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ALLEE AND FEAR EFFECTS IN PREY-PREDATOR DYNAMICS: STABILITY ANALYSIS AND OPTIMAL HARVESTING

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Abstract. This study incorporates the Allee effect and the fear effect into a predator-prey model. We investigate how these factors, along with fishing pressure, influence the population dynamics of both predator and prey. The analysis begins by establishing the positivity and boundedness of solutions, ensuring the populations remain within realistic ranges. Stability analysis of the interior equilibrium point, where both populations coexist, reveals the system's resilience to disturbances. Furthermore, we explore optimal fishing strategies that balance maximizing long-term profit with maintaining sustainable populations. Finally, numerical simulations conducted validate the theoretical predictions.

Keywords: predator-prey model; stability analysis; Allee effect; fear effect; optimal harvesting.

2020 AMS Subject Classification: 91B05, 91A06, 91B02, 91B50.

1. Introduction

The groundbreaking work of Lotka [1] and Volterra [2] in the 20th century provided fresh insights on biological species. Numerous researchers have accomplished a great deal in this field [3]-[5]. These days, a lot of study is done on the dynamics of interacting prey-predator

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models in order to understand the long-term behavior of a population. A variety of nonlinear ODE models are examined, and the prey-predator interaction is examined.

Recently, harvesting has drawn the attention of ecologists and economists alike. Thus, harvesting models are becoming increasingly important to understand the dynamics of the predator-prey relationship. The bioeconomic harvesting of a prey–predator fishery, in which both species are afflicted by some toxicants emitted by some other species, is covered by Tapasi Das et al. in 2009 [6]. Here, both species are harvested, and the standard catch-per-unit-effort hypothesis is applied. An analysis is conducted on the exploited system's dynamic behavior. The potential for a bionomic equilibrium to exist is taken into account. The maximal principle of Pontryagin is used to study the best harvesting strategy. A. Rojas-Palma and E. González-Olivares tackled the problem of determining the best harvest strategy in an open-access fishery in 2011 when non-selective harvesting is applied to both predator and prey species [7].

Allee found in 1931 that although the cluster's living condition supports population development, its great density will impede it and possibly cause it to go extinct owing to resource competition. The Allee effect is the process that states there must be a unique optimal density for each population's development and reproduction [8]. Numerous researchers are also examining the predator-prey paradigm with Allee effect in prey growth [9]-[11]. The stability of a predator-prey model incorporating the Allee effect on prey populations was the main focus of the authors' 2016 study, which also considered the effects of human predation on both predators and prey. The study used numerical modeling to analyze each population's evolutionary process and successfully demonstrated the existence and stability of a positive equilibrium point [12]. A prey-predator model with the Allee effect and Holling type-I functional response was established by Yong Ye et al. in 2019, and its dynamical behaviors have been thoroughly examined. There is a qualitative discussion of the model's existence, boundedness, and stability [13].

As a predator that satisfies the conditions of the Allee effect, we find the Atlantic bluefin tuna (Thunnus thynnus) is a species of tuna in the family Scombridae. It is variously known as the northern bluefin tuna (mainly when including Pacific bluefin as a subspecies), giant bluefin tuna [for individuals exceeding 150 kg (330 lb)], and formerly as the tunny. The Thunnus thynnus is distributed throughout the Atlantic and Pacific Oceans in subtropical and temperate waters. In the western Atlantic Ocean, it is found from Labrador, Canada, to northern Brazil, including the Gulf of Mexico. In the eastern Atlantic Ocean, it is found from Norway to the Canary Islands. In the western Pacific Ocean, it is distributed from Japan to the Philippines. In the eastern Pacific Ocean, it is distributed from the southern coast of Alaska, USA to Baja California, Mexico (see Figure 1).

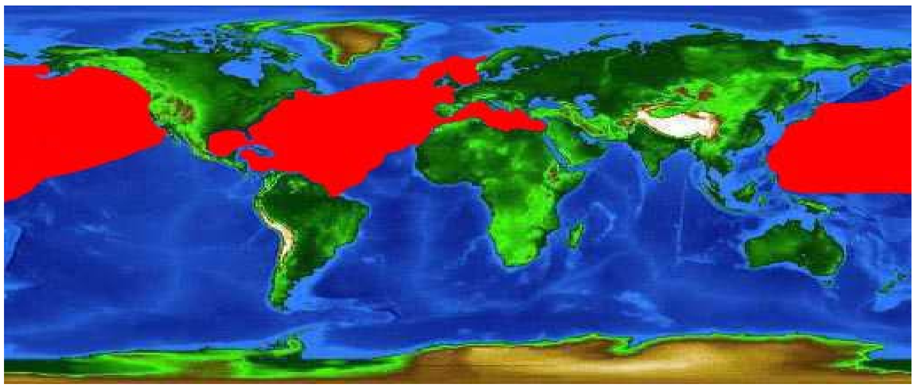

FIGURE 1. World distribution map for the Thunnus thynnus

Source: https://www.floridamuseum.ufl.edu/discover-fish/species-profiles/thunnus-thynnus/

Predators have a significant impact on prey reproduction in an ecosystem, according to experimental research. In a study on song sparrows, for instance, Zanette et al. found that fear-induced reactions to predators resulted in a 40% drop in offspring during the breeding season, underscoring the negative effect on prey reproductive success [14]. A mathematical model that included predator fear in a prey-predator system was presented by Wang et al. in 2016. It was discovered that integrating fear could stabilize the relationship between prey and predator, thereby preventing oscillatory behavior and fostering harmonious coexistence between the two species [15]. In [16], the authors explore how fear of predators impacts prey and intermediate predator behavior in a three-species model. They show fear can stabilize the system, with top predators having diverse food sources playing a key role. This highlights the importance of fear dynamics in shaping food web stability. In [17], S. Mandal et. al. study a predator-prey model with fear, refuge use, cooperative hunting, and environmental variability. They analyze equilibria and stability, finding that predator food sources and cooperation can enhance their sustainability. They also show how fear and refuge use in prey can impact predator populations. This research incorporates noise to understand how random fluctuations affect predator-prey dynamics. In [18], The authors investigate a predator-prey model where fear reduces prey birth rate and increases competition. Prey also uses refuges to avoid predators. The model exhibits complex dynamics with multiple stable states depending on fear levels and food availability. They then introduce environmental noise to the system and show it can cause transitions between these states. This highlights the interplay between fear, refugees, and environmental variability in shaping predator-prey interactions.

As a predator, Thunnus thynnus can induce a fear effect on Engraulis encrasicolus, which is a forage fish somewhat related to the herring. It is a type of anchovy; anchovies are placed in the family Engraulidae. It lives off the coasts of Europe and Africa, including in the Mediterranean Sea, the Black Sea, and the Sea of Azov. It is fished by humans throughout much of its range (see Figure 2)

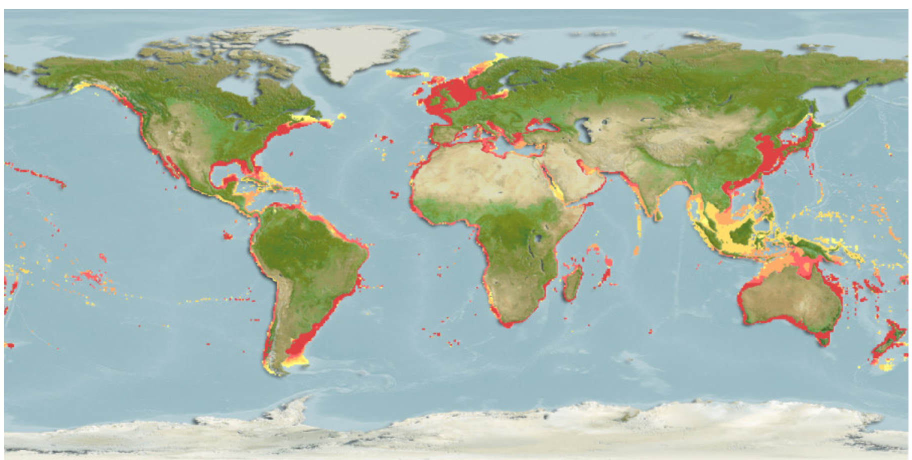

FIGURE 2. World distribution map for the Engraulis encrasicolus

Source: https://www.aquamaps.org/receive.php?type_of_map=regular&map=cached

This work takes a novel approach to marine conservation by focusing on predicting species behavior to safeguard biodiversity. It utilizes a dynamic model to simulate predator-prey interactions, incorporating the critical Allee effect (where population decline hinders reproduction) and the Fear effect (where prey behavior alters predator success). By analyzing the system's stability, the study aims to identify optimal fishing policies. The innovation lies in recognizing how the Allee and Fear effects influence the ecological balance. By modeling these factors, we can create sustainable fishing regulations that consider both predator and prey populations, ultimately promoting biodiversity conservation. This work enhances our understanding of marine ecosystem dynamics and emphasizes the importance of considering species interactions in resource management plans.

This article takes a step-by-step approach to explore the dynamics between predator and prey populations in a marine ecosystem. It starts by introducing a biological model in Section 2 that simulates this interaction. Following that (Section 3), the model's properties are analyzed to ensure the results are realistic (the boundedness and positivity of the system,). Sections 4 and 5 delve deeper into the model's behavior. Section 4 examines the points where predator and prey populations stabilize, while Section 5 explores how to develop the best fishing policy to maximize sustainable resource use without disrupting this balance. Finally, Section 6 uses computer simulations to test the model's predictions, and the article concludes by summarizing the key findings from this entire analysis.

2. Description of the Model

When modeling marine species, predator-prey models are essential. They aid in our comprehension of the intricate relationships that exist between populations of prey and predators as well as how these relationships affect population dynamics. However, there is a phenomenon with major biological significance known as the Allee effect. The Allee effect happens when a population's density drops below a particular threshold, which lowers population viability. This may cause the population to remain extremely low or to be on the verge of extinction. The Allee effect is especially concerning for endangered species because these populations are more susceptible to this detrimental dynamic. In light of this impact on the prey, the growth of the latter is expressed as

$$\dot{T}(t) = T\left(\frac{r_2 T}{a+y} - d_2 - k_2 T\right)$$

In the natural world, animals' fear of predators affects their use of habitat, feeding habits, ability to reproduce, and physiological changes. Prey species change how they behave to stay away from high-risk areas, which can affect how populations are distributed geographically. Fear can also cause one to eat less, which can have an impact on survival, reproduction, and growth. Additionally, it can lessen the success of reproduction and suppress courtship behaviors. Stress hormone release is one example of a physiological reaction that can affect an animal's health and well-being. It is essential to comprehend these intricate effects in order to manage and conserve animal populations. Our goal is to investigate the system's dynamic behavior when the prey population experiences the fear effect. Taking this into account, we can represent the prey's growth as

$$\dot{E}(t) = E\left(\frac{r_1 E}{1+fT} - d_1 - k_1 E\right)$$

We additionally take into account how fishing affects the prey and predator species. The balance of ecosystems and population dynamics can be significantly impacted by fishing. We can evaluate sustainable fishing methods and make well-informed decisions to preserve the marine ecosystem's general health and population balance by taking this effect into account. Each species' captured quantity is written as "qEB^2", where q represents the catchability rate, E represents the fishing effort, and B represents the species biomass.
Consequently, we can obtain the following shema by utilizing all of the previous data (see 3)

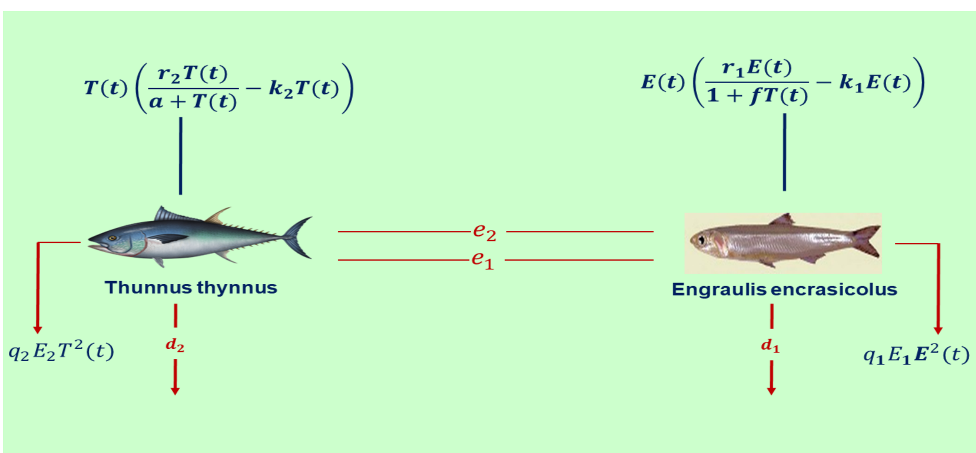

FIGURE 3. Interaction resulting between Thunnus thynnus and Engraulis encrasicolus populations

To summarize, the representation of our model is as follows

(1) $$\begin{cases} \dot{E}(t) = E\left(\frac{r_1 E}{1+fT} - d_1 - k_1 E\right) - e_1 ET - q_1 E_1 E^2 \\ \dot{T}(t) = T\left(\frac{r_2 T}{a+T} - d_2 - k_2 T\right) + e_2 ET - q_2 E_2 T^2. \end{cases}$$

Where T represents the predator's biomass, and E represents the prey's biomass. Below, you'll find a table that summarizes the parameters along with their corresponding explanations (see 1).

Parameter	Signification
r_1	The maximum specific growth rate of the prey population;
r_2	The maximum specific growth rate of the predator population;
k_1	Carrying capacities for the prey population
k_2	Carrying capacities for the predator population
e_1	Rates of mortality resulting from the effects of predation.
e_2	Reproductive rates of predators determined by encountered prey.
d_1	Prey death rate
d_2	Predator death rate
a	Strength of Allee effect
f	Fear induced by predator population
q_i	Catchability rate
E_i	Fishing effort

TABLE 1. The signification of bioeconomic parameters

Let $x = E$ and $y = T$, the system 1 becomes as follow

$$\begin{cases} \dot{x}(t) = x\left(\frac{r_1 x}{1+fy} - d_1 - k_1 x\right) - e_1 xy - q_1 E_1 x^2 \\ \dot{y}(t) = y\left(\frac{r_2 y}{a+y} - d_2 - k_2 y\right) + e_2 xy - q_2 E_2 y^2. \end{cases} \tag{2}$$

With $x(0) = x_0$, $y(0) = y_0$, and $x_0, y_0 > 0$

3. Existence and Uniqueness, Positivity and Boundedness of Solution

This section will deal with the existence and uniqueness of solutions, as well as their positivity and boundedness.

3.1. Existence and uniqueness of solution.

The system 2 can be represented in the following form

$$\dot{v}(t) = (f(v(t)), g(v(t)))$$

with $v=(x,y)$ and

$$f = x\left(\frac{r_1 x}{1+fy} - d_1 - k_1 x\right) - e_1 xy - q1E_1 x^2$$
$$g = y\left(\frac{r_2 y}{a+y} - d_2 - k_2 y\right) + e_2 xy - q2E_2 y^2$$

The functions f and g are continuous, and their partial derivatives are continuous and bounded. Consequently, the conditions of the Cauchy-Lipschitz theorem are satisfied. According to the fundamental theorem of functional differential equations, system 2 has a unique solution.

3.2. Positivity of Solutions. Consider the following theorem regarding the positivity of the system 2

Theorem 1. *The set $\{(x,y)\in\mathbb{R}_+^2 : x\geq 0$ and $y\geq 0\}$ is invariant under the system.*

Proof.

$$\begin{cases} x(t) = x(0)\exp\int_0^t \frac{r_1 x}{1+fy} - d_1 - k_1 x - e_1 y - q1E_1 x\, dx > 0 \\ y(t) = y(0)\exp\int_0^t \frac{r_2 y}{a+y} - d_2 - k_2 y + e_2 x - q2E_2 y\, dy > 0 \end{cases} \tag{3}$$

So the set $\{(x,y)\in\mathbb{R}^2 : x\geq 0 \quad \text{and} \quad y\geq 0\}$ is positively invariant. □

3.3. Boundedness of solutions.

Theorem 2. *The solutions of system 2 are bounded.*

Proof. We consider the following inequality

$$\begin{aligned} \dot{x}(t) &= x\left(\frac{r_1 x}{1+fy} - d_1 - k_1 x\right) - e_1 xy - q_1 E_1 x^2 \\ &\leq r_1 x - d_1 x - k_1 x^2 - q_1 E_1 x^2 \\ &\leq x(r_1 - d_1) - x^2 (k_1 + q_1 E_1) \end{aligned}$$

By integrating, we have

$$x(t) \leq \frac{r_1 - d_1}{k_1 + q_1 E_1} = M_1$$

For y, we have

$$\begin{aligned}\dot{y}(t) &= y\left(\frac{r_2 y}{a+y} - d_2 - k_2 y\right) + e_2 xy - q_2 E_2 y^2 \\ &\leq y\left(r_2 - d_2 - k_2 y - q_2 E_2 y\right) + e_2 xy \\ &\leq y\left(r_2 - k_2 y\right) + e_2 M_1 y \\ &\leq y\left(r_2 + e_2 M_1 - k_2 y\right)\end{aligned}$$

By integrating, we have

$$y(t) \leq \frac{r_2 - e_2 M_1}{k_2} = M_2$$

□

4. Equilibrium Point and Stability Analysis

In this section, we will focus on calculating the positive equilibrium point and conducting stability analysis.

4.1. Equilibrium Points. We solve the following two equations in order to find the positive equilibrium point:

$$\dot{x}(t) = 0 \quad \text{and} \quad \dot{y}(t) = 0$$

The system 2 has a unique strictly positive equilibrium point $E^*(x^*, y^*)$ where

$$x^* = \frac{A_1 y^{*2} + A_2 y^* + d_2}{e_2 y^* + a e_2}$$

and y^* is given from the cubic equation

$$G_3 y^3 + G_2 y^2 + G_1 y + G_0 = 0 \tag{4}$$

Where

$$A_1 = k_2 + a q_2 E_2$$

$$A_2 = d_2 + a k_2 + a q_2 E_2 - r_2$$

$$G_0 = a e_2 d_1 + k_1 d_2 + q_1 E_1 d_2 - r_1 d_2$$

$$G_1 = e_2 d_1 + a e_2 d_1 f + k_1 f d_2 + a e_1 e_2 + q_1 E_1 A_2$$

$$G_2 = e_2d_1f + k_1A_1 + k_1A_2f + e_1e_2 + ae_1e_2f + q_1E_1A_1 + q_1fE_1A_2 - r_1A_1$$

$$G_3 = k_1fA_1 + e_1e_2f + q_1fE_1A_1$$

Now let's look into the conditions in which we can establish that Equation 4 has at least one positive solution.

According to [16], we have the following theorem

Theorem 3. *Let* $\Delta = -27G_3^2G_0^2 + 18G_3G_2G_1G_0 - 4G_3G_1^3 - 4G_2^3G_0 + G_1^2G_2^2$ *denote the discriminant of 4. And let* $B = (1+af)(k_1k_2 + e_1e_2k_2q_1E_1) + (k_1+q_1E_1)(aq_2E_2 + afq_2E_2 + d_2f) + e_2d_1f - r_2f(k_1+q_1E_1) - r_1(k_2 + aq_2E_2)$ *Now, if* $\Delta > 0$, *and*

- *if* $B > 0, d_2(1+f) - r_2 > 0, k_1 + q_1E_1 - r_1 > 0, ae_1e_2 - r_1(d_1+r_2) > 0$ *and* $G_0 > 0$ *or* $B < 0$, $d_2(1+f) - r_2 > 0$, $k_1 + q_1E_1 - r_1 > 0$, $ae_1e_2 - r_1(d_1+r_2) > 0$ *and* $G_0 > 0$ *or* $B < 0$, $d_2(1+f) - r_2 < 0$,$k_1 + q_1E_1 - r_1 < 0$, $ae_1e_2 - r_1(d_1+r_2) < 0$ *and* $G_0 > 0$, *then 4 have a single positive root.*
- *if* $B > 0$, $d_2(1+f) - r_2 < 0$, $k_1 + q_1E_1 - r_1 < 0$, $ae_1e_2 - r_1(d_1+r_2) < 0$ *and* $G_0 < 0$ *or* $B < 0, d_2(1+f) - r_2 < 0, k_1 + q_1E_1 - r_1 < 0, ae_1e_2 - r_1(d_1+r_2) < 0$ *and* $G_0 > 0$, *then 4 have two positive roots.*
- *if* $B > 0$, $d_2(1+f) - r_2 < 0$, $k_1 + q_1E_1 - r_1 < 0$, $ae_1e_2 - r_1(d_1+r_2) < 0$ *and* $G_0 > 0$, *then 4 have three positive roots.*

4.2. Stability analysis.

It is necessary to first define the Jacobian matrix at the positive equilibrium point and determine its characteristic equation in order to examine the stability of the system 2..

Let's define the Jacobian matrix at the positive equilibrium point (x^*, y^*) as follows:

$$J^* = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

Where

$$J_{11} = \frac{2r_1x^*}{1+fy^*} - 2k_1x^* - d_1 - e_1y^* - 2q_1E_1x^*$$

$$J_{12} = \frac{-r_1fx^{*2}}{(1+fy^*)^2} - e_1x^*$$

$$J_{21} = e_2 y^*$$

$$J_{22} = \frac{2ar_2y^* + r_2y^{*2}}{(a+y^*)^2} - 2k_2y^* + e_2x^* - 2q_2E_2y^* - d_2$$

The characteristic equation of the Jacobian matrix J^* will be expressed as follows:

$$\lambda^2 + A\lambda + B = 0 \tag{5}$$

Where

$$A = \frac{-2r_1x^*}{1+fy^*} + 2k_1x^* + d_1 + e_1y^* + 2q_1E_1x^* - \frac{2ar_2y^* + r_2y^{*2}}{(a+y^*)^2} + 2k_2y^* - e_2x^* + 2q_2E_2y^* + d_2$$

$$B = \left(\frac{2r_1x^*}{1+fy^*} - 2k_1x^* - d_1 - e_1y^* - 2q_1E_1x^*\right)\left(\frac{2ar_2y^* + r_2y^{*2}}{(a+y^*)^2} - 2k_2y^* + e_2x^* - 2q_2E_2y^* - d_2\right)$$
$$- e_2y^*\left(\frac{-r_1fx^{*2}}{(1+fy^*)^2} - e_1x^*\right)$$

According to the Routh-Hurwitz stability criterion, if $A > 0$ and $B > 0$, then the positive equilibrium point (x^*, y^*) is locally asymptotically stable

5. Optimal Harvesting Policy

To ascertain an optimal harvesting policy, to consequently maximize the total return produced by the exploitation of the resources, we consider a performance measure or objective function defined by the present value J of a continuous time-stream of revenues

$$J = \int_0^\infty e^{-\delta t}\pi(x, y, E_1, E_2)\,dt$$

In this formulation, we consider π as the net revenue, which represents the earnings obtained after subtracting costs, at any time t in the future. It is expressed as follows

$$\pi(x, y, E_1, E_2) = (p_1q_1x^2 - c_1)E_1 + (p_2q_2y^2 - c_2)E_2$$

On the other hand, δ represents the instantaneous annual rate of discount. This rate is used to convert the future value of revenues into an equivalent present value.
Our optimal control problem is.

$$\begin{cases} \text{Maximize} J = \int_0^\infty e^{-\delta t}[(p_1q_1x^2 - c_1)E_1 + (p_2q_2y^2 - c_2)E_2]\,dt \\ \text{Subject to:} \\ \quad \begin{cases} \dot{x}(t) = x\left(\frac{r_1x}{1+fy} - d_1 - k_1x\right) - e_1xy - q_1E_1x^2 \\ \dot{y}(t) = y\left(\frac{r_2y}{a+y} - d_2 - k_2y\right) + e_2xy - q_2E_2y^2. \\ 0 \le E_i \le E_i^{max} \end{cases} \end{cases}$$

Where the control variable $E_i(t)$ belongs to the control set $V = [0, E_i^{max}]$.

This is obviously an infinite horizon linear control problem. Consequently, a mix of singular controls and bang-bang controls will be used to solve the problem. Only the singular control for the optimization problem will be found in this scenario.

Theorem 4. *The optimal fishing effort to maximize the profit for the fisherman is written as follows:*

$$E_1^* = \frac{\delta(p_1q_1x^2 - c_1)}{2c_1q_1x} - \frac{r_1(p_1q_1x^2 - c_1)}{c_1q_1(1+fy)} + \frac{k_1(p_1q_1x^2 - c_1)}{c_1q_1} + \frac{d_1(p_1q_1x^2 - c_1)}{2c_1q_1x} + \frac{e_1y(p_1q_1x^2 - c_1)}{2c_1q_1x} - \frac{x(p_2q_2y^2 - c_2)}{2c_1q_2y}e_2$$

$$E_2^* = \frac{\delta(p_2q_2y^2 - c_2)}{2c_2q_2y} - \frac{(2ar_2 + r_2y)(p_2q_2y^2 - c_2)}{2c_2q_2(a+y)^2} + \frac{d_2(p_2q_2y^2 - c_2)}{2c_2q_2y} + \frac{k_2(p_2q_2y^2 - c_2)}{c_2q_2} - \frac{e_2x(p_2q_2y^2 - c_2)}{2c_2q_2y} - \frac{r_1fy(p_1q_1x^2 - c_1)}{2c_2q_1} + \frac{e_1y(p_1q_1x^2 - c_1)}{2c_2q_1x}$$

Proof. The problem's Hamiltonian is provided by

$$H = e^{-\delta(t)}[(p_1q_1x^2 - c_1)E_1 + (p_2q_2y^2 - c_2)E_2] + Q_1\left[x\left(\frac{r_1x}{1+fy} - d_1 - k_1x\right) - e_1xy - q1E_1x^2\right] + Q_2\left[y\left(\frac{r_2y}{a+y} - d_2 - k_2y\right) + e_2xy - q2E_2y^2\right]$$

We set

$$\frac{\partial H}{\partial E_1} = 0 \quad \text{and} \quad \frac{\partial H}{\partial E_2} = 0$$

As the necessary conditions for the control variables $E1$ and E_2 to be optimal. Then we get

$$Q_1 = \frac{e^{-\delta t}(p_1q_1x^2 - c_1)}{q_1x^2}$$

$$Q_2 = \frac{e^{-\delta t}(p_2q_2y^2 - c_2)}{q_2y^2}$$

Using the Pontryagin's maximum principle, we have:

$$\dot{Q}_1 = -\frac{\partial H}{\partial x} = -2p_1q_1E_1xe^{-\delta t} - Q_1\left[\frac{2r_1x}{1+fy} - 2k_1x - d_1 - e_1y - 2q_1E_1x\right] - Q_2e_2y$$
$$\dot{Q}_2 = -\frac{\partial H}{\partial y} = -2p_2q_2E_2ye^{-\delta t} - Q_2\left[\frac{r_2y}{a+y} - d_2 - 2k_2y + \frac{ar_2y}{(a+y)^2} + e_2x - 2q_2E_2y\right]$$
$$- Q_1\left[\frac{r_1fx^2}{(1+fy)^2} - e_1x\right]$$

by replacing Q_1 and Q_2 with their expressions we get

$$\dot{Q}_1 = -2p_1q_1E_1xe^{-\delta t} - \frac{e^{-\delta t}(p_1q_1x^2 - c_1)}{q_1x^2}\left[\frac{2r_1x}{1+fy} - 2k_1x - d_1 - e_1y - 2q_1E_1x\right]$$
$$- \frac{e^{-\delta t}(p_2q_2y^2 - c_2)e_2y}{q_2y^2}$$
$$\dot{Q}_2 = -2p_2q_2E_2ye^{-\delta t} - \frac{e^{-\delta t}(p_1q_1x^2 - c_1)}{q_1x^2}\left[\frac{r_1fx^2}{(1+fy)^2} - e_1x\right]$$
$$- \frac{e^{-\delta t}(p_2q_2y^2 - c_2)}{q_2y^2}\left[\frac{r_2y}{a+y} - d_2 - 2k_2y + \frac{ar_2y}{(a+y)^2} + e_2x - 2q_2E_2y\right]$$

After integration of the previous equations

$$Q_1 = \frac{e^{-\delta t}}{\delta}\left[2p_1q_1E_1x + \frac{p_1q_1x^2 - c_1}{q_1x^2}\left[\frac{2r_1x}{1+fy} - 2k_1x - d_1 - e_1y - 2q_1E_1x\right]\right.$$
$$\left. + \frac{(p_2q_2y^2 - c_2)e_2y}{q_2y^2}\right]$$
$$Q_2 = \frac{e^{-\delta t}}{\delta}\left[2p_2q_2E_2y + \frac{(p_1q_1x^2 - c_1)}{q_1x^2}\left[\frac{r_1fx^2}{(1+fy)^2} - e_1x\right]\right.$$
$$\left. + \frac{(p_2q_2y^2 - c_2)}{q_2y^2}\left[\frac{r_2y}{a+y} - d_2 - 2k_2y + \frac{ar_2y}{(a+y)^2} + e_2x - 2q_2E_2y\right]\right]$$

Consequently, we find

$$E_1^* = \frac{\delta(p_1q_1x^2 - c_1)}{2c_1q_1x} - \frac{r_1(p_1q_1x^2 - c_1)}{c_1q_1(1+fy)} + \frac{k_1(p_1q_1x^2 - c_1)}{c_1q_1} + \frac{d_1(p_1q_1x^2 - c_1)}{2c_1q_1x}$$
$$+ \frac{e_1y(p_1q_1x^2 - c_1)}{2c_1q_1x} - \frac{x(p_2q_2y^2 - c_2)}{2c_1q_2y}e_2$$
$$E_2^* = \frac{\delta(p_2q_2y^2 - c_2)}{2c_2q_2y} - \frac{(2ar_2 + r_2y)(p_2q_2y^2 - c_2)}{2c_2q_2(a+y)^2} + \frac{d_2(p_2q_2y^2 - c_2)}{2c_2q_2y} + \frac{k_2(p_2q_2y^2 - c_2)}{c_2q_2}$$

$$-\frac{e_2x(p_2q_2y^2-c_2)}{2c_2q_2y}-\frac{r_1fy(p_1q_1x^2-c_1)}{2c_2q_1}+\frac{e_1y(p_1q_1x^2-c_1)}{2c_2q_1x}$$

□

6. DISCUSSION

To substantiate the theoretical insights discussed in the preceding sections, this section will employ numerical simulations. We set the parameters as follows: the growth rate of the prey, r_1, is 0.07, the death rate of prey due to other causes, d_1, is 0.001, the mortality rate resulting from predation, e_1, is 0.01, the carrying capacity of the habitat for prey, k_1, is 0.004, the catchability rate for prey, q_1, is 0.001, and the fishing effort for prey, E_1, is 6. Additionally, we set the parameter f to 0.01 and the Allee effect coefficient a to 0.1 for the prey species. For the predator species, we set the growth rate, r_2, to 0.5, the death rate, d_2, to 0.8, the reproductive rate determined by encountered prey, e_2, to 0.013, the carrying capacity for predators, k_2, to 0.001, the catchability rate for predators, q_2, to 0.001, and the fishing effort for predators, E_2, to 9. These simulations will aid in visualizing and comprehending the phenomena under study.

The system reaches a positive equilibrium point $(x^*,y^*) = (62.36,44.17)$ when the parameters are set to the values listed in the preceding table. The sizes of the predator and prey species are represented by the values of x and y, respectively. The fact that there are no significant fluctuations or extinctions indicates that both populations can coexist and maintain their respective sizes. This is known as the positive equilibrium. Starting at the initial point (100, 100), we see that our system's solution converges to the interior equilibrium point. This suggests that the system is asymptotically stable locally. The equilibrium point's stability suggests that the interactions between the species—which are impacted by the specified parameter values—have stabilized. This shows that maintaining the ecosystem's health and stability and controlling population dynamics are important tasks for the Allee and Fear Effects. It can be seen that the biomass of these marine populations converges towards the values of the interior equilibrium point, this can be biologically interpreted that the interior point of equilibrium ensures the existence of predefined species(see Figures 4 and 5).

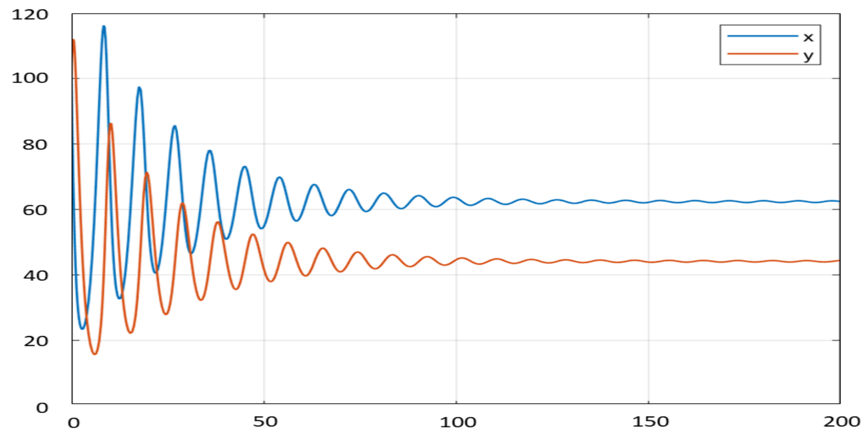

FIGURE 4. Biomass of the prey and predator

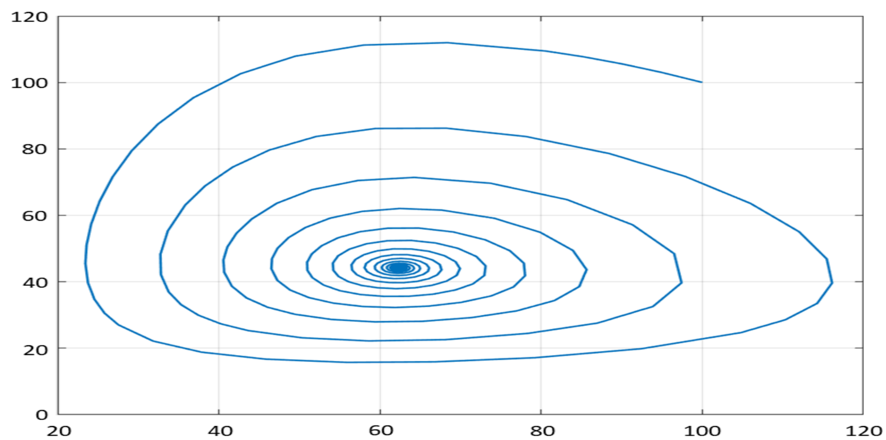

FIGURE 5. Phase portrait of the predator-prey model

The particular parameter values were examined in the analysis of the system in question, and it was found that the Routh-Hurwitz method's demands were reached. This indicates that $A = 3.993 > 0$ and $B = 0.5171 > 0$ satisfy the necessary conditions. According to control theory, verifying these conditions is essential to guarantee the system's stability. The system's resilience under the specified parameterization is highlighted by its adherence to the Routh-Hurwitz criterion, increasing confidence in its dependability and performance. This thorough evaluation is an essential first step toward ensuring the efficacy and dependability of the system under analysis.

Furthermore, we have determined that $E_1 = 8.1495$ and $E_2 = 14.1675$ represent the optimal effort levels. These numbers represent the optimal effort to maximize total revenue and minimize total costs, which leads to the achievement of the maximum profit that can be made. To maximize profit, it is imperative to strategically use effort at these exact levels to optimize financial outcomes and guarantee the best possible cost-income balance.

7. Conclusion

This study examined the combined influence of the Allee effect and fear effect on a predator-prey system under harvesting pressure. By formulating and analyzing a dynamic bioeconomic model, we demonstrated the existence, positivity, and boundedness of solutions. The local stability of the coexistence equilibrium was established through the Routh-Hurwitz criterion, confirming the potential for stable coexistence between the predator and prey populations.

An optimal harvesting strategy was derived using Pontryagin's Maximum Principle. Analytical expressions for the optimal fishing efforts were obtained, aiming to maximize long-term profit while ensuring ecological sustainability. Numerical simulations supported the theoretical findings, showing convergence toward a stable interior equilibrium and validating the effectiveness of the proposed harvesting policy.

Beyond providing insights into the complex interactions between biological and economic dynamics, this work highlights the critical roles of behavioral ecology (fear effect) and demographic thresholds (Allee effect) in shaping sustainable resource management strategies. Future research may explore the effects of environmental variability, time delays, or spatial heterogeneity to further enhance the model's applicability to real-world marine ecosystems.

Conflict of Interests

The authors declare that there is no conflict of interests.

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