigma\beta_1I_1^*-\frac{a_2\sigma a_3I_1^*}{I_2^*}\right)+\frac{a_2\sigma\beta_1S^*I_1^*}{I_2^*}\\
= \;&\mu a_1(a_3-\beta_1S^*)+\mu a_2(a_3-\beta_1S^*)+a_1a_2(a_3-\beta_1S^*)+a_1a_2P+a_1a_2\mu+\delta\sigma\beta_1I_1^*\\
&+\delta\mu\beta_1I_1^*+\delta\phi_1\beta_1I_1^*+\mu^2\beta_1I_1^*+\mu\phi_1\beta_1I_1^*+\phi_2\sigma\beta_1I_1^*+\phi_2\mu\beta_1I_1^*+\phi_1\phi_2\beta_1I_1^*\\
&+\alpha\mu\beta_1I_1^*+\alpha\phi_1\beta_1I_1^*+\delta\sigma\beta_2I_2^*+\delta\mu\beta_2I_2^*+\delta\phi_1\beta_2I_2^*+\mu\sigma\beta_2I_2^*+\mu^2\beta_2I_2^*\\
&+\mu\phi_1\beta_2I_2^*+\phi_2\sigma\beta_2I_2^*+\phi_2\mu\beta_2I_2^*+\phi_1\phi_2\beta_2I_2^*+\alpha\mu\beta_2I_2^*+\alpha\phi_1\beta_2I_2^*+a_2\mu\beta_1I_1^*
\end{aligned}
$$

$$+a_2\phi_1\beta_1 I_1^* + a_2\sigma\beta_2 I_2^* + a_2\mu\beta_2 I_2^* + a_2\phi_1\beta_2 I_2^* + \mu\sigma I_1^*\left(\beta_1 - \frac{a_3}{I_2^*}\right) + \frac{\mu\sigma\beta_1 S^* I_1^*}{I_2^*}$$
$$+a_2\sigma I_1^*\left(\beta_1 - \frac{a_3}{I_2^*}\right) + \frac{a_2\sigma\beta_1 S^* I_1^*}{I_2^*}\ .$$

Thus $b_3 > 0$. Now for b_4, we write

$$\begin{aligned} b_4 &= a_1a_2a_3P + \mu a_1a_2a_3 - \sigma\alpha a_2 P - \mu\beta_1 S^* a_1a_2 - \mu\sigma\beta_2 S^* a_2 - \mu\sigma\alpha a_2 \\ &= a_1a_2\mu(a_3 - \beta_1 S^*) + a_1a_2a_3P - \sigma\alpha a_2P - \mu\sigma a_2\beta_2 S^* - \mu\sigma a_2\alpha \end{aligned}$$

and from

$$\sigma\beta_2 S^* = \frac{\sigma(a_3 I_1^* - \beta_1 S^* I_1^*)}{I_2^*} - \sigma\alpha$$

we then have

$$\begin{aligned} b_4 &= a_1a_2\mu(a_3 - \beta_1 S^*) + a_1a_2a_3P - \sigma\alpha a_2P - \mu\sigma a_2\beta_2 S^* - \mu\sigma a_2\alpha \\ &= a_1a_2\mu(a_3 - \beta_1 S^*) + a_1a_2a_3P - \sigma\alpha a_2P - \mu a_2\left(\frac{\sigma(a_3 I_1^* - \beta_1 S^* I_1^*)}{I_2^*} - \sigma\alpha\right) - \mu\sigma\alpha a_2 \\ &= a_1a_2\mu(a_3 - \beta_1 S^*) + a_2(\delta + \mu + \phi_2 + \alpha)(\sigma + \mu + \phi_1)(\beta_1 I_1^* + \beta_2 I_2^*) - \sigma\alpha P a_2 \\ &\quad - \frac{\mu\sigma a_2 a_3 I_1^*}{I_2^*} + \frac{\mu\sigma a_2\beta_1 S^* I_1^*}{I_2^*} + \mu\sigma\alpha a_2 - \mu\sigma\alpha a_2 \\ &= a_1a_2\mu(a_3 - \beta_1 S^*) + \delta\sigma a_2\beta_1 I_1^* + \delta\mu a_2\beta_1 I_1^* + \delta\phi_1 a_2\beta_1 I_1^* + \mu\sigma a_2\beta_1 I_1^* + \mu^2 a_2\beta_1 I_1^* \\ &\quad + \mu\phi_1 a_2\beta_1 I_1^* + \sigma\phi_2 a_2\beta_1 I_1^* + \mu\phi_2 a_2\beta_1 I_1^* + \phi_1\phi_2 a_2\beta_1 I_1^* + \sigma\alpha a_2\beta_1 I_1^* + \mu\alpha a_2\beta_1 I_1^* \\ &\quad + \alpha\phi_1 a_2\beta_1 I_1^* + \delta\sigma a_2\beta_2 I_2^* + \delta\mu a_2\beta_2 I_2^* + \delta\phi_1 a_2\beta_2 I_2^* + \mu\sigma a_2\beta_2 I_2^* + \mu^2 a_2\beta_2 I_2^* \\ &\quad + \mu\phi_1 a_2\beta_2 I_2^* + \sigma\phi_2 a_2\beta_2 I_2^* + \mu\phi_2 a_2\beta_2 I_2^* + \phi_1\phi_2 a_2\beta_2 I_2^* + \sigma\alpha a_2\beta_2 I_2^* + \mu\alpha a_2\beta_2 I_2^* \\ &\quad + \alpha\phi_1 a_2\beta_2 I_2^* - \sigma\alpha P a_2 - \frac{\mu\sigma a_2 a_3 I_1^*}{I_2^*} + \frac{\mu\sigma a_2\beta_1 S^* I_1^*}{I_2^*} \\ &= a_1a_2\mu(a_3 - \beta_1 S^*) + \delta\sigma a_2\beta_1 I_1^* + \delta\mu a_2\beta_1 I_1^* + \delta\phi_1 a_2\beta_1 I_1^* + \mu^2 a_2\beta_1 I_1^* + \mu\phi_1 a_2\beta_1 I_1^* \\ &\quad + \sigma\phi_2 a_2\beta_1 I_1^* + \mu\phi_2 a_2\beta_1 I_1^* + \phi_1\phi_2 a_2\beta_1 I_1^* + \mu\alpha a_2\beta_1 I_1^* + \alpha\phi_1 a_2\beta_1 I_1^* + \delta\sigma a_2\beta_2 I_2^* \\ &\quad + \delta\mu a_2\beta_2 I_2^* + \delta\phi_1 a_2\beta_2 I_2^* + \mu\sigma a_2\beta_2 I_2^* + \mu^2 a_2\beta_2 I_2^* + \mu\phi_1 a_2\beta_2 I_2^* + \sigma\phi_2 a_2\beta_2 I_2^* \\ &\quad + \mu\phi_2 a_2\beta_2 I_2^* + \phi_1\phi_2 a_2\beta_2 I_2^* + \mu\alpha a_2\beta_2 I_2^* + \alpha\phi_1 a_2\beta_2 I_2^* + (\sigma\alpha a_2\beta_1 I_1^* + \sigma\alpha a_2\beta_2 I_2^*) \\ &\quad - \sigma\alpha a_2(\beta_1 I_1^* + \beta_2 I_2^*) + \left(\mu\sigma a_2\beta_1 I_1^* - \frac{\mu\sigma a_2 a_3 I_1^*}{I_2^*}\right) + \frac{\mu\sigma a_2\beta_1 S^* I_1^*}{I_2^*} \end{aligned}$$

$$
\begin{aligned}
&= a_1a_2\mu(a_3-\beta_1S^*)+\delta\sigma a_2\beta_1I_1^*+\delta\mu a_2\beta_1I_1^*+\delta\phi_1a_2\beta_1I_1^*+\mu^2a_2\beta_1I_1^*+\mu\phi_1a_2\beta_1I_1^*\\
&\quad+\sigma\phi_2a_2\beta_1I_1^*+\mu\phi_2a_2\beta_1I_1^*+\phi_1\phi_2a_2\beta_1I_1^*+\mu\alpha a_2\beta_1I_1^*+\alpha\phi_1a_2\beta_1I_1^*+\delta\sigma a_2\beta_2I_2^*\\
&\quad+\delta\mu a_2\beta_2I_2^*+\delta\phi_1a_2\beta_2I_2^*+\mu\sigma a_2\beta_2I_2^*+\mu^2a_2\beta_2I_2^*+\mu\phi_1a_2\beta_2I_2^*+\sigma\phi_2a_2\beta_2I_2^*\\
&\quad+\mu\phi_2a_2\beta_2I_2^*+\phi_1\phi_2a_2\beta_2I_2^*+\mu\alpha a_2\beta_2I_2^*+\alpha\phi_1a_2\beta_2I_2^*+\mu\sigma a_2I_1^*\left(\beta_1-\frac{a_3}{I_2^*}\right)\\
&\quad+\frac{\mu\sigma a_2\beta_1S^*I_1^*}{I_2^*}.
\end{aligned}
$$

Therefore $b_4>0$. Similarly, we will show that $b_1b_2-b_3>0$. Consider that

$$
\begin{aligned}
b_1b_2-b_3 &= (a_3-\beta_1S)\big(a_1(a_3-\beta_1S^*)+a_2(a_3-\beta_1S^*)+\mu(a_3-\beta_1S^*)+a_1a_2\\
&\quad+\mu a_1+\mu a_2+a_1P+a_2P+\beta_1I_1^*\mu+\beta_1I_1^*\phi_1+\beta_2I_2^*\sigma+\beta_2I_2^*\mu+\beta_2I_2^*\phi_1\\
&\quad+\sigma I_1^*\left(\beta_1-\frac{a_3}{I_2^*}\right)+\frac{\sigma\beta_1S^*I_1^*}{I_2^*}\big)-\big(\mu a_1(a_3-\beta_1S^*)+\mu a_2(a_3-\beta_1S^*)\\
&\quad+a_1a_2(a_3-\beta_1S^*)+a_1a_2P+a_1a_2\mu+\delta\sigma\beta_1I_1^*+\delta\mu\beta_1I_1^*+\delta\phi_1\beta_1I_1^*\\
&\quad+\mu^2\beta_1I_1^*+\mu\phi_1\beta_1I_1^*+\phi_2\sigma\beta_1I_1^*+\phi_2\mu\beta_1I_1^*+\phi_1\phi_2\beta_1I_1^*+\alpha\mu\beta_1I_1^*\\
&\quad+\alpha\phi_1\beta_1I_1^*+\delta\sigma\beta_2I_2^*+\delta\mu\beta_2I_2^*+\delta\phi_1\beta_2I_2^*+\mu\sigma\beta_2I_2^*+\mu^2\beta_2I_2^*+\mu\phi_1\beta_2I_2^*\\
&\quad+\phi_2\sigma\beta_2I_2^*+\phi_2\mu\beta_2I_2^*+\phi_1\phi_2\beta_2I_2^*+\alpha\mu\beta_2I_2^*+\alpha\phi_1\beta_2I_2^*+a_2\mu\beta_1I_1^*\\
&\quad+a_2\phi_1\beta_1I_1^*+a_2\sigma\beta_2I_2^*+a_2\mu\beta_2I_2^*+a_2\phi_1\beta_2I_2^*+\mu\sigma I_1^*\left(\beta_1-\frac{a_3}{I_2^*}\right)\\
&\quad+\frac{\mu\sigma\beta_1S^*I_1^*}{I_2^*}+a_2\sigma I_1^*\left(\beta_1-\frac{a_3}{I_2^*}\right)+\frac{a_2\sigma\beta_1S^*I_1^*}{I_2^*}\big)\\
&= (a_3-\beta_1S)\big(a_1(a_3-\beta_1S^*)+a_2(a_3-\beta_1S^*)+\mu(a_3-\beta_1S^*)+a_1a_2\\
&\quad+\mu a_1+\mu a_2+a_1P+a_2P+\beta_1I_1^*\mu+\beta_1I_1^*\phi_1+\beta_2I_2^*\sigma+\beta_2I_2^*\mu+\beta_2I_2^*\phi_1\\
&\quad+\sigma I_1^*\left(\beta_1-\frac{a_3}{I_2^*}\right)+\frac{\sigma\beta_1S^*I_1^*}{I_2^*}\big)-\mu a_1(a_3-\beta_1S^*)-\mu a_2(a_3-\beta_1S^*)\\
&\quad-a_1a_2(a_3-\beta_1S^*)-a_1a_2P-a_1a_2\mu-\delta\sigma\beta_1I_1^*-\delta\mu\beta_1I_1^*-\delta\phi_1\beta_1I_1^*\\
&\quad-\mu^2\beta_1I_1^*-\mu\phi_1\beta_1I_1^*-\phi_2\sigma\beta_1I_1^*-\phi_2\mu\beta_1I_1^*-\phi_1\phi_2\beta_1I_1^*-\alpha\mu\beta_1I_1^*\\
&\quad-\alpha\phi_1\beta_1I_1^*-\delta\sigma\beta_2I_2^*-\delta\mu\beta_2I_2^*-\delta\phi_1\beta_2I_2^*-\mu\sigma\beta_2I_2^*-\mu^2\beta_2I_2^*-\mu\phi_1\beta_2I_2^*\\
&\quad-\phi_2\sigma\beta_2I_2^*-\phi_2\mu\beta_2I_2^*-\phi_1\phi_2\beta_2I_2^*-\alpha\mu\beta_2I_2^*-\alpha\phi_1\beta_2I_2^*-a_2\mu\beta_1I_1^*
\end{aligned}
$$

$$
\begin{aligned}
& -a_2\phi_1\beta_1 I_1^* - a_2\sigma\beta_2 I_2^* - a_2\mu\beta_2 I_2^* - a_2\phi_1\beta_2 I_2^* - \mu\sigma I_1^*\left(\beta_1 - \frac{a_3}{I_2^*}\right) \\
& -\frac{\mu\sigma\beta_1 S^* I_1^*}{I_2^*} - a_2\sigma I_1^*\left(\beta_1 - \frac{a_3}{I_2^*}\right) - \frac{a_2\sigma\beta_1 S^* I_1^*}{I_2^*} \\
= \;& a_1(a_3-\beta_1 S^*)^2 + a_2(a_3-\beta_1 S^*)^2 + \mu(a_3-\beta_1 S^*)^2 + a_1a_2(a_3-\beta_1 S^*) \\
& +\mu a_1(a_3-\beta_1 S^*) + \mu a_2(a_3-\beta_1 S^*) + a_1P(a_3-\beta_1 S^*) + a_2P(a_3-\beta_1 S^*) \\
& +\beta_1 I_1^*\mu(a_3-\beta_1 S^*) + \beta_1 I_1^*\phi_1(a_3-\beta_1 S^*) + \beta_2 I_2^*\sigma(a_3-\beta_1 S^*) \\
& +\beta_2 I_2^*\mu(a_3-\beta_1 S^*) + \beta_2 I_2^*\phi_1(a_3-\beta_1 S^*) + \sigma I_1^*(a_3-\beta_1 S^*)\left(\beta_1 - \frac{a_3}{I_2^*}\right) \\
& +\frac{\sigma\beta_1 S^* I_1^*}{I_2^*}(a_3-\beta_1 S^*) + a_1^2(a_3-\beta_1 S^*) + \mu a_1^2 + \mu a_1a_2 + a_1^2a_2 + a_1^2P \\
& +\beta_1 I_1^*\mu a_1 + \beta_1 I_1^*\phi_1 a_1 + \beta_2 I_2^*\sigma a_1 + \beta_2 I_2^*\mu a_1 + \beta_2 I_2^*\phi_1 a_1 + \sigma I_1^* a_1\left(\beta_1 - \frac{a_3}{I_2^*}\right) \\
& +\frac{a_1\sigma\beta_1 S^* I_1^*}{I_2^*} + a_1a_2(a_3-\beta_1 S^*) + a_2^2(a_3-\beta_1 S^*) + a_1a_2^2 + \mu a_2^2 + a_1a_2P \\
& +a_2^2P + a_1P(a_3-\beta_1 S^*) + a_2P(a_3-\beta_1 S^*) + \mu P(a_3-\beta_1 S^*) + a_1a_2P \\
& +\mu a_1P + \mu a_2P + a_1P^2 + a_2P^2 + \beta_1 I_1^*\mu P + \beta_1 I_1^*\phi_1 P + \beta_2 I_2^*\sigma P + \beta_2 I_2^*\mu P \\
& +\beta_2 I_2^*\phi_1 P + \sigma I_1^* P\left(\beta_1 - \frac{a_3}{I_2^*}\right) + \frac{P\sigma\beta_1 S^* I_1^*}{I_2^*} + \mu a_1(a_3-\beta_1 S^*) \\
& +\mu^2(a_3-\beta_1 S^*) + \mu a_2(a_3-\beta_1 S^*) + \mu a_1a_2 + \mu^2 a_1 + \mu^2 a_2 + a_1\mu P + a_2\mu P \\
& +\sigma I_1^*\mu\left(\beta_1 - \frac{a_3}{I_2^*}\right) \\
= \;& a_1(a_3-\beta_1 S^*)^2 + a_2(a_3-\beta_1 S^*)^2 + \mu(a_3-\beta_1 S^*)^2 + a_1a_2(a_3-\beta_1 S^*) \\
& +\mu a_1(a_3-\beta_1 S^*) + \mu a_2(a_3-\beta_1 S^*) + a_1P(a_3-\beta_1 S^*) + a_2P(a_3-\beta_1 S^*) \\
& +\beta_1 I_1^*\mu(a_3-\beta_1 S^*) + \beta_1 I_1^*\phi_1(a_3-\beta_1 S^*) + \beta_2 I_2^*\sigma(a_3-\beta_1 S^*) \\
& +\beta_2 I_2^*\mu(a_3-\beta_1 S^*) + \beta_2 I_2^*\phi_1(a_3-\beta_1 S^*) + \sigma I_1^*(a_3-\beta_1 S^*)\left(\beta_1 - \frac{a_3}{I_2^*}\right) \\
& +\frac{\sigma\beta_1 S^* I_1^*}{I_2^*}(a_3-\beta_1 S^*) + a_1^2(a_3-\beta_1 S^*) + \mu a_1^2 + \mu a_1a_2 + a_1^2a_2 + a_1^2P \\
& +\beta_1 I_1^*\mu^2 + \beta_1 I_1^*\phi_1\mu + \beta_2 I_2^*\sigma\mu + \beta_2 I_2^*\mu^2 + \beta_2 I_2^*\phi_1\mu + \sigma I_1^* a_1\left(\beta_1 - \frac{a_3}{I_2^*}\right) \\
& +\frac{a_1\sigma\beta_1 S^* I_1^*}{I_2^*} + a_1a_2(a_3-\beta_1 S^*) + a_2^2(a_3-\beta_1 S^*) + a_1a_2(a_3-\beta_1 S^*)
\end{aligned}
$$

$$
\begin{aligned}
& +a_2^2(a_3-\beta_1S^*)+a_1a_2^2+a_2^2\mu+a_1a_2P+a_2^2P+a_1P(a_3-\beta_1S^*)+a_1a_2P \\
& +\mu P(a_3-\beta_1S^*)+a_2P(a_3-\beta_1S^*)+\mu a_1P+\mu a_2P+a_1P^2+a_2P^2 \\
& +\beta_1I_1^*\mu P+\beta_1I_1^*\phi_1P+\beta_2I_2^*\sigma P+\beta_2I_2^*\mu P+\beta_2I_2^*\phi_1P+\sigma I_1^*P\left(\beta_1-\frac{a_3}{I_2^*}\right) \\
& +\frac{P\sigma\beta_1S^*I_1^*}{I_2^*}+\mu a_1(a_3-\beta_1S^*)+\mu^2(a_3-\beta_1S^*)+\mu a_2(a_3-\beta_1S^*) \\
& +\mu a_1a_2+\mu^2a_1+\mu^2a_2+a_1\mu P+a_2\mu P+\sigma I_1^*\mu\left(\beta_1-\frac{a_3}{I_2^*}\right)+\frac{\mu\sigma\beta_1S^*I_1^*}{I_2^*} \\
& -\delta\sigma\beta_1I_1^*-\phi_2\sigma\beta_1I_1^*-\alpha\sigma\beta_1I_1^*-\mu\sigma\beta_1I_1^* \\
= \; & a_1(a_3-\beta_1S^*)^2+a_2(a_3-\beta_1S^*)^2+\mu(a_3-\beta_1S^*)^2+a_1a_2(a_3-\beta_1S^*) \\
& +\mu a_1(a_3-\beta_1S^*)+\mu a_2(a_3-\beta_1S^*)+a_1P(a_3-\beta_1S^*)+a_2P(a_3-\beta_1S^*) \\
& +\beta_1I_1^*\mu(a_3-\beta_1S^*)+\beta_1I_1^*\phi_1(a_3-\beta_1S^*)+\beta_2I_2^*\sigma(a_3-\beta_1S^*) \\
& +\beta_2I_2^*\mu(a_3-\beta_1S^*)+\beta_2I_2^*\phi_1(a_3-\beta_1S^*)+\sigma I_1^*(a_3-\beta_1S^*)\left(\beta_1-\frac{a_3}{I_2^*}\right) \\
& +\frac{\sigma\beta_1S^*I_1^*}{I_2^*}(a_3-\beta_1S^*)+a_1^2(a_3-\beta_1S^*)+\mu a_1^2+\mu a_1a_2+a_1^2a_2+a_1^2P \\
& +\beta_1I_1^*\mu^2+\beta_1I_1^*\phi_1\mu+\beta_2I_2^*\sigma\mu+\beta_2I_2^*\mu^2+\beta_2I_2^*\phi_1\mu+\sigma I_1^*a_1\left(\beta_1-\frac{a_3}{I_2^*}\right) \\
& +\frac{a_1\sigma\beta_1S^*I_1^*}{I_2^*}+a_1a_2(a_3-\beta_1S^*)+a_2^2(a_3-\beta_1S^*)+a_1a_2(a_3-\beta_1S^*) \\
& +a_2^2(a_3-\beta_1S^*)+a_1a_2^2+a_2^2\mu+a_1a_2P+a_2^2P+a_1P(a_3-\beta_1S^*)+a_1a_2P \\
& +\mu P(a_3-\beta_1S^*)+a_2P(a_3-\beta_1S^*)+\mu a_1P+\mu a_2P+a_1P^2+a_2P^2 \\
& +\beta_1I_1^*\mu P+\beta_1I_1^*\phi_1P+\beta_2I_2^*\sigma P+\beta_2I_2^*\mu P+\beta_2I_2^*\phi_1P+\sigma I_1^*P\left(\beta_1-\frac{a_3}{I_2^*}\right) \\
& +\frac{P\sigma\beta_1S^*I_1^*}{I_2^*}+\mu a_1(a_3-\beta_1S^*)+\mu^2(a_3-\beta_1S^*)+\mu a_2(a_3-\beta_1S^*)+\mu a_1a_2 \\
& +\mu^2a_1+\mu^2a_2+a_1\mu P+a_2\mu P+\sigma I_1^*\mu\left(\beta_1-\frac{a_3}{I_2^*}\right)+\frac{\mu\sigma\beta_1S^*I_1^*}{I_2^*}-\beta_1I_1^*\sigma a_1 \\
= \; & a_1(a_3-\beta_1S^*)^2+a_2(a_3-\beta_1S^*)^2+\mu(a_3-\beta_1S^*)^2+a_1a_2(a_3-\beta_1S^*) \\
& +\mu a_1(a_3-\beta_1S^*)+\mu a_2(a_3-\beta_1S^*)+a_1P(a_3-\beta_1S^*)+a_2P(a_3-\beta_1S^*) \\
& +\beta_1I_1^*\mu(a_3-\beta_1S^*)+\beta_1I_1^*\phi_1(a_3-\beta_1S^*)+\beta_2I_2^*\sigma(a_3-\beta_1S^*)
\end{aligned}
$$

$$
\begin{aligned}
&+\beta_2 I_2^*\mu(a_3-\beta_1 S^*)+\beta_2 I_2^*\phi_1(a_3-\beta_1 S^*)+\sigma I_1^*(a_3-\beta_1 S^*)\left(\beta_1-\frac{a_3}{I_2^*}\right)\\
&+\frac{\sigma\beta_1 S^* I_1^*}{I_2^*}(a_3-\beta_1 S^*)+a_1^2(a_3-\beta_1 S^*)+\mu a_1^2+\mu a_1 a_2+a_1^2 a_2+a_1^2 P\\
&+\beta_1 I_1^*\mu^2+\beta_1 I_1^*\phi_1\mu+\beta_2 I_2^*\sigma\mu+\beta_2 I_2^*\mu^2+\beta_2 I_2^*\phi_1\mu+\sigma I_1^* a_1\left(\beta_1-\frac{a_3}{I_2^*}\right)\\
&+a_1 a_2(a_3-\beta_1 S^*)+a_2^2(a_3-\beta_1 S^*)+a_1 a_2(a_3-\beta_1 S^*)+a_2^2(a_3-\beta_1 S^*)\\
&+a_1 a_2^2+a_2^2\mu+a_1 a_2 P+a_2^2 P+a_1 P(a_3-\beta_1 S^*)+a_1 a_2 P+\mu P(a_3-\beta_1 S^*)\\
&+a_2 P(a_3-\beta_1 S^*)+\mu a_1 P+\mu a_2 P+a_1 P^2+a_2 P^2+\beta_1 I_1^*\mu P+\beta_1 I_1^*\phi_1 P\\
&+\beta_2 I_2^*\sigma P+\beta_2 I_2^*\mu P+\beta_2 I_2^*\phi_1 P+\sigma I_1^* P\left(\beta_1-\frac{a_3}{I_2^*}\right)+\frac{P\sigma\beta_1 S^* I_1^*}{I_2^*}\\
&+\mu a_1(a_3-\beta_1 S^*)+\mu^2(a_3-\beta_1 S^*)+\mu a_2(a_3-\beta_1 S^*)+\mu a_1 a_2+\mu^2 a_1+\mu^2 a_2\\
&+a_1\mu P+a_2\mu P+\sigma I_1^*\mu\left(\beta_1-\frac{a_3}{I_2^*}\right)+\frac{\mu\sigma\beta_1 S^* I_1^*}{I_2^*}+\beta_1 I_1^*\sigma a_1\left(\frac{S^*}{I_2^*}-1\right).
\end{aligned}
$$

Clearly, we have $b_1b_2-b_3>0$. Similarly, we can also conclude that $b_3(b_1b_2-b_3)-b_1^2b_4>0$. We complete the proof.

5. Optimal Control Study

We now turn to the more general model with time-dependent controls $\phi_1(t)$, $\phi_2(t)$, and $\phi_3(t)$. Here, $\phi_1(t)$ represents the rate at which individuals who are aware of their HIV status begin taking antiretroviral therapy (ART). We assume that these individuals start ART as soon as they are infected, and we classify them as asymptomatic patients. $\phi_2(t)$ represents the medication rate for symptomatic HIV patients. This group of individuals may also suffer from co-infections such as hepatitis or tuberculosis, resulting in additional treatment costs for both these diseases and ART. Additionally, individuals with weakened immune systems may progress to AIDS. This group experiences more severe symptoms due to their co-infections, leading to higher treatment costs. Therefore, $\phi_3(t)$ represents the additional medical costs for AIDS patients.

In this section, we consider the system over a time interval $[0,T]$. The functions $\phi_1(t)$, $\phi_2(t)$, and $\phi_3(t)$ are assumed to be at least Lebesgue measurable on $[0,T]$. The control set is defined as:

$$\Omega=\{\phi_1(t),\phi_2(t),\phi_3(t)|0\le\phi_1(t)\le\phi_{1\max},0\le\phi_2(t)\le\phi_{2\max},0\le\phi_3(t)\le\phi_{3\max}\}$$

where $\phi_{1,2,3max}$ denotes the upper bounds for the effort of controls. The bound reflects practical limitation on the maximum rate of control in given time period.

The inclusion of time-dependent controls complicates the analysis of our system, as the disease dynamics now depend on the evolution of these controls. In the following, we conduct an optimal control analysis of this problem, aiming to minimize both the total number of infections and the control costs over the time interval $[0,T]$ i.e.,

$$\begin{aligned} \min_{(\phi_{1,2,3})\in\Omega} \int_0^T & [I_1(t)+I_2(t)+A(t)+c_{11}\phi_1(t)I_1(t)+c_{12}\phi_2(t)I_2+c_{13}\phi_3(t)A \\ & +c_{21}\phi_1^2(t)+c_{22}\phi_2^2(t)+c_{23}\phi_3^2(t)]dt \end{aligned} \tag{9}$$

Here, the parameters c_{11}, c_{12}, c_{13}, c_{21}, c_{22}, and c_{23}, with their respective units, represent the costs associated with each control. Quadratic terms are included to account for nonlinear costs that may arise at higher intervention levels.

The minimization process is subject to the differential equations of our system, which are referred to as the state equations. The unknowns I_1, I_2, and A are now considered state variables, in contrast to the control variables ϕ_1, ϕ_2, and ϕ_3. Our objective is to determine the optimal controls $\phi_1^*(t)$, $\phi_2^*(t)$, and $\phi_3^*(t)$ that minimize the objective functional in (9).

We first establish the following theorem on the existence of optimal control.

Theorem There exists $\phi_1^*(t), \phi_2^*$ and $\phi_3^* \in \Omega$ such that the objective functional in (9) is minimized.

Proof. The control set Ω is closed and convex, and the integrand of the objective functional in (9) is also convex. Therefore, according to the standard optimal control theorems outlined in [10], the conditions for the existence of an optimal control are satisfied, as the model is linear with respect to the control variables. Furthermore, the optimal control is unique for small T due to the Lipschitz continuity of the state equations and the boundedness of the state variables [13].

We will apply the method outlined in [2] and [15] to determine the optimal control solution. This method is based on Pontryagin's Maximum Principle [24], which introduces adjoint functions and expresses the optimal control in terms of both the state and adjoint functions. Essentially, this approach transforms the problem of minimizing the objective functional (subject to the state equations) into minimizing the Hamiltonian with respect to the controls.

$$\frac{dS}{dt} = \mu N - \beta_1 S I_1 - \beta_2 S I_2 - \mu S \tag{10}$$
$$\frac{dI_1}{dt} = \beta_1 S I_1 + \beta_2 S I_2 - \sigma I_1 - \mu I_1 - \phi_1(t) I_1 + \alpha I_2 \tag{11}$$
$$\frac{dI_2}{dt} = \sigma I_1 - \delta I_2 - \mu I_2 - \phi_2(t) I_2 - \alpha I_2 \tag{12}$$
$$\frac{dA}{dt} = \delta I_2 - \kappa A - \mu A - \phi_3(t) A \tag{13}$$
$$\frac{dR}{dt} = \phi_1(t) I_1 + \phi_2(t) I_2 + \phi_3(t) A - \mu R \tag{14}$$

Let us first define the adjoint functions $\lambda_S, \lambda_{I_1}, \lambda_{I_2}$ and λ_A associated with the state equations for S, I_1, I_2 and A, respectively. We then form the Hamiltonian, H, by corresponding state equations, and adding each of these products to the integrand of the objective functional. As a result, we obtain

$$\begin{aligned} H = {} & I_1(t) + I_2(t) + A(t) + c_{11}\phi_1(t) I_1(t) + c_{12}\phi_2(t) I_2(t) + c_{13}\phi_3(t) A(t) + c_{21}\phi_1^2(t) \\ & + c_{22}\phi_2^2(t) + c_{23}\phi_3^2(t) \\ & + \lambda_S(\mu N - \beta_1 S I_1 - \beta_2 S I_2 - \mu S) \\ & + \lambda_{I_1}(\beta_1 S I_1 + \beta_2 S I_2 - \sigma I_1 - \mu I_1 - \phi_1(t0) I_1 + \alpha I_2) \\ & + \lambda_{I_2}(\sigma I_1 - \delta I_2 - \mu I_2 - \phi_2(t) I_2 - \alpha I_2) \\ & + \lambda_A(\delta I_2 - \kappa A - \mu A - \phi_3(t) A) \end{aligned}$$

To achieve the optimal control, the adjoint functions must satisfy

$$\frac{d\lambda_S}{dt} = -\frac{\partial H}{\partial S} = -\left(\lambda_S(-\beta_1 I_1 - \beta_2 I_2 - \mu) + \lambda_{I_1}(\beta_1 I_1 + \beta_2 I_2)\right)$$
$$\frac{d\lambda_{I_1}}{dt} = -\frac{\partial H}{\partial I_1} = -\left(1 + c_{11}\phi_1(t) + \lambda_S(-\beta_1 S) + \lambda_{I_1}(\beta_1 S - \sigma - \mu - \phi_1(t)) + \lambda_{I_2}\sigma\right)$$

$$\frac{d\lambda_{I_2}}{dt} = -\frac{\partial H}{\partial I_2} = -\left(1 + c_{12}\phi_2(t) + \lambda_{I_1}\alpha + \lambda_{I_2}(-\delta - \mu - \phi_2(t) - \alpha) + \lambda_A\delta\right)$$

$$\frac{d\lambda_A}{dt} = -\frac{\partial H}{\partial A} = -\left(1 + c_{13}\phi_3(t) + \lambda_A(-\kappa - \mu - \phi_3(t))\right)$$

with transversality conditions (or final time conditions):

$$\lambda_S(T) = 0, \qquad \lambda_{I_1}(T) = 0, \qquad \lambda_{I_2}(T) = 0 \qquad \text{and} \qquad \lambda_A(T) = 0$$

The characterization of the optimal control $\phi_1^*(t)$, $\phi_2^*(t)$ and $\phi_3^*(t)$ based on the condition

$$\frac{\partial H}{\partial \phi_1} = 0, \qquad \frac{\partial H}{\partial \phi_2} = 0 \qquad \text{and} \qquad \frac{\partial H}{\partial \phi_3} = 0$$

respectively, subject to the constraint $0 \le \phi_1 \le \phi_{1\max}$, $0 \le \phi_2 \le \phi_{2\max}$ and $0 \le \phi_3 \le \phi_{3\max}$.Specifically, we have

$$\phi_1^*(t) = \max\left(0, \min(\phi_1(t), \phi_{1\max})\right)$$

$$\phi_2^*(t) = \max\left(0, \min(\phi_2(t), \phi_{2\max})\right)$$

$$\phi_3^*(t) = \max\left(0, \min(\phi_3(t), \phi_{3\max})\right)$$

where

$$\phi_1(t) = \left((\lambda_{I_1} I_1 - c_{11} I_1(t))\right)/(2c_{21})$$

$$\phi_2(t) = \left((\lambda_{I_2} I_2 - c_{12} I_2(t))\right)/(2c_{22})$$

$$\phi_3(t) = \left((\lambda_{I_3} A - c_{13} A(t))\right)/(2c_{23})$$

Given the presence of both initial conditions (for the state equations) and final time conditions (for the adjoint equations), along with the nonlinearity of most models of interest, the optimal control system must be solved numerically. For this purpose, we will employ the Forward-Backward Sweep Method to perform the numerical simulations.

Parameter	Symbol	Value	Reference
Total population	N	61,340,000	[30]
Initial susceptible population	S(0)	60,000,000	[30]
HIV asymptotic patients	$I_1(0)$	580,000	[30]
HIV symptomatic patients	$I_2(0)$	110,000	[30]
AIDS patients	$A(0)$	180,000	[30]
Patients living with HIV and know their status	$R(0)$	470,000	[30]
Natural human birth and death rate	μ	0.01470588	[7]
The transmission rate from asymptotic HIV patients to susceptible population	β_1	1×10^{-11}	Observed
The transmission rate from symptomatic HIV patients to susceptible population	β_2	1.2×10^{-8}	Observed
The progression rate of becoming I_2 for I_1	σ	0.0002	[7]
The rate at which I_2 progressing to I_1	α	0.0001	[25]
The progression rate to A for I_2	δ	0.1	[25]
The rate of the disease-related death	κ	0.1630045919	[25]

TABLE 1. Model parameters.

6. RESULTS

In this section, we use HIV/AIDS data from UNAIDS [30] for Thailand to simulate our model. Based on the data, the initial values for each state are provided in Table 1. According to the data, Thailand records approximately 9,100 new infections annually. To align with this, we set $\beta_1 = 1 \times 10^{-11}$ and $\beta_2 = 1.2 \times 10^{-8}$. The natural birth and death rate is assumed to be 0.01470588, while other parameter values are listed in Table 1.

First, we use Thailand's data to analyze the trend of infections and adjust our parameters accordingly. Thailand has been highly effective in maintaining a low number of new infections and successfully implementing an ART distribution program for HIV patients. However, the cost of care for HIV and AIDS patients remains significantly high.

In our first simulation, we assume that the additional medical costs for symptomatic and AIDS patients are considerably higher, setting $c_2 = 30$ and $c_3 = 40$, while keeping $c_1 = 10$. Figure 2 illustrates the dynamics of HIV/AIDS patients under this scenario. Figure 2(a) shows that if HIV patients consistently take ART, regardless of whether they exhibit symptoms or remain asymptomatic, the trend of HIV infections will decline over time. This leads to longer lifespans for patients, as depicted in Figure 2(d), and encourages awareness, reducing the likelihood of transmitting HIV to others.

In contrast, if patients do not take ART, the infection rate will increase, as indicated by the solid line. This growth can lead to community collapse, with a significant rise in asymptomatic HIV patients and a sharp decline in the number of individuals in the HIV awareness group, ultimately causing severe challenges for the entire nation. This scenario closely resembles the current situation in Thailand, where most HIV-infected individuals with symptoms are receiving ART. Additionally, our numerical simulation in Figure 4 aligns with the trend of new HIV infections observed in the actual data from Thailand, as shown in Figure 3.

In the first simulation, the model indicates a decreasing trend in the number of HIV patients, which corresponds to the actual data. Furthermore, the number of individuals who are aware of their HIV status and take precautions to prevent transmission is increasing. This allows us to predict future trends in new infections, as shown in Figure 5.

According to the simulation in Figure 5, Thailand could see as few as 200 new HIV infections by 2050 if current strategies are maintained. However, various factors, such as an increase in infections among young people, could potentially lead to higher numbers.

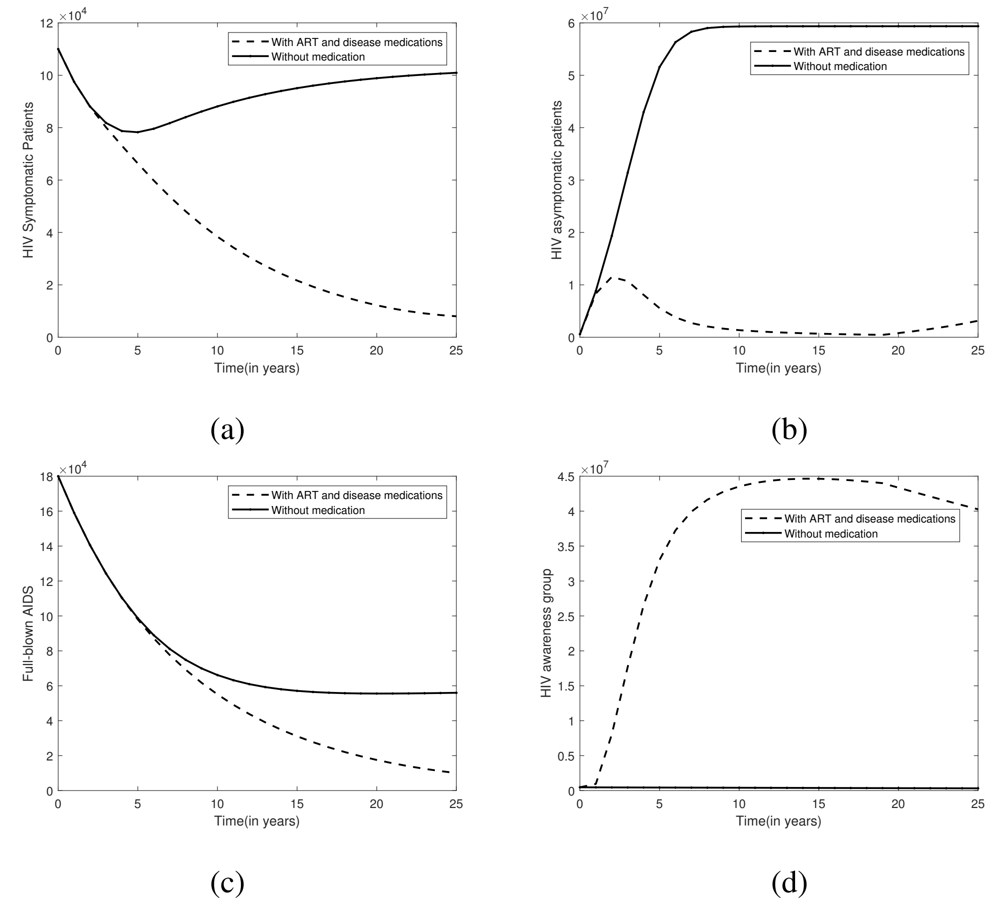


FIGURE 2. Simulation results HIV infections: (a) the HIV infection with symptomatic individuals, (b) the HIV asymptomatic patients, (c) AIDS people, (d) the HIV Awareness Group

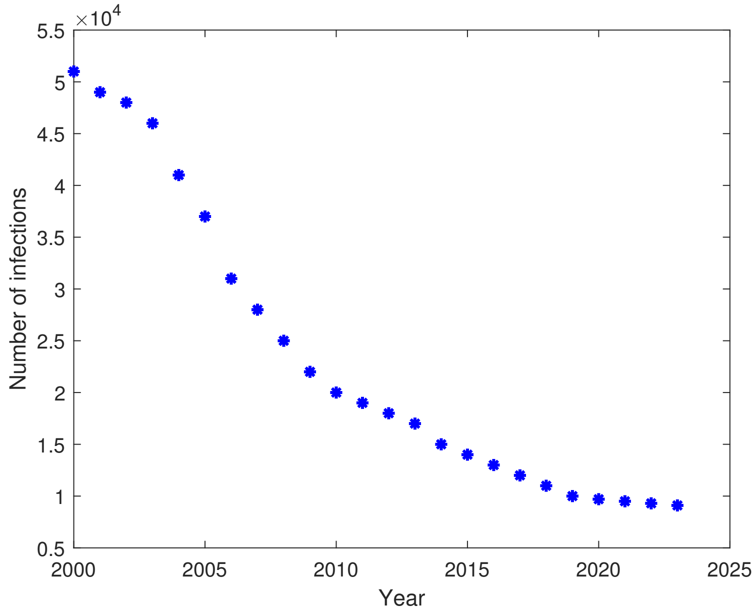


FIGURE 3. Thailand data for newly infected from years 2000 to 2023 [30].

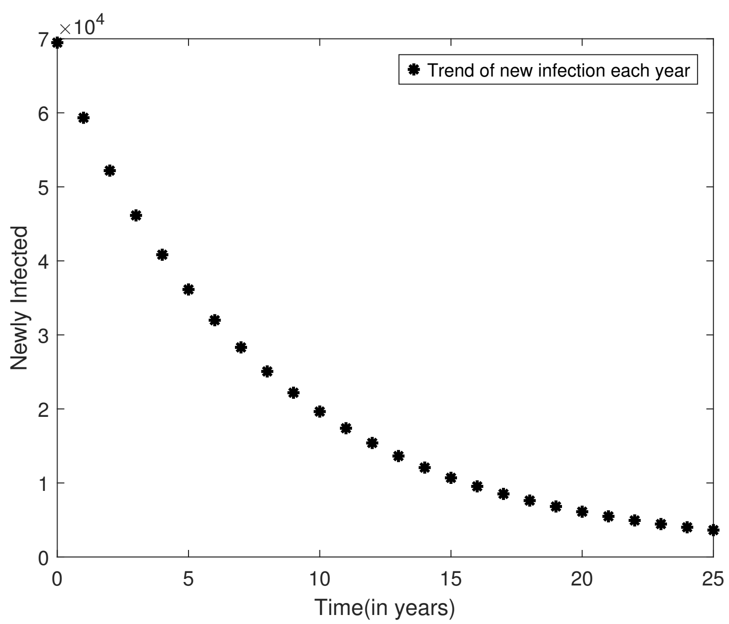


FIGURE 4. The simulated data from our model for the years 2000 to 2023.

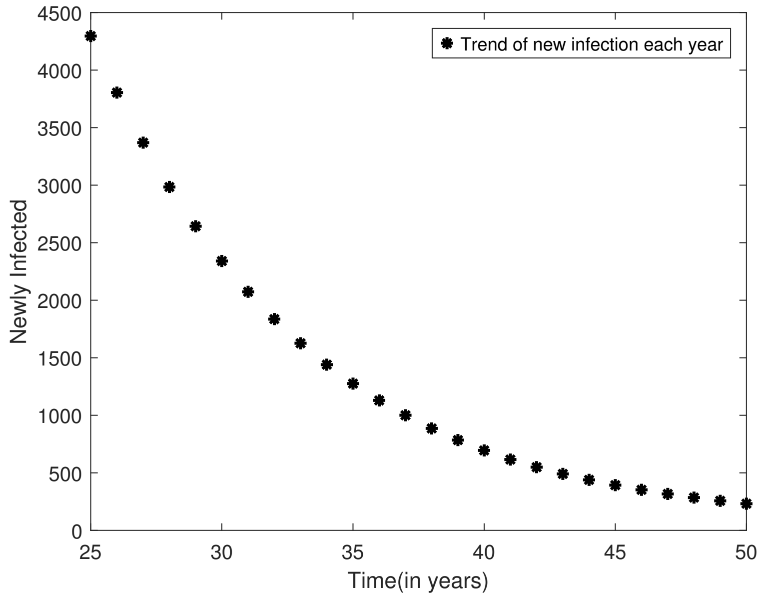
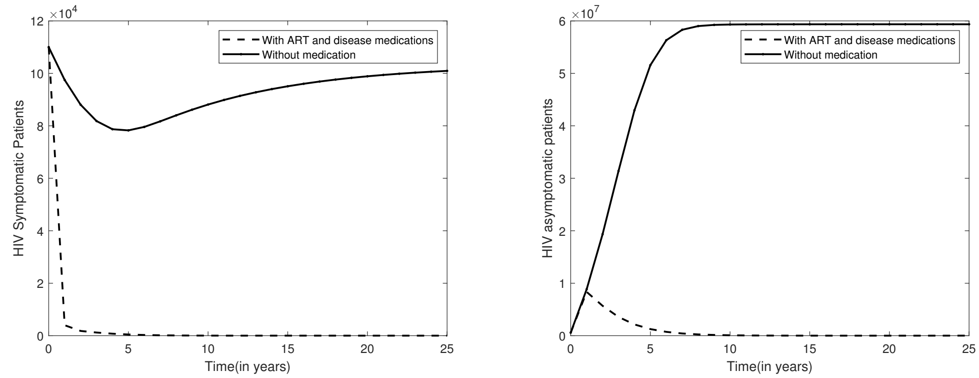


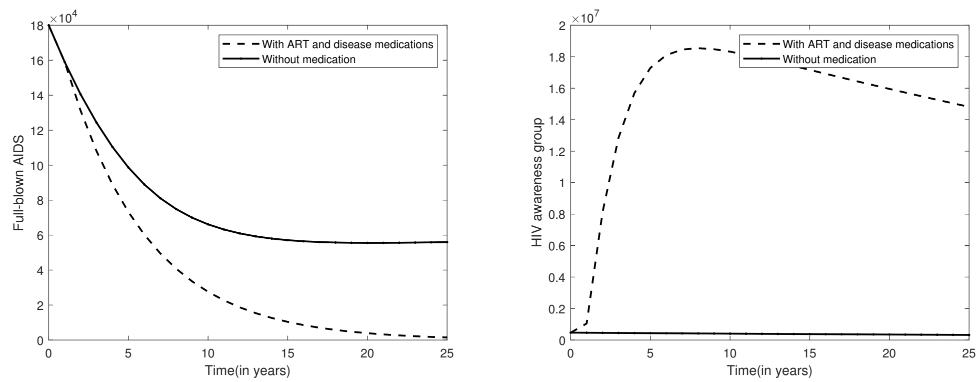
FIGURE 5. The projected number of new infections in Thailand based on our model for the years 2025 to 2050.

Next, we assume that Thailand could have improved outcomes by reducing the costs of medications for other diseases that individuals with HIV commonly experience, thereby lowering the overall cost of treating HIV. We implement this assumption by reducing medication costs as follows: $c_1 = 10$, $c_2 = 10$, $c_3 = 10$.

As illustrated in Figure 6 (dashed lines), lower treatment costs would improve patient access to medical care, potentially leading to reduced infections across all stages of the disease. In contrast, the solid lines represent the number of infections without a treatment plan. In this scenario, the infection rates could overwhelm the population over time, ultimately leading to the collapse of the community as the numbers continue to rise unchecked.



(a) (b)



(c) (d)

FIGURE 6. Simulation results of HIV infections: (a) the HIV infection with symptomatic individuals, (b) the HIV asymptomatic patients, (c) AIDS people, (d) the HIV Awareness Group

To explore the impact of reducing additional medical care costs beyond ART, we lower the treatment costs for symptomatic HIV patients and individuals with AIDS to minimal levels ($c_1 = 1$ and $c_2 = 1$), while keeping the cost of ART unchanged. The simulation results for this scenario are presented in Figure 7.

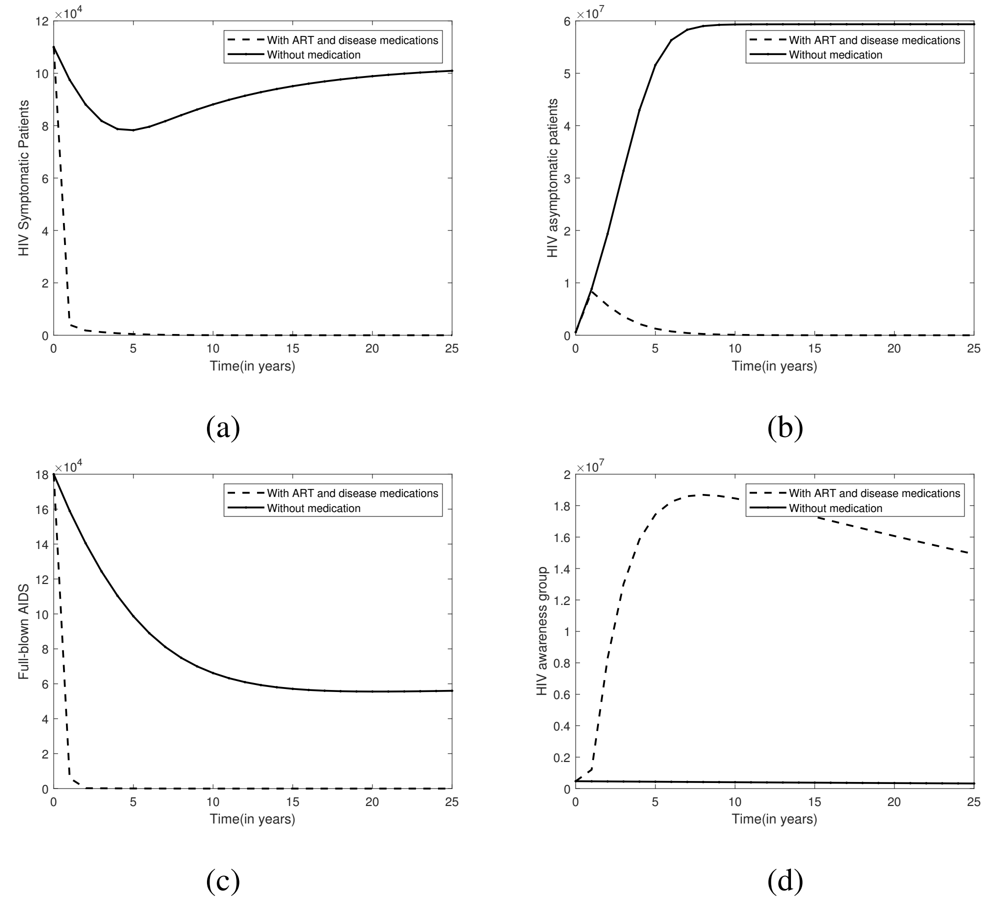


FIGURE 7. Simulation results of HIV infections: (a) the HIV infection with symptomatic individuals, (b) the HIV asymptomatic patients, (c) AIDS people, (d) the HIV Awareness Group

With lower costs for additional medical care, individuals with HIV and AIDS can access more treatments, leading to a decrease in the number of people with AIDS and an increase in the HIV awareness group, as shown in Figure 7(d). This latter group plays a crucial role in controlling the spread of the disease because they are aware of their HIV status and take measures to avoid transmitting it to others. Therefore, a higher number of patients in this group contributes significantly to disease control.

For Thailand, it is essential to provide basic healthcare that includes additional treatments for HIV patients, such as symptom-based care plans or reduced costs for doctor visits. By doing so, individuals can live normal lives while remaining aware of their condition and preventing the transmission of HIV to others.

7. Conclusion

We have developed a mathematical model for HIV infection based on data from Thailand, using both theoretical and numerical approaches. The model captures the dynamics of the disease through a system of five nonlinear differential equations.

Additionally, we analyzed the epidemic and endemic dynamics of the model, focusing on the local and global stability properties determined by the basic reproductive number (R_0). Specifically, when $R_0 < 1$, the disease dies out, and the disease-free equilibrium is stable.

When $R_0 > 1$, the disease persists, and the disease-free equilibrium becomes unstable. We conducted numerical simulations using two sets of control costs. The first set represents the current situation in Thailand, and the results indicate that the number of new infections predicted by our model closely aligns with actual data.

Building on this, we extended our simulations to predict the number of new infections from 2025 to 2050. The results suggest a significant decline in yearly new infections, with Thailand projected to have only about 200 new HIV cases by 2050. This trend is expected to continue as long as no major factors, such as increased infections among young people who lack awareness of prevention methods, trigger a resurgence.

Nevertheless, both our simulations and the real-world data demonstrate a positive outlook for Thailand. The country is on track to maintain the current infection rate and could potentially reduce it to approximately 100 new cases annually by 2050.

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Conflict of Interests

The author declares that there is no conflict of interests.

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