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A COMPARTMENTAL MODELING OF CRIME INFLUENCED BY DRUG CONSUMPTION

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Abstract. This article proposes a discrete compartmental model to analyze the influence of drug consumption on crime dynamics and to assess the effectiveness of different intervention strategies. The population is divided into five categories: susceptible individuals, drug users, active criminals, prisoners, and rehabilitated individuals. Three control functions are introduced: rehabilitation of drug users, social reintegration of rehabilitated individuals, and awareness campaigns. Based on optimal control theory and Pontryagin's maximum principle, the necessary conditions for identifying optimal strategies are established. Numerical simulations carried out using MATLAB complement the theoretical analysis and illustrate the behavior of the system with and without intervention.

Keywords: crime dynamics; compartmental modeling; simulation; optimal control theory; Pontryagin's maximum principle.

2020 AMS Subject Classification: 91D10, 92D25, 65L20, 49K15, 49K21.

1. Introduction

Crime and drug use are two major issues in contemporary societies, whose interaction deserves particular attention. Drug consumption today represents not only a public health problem but also a key factor in marginalization and criminalization. Beyond its medical consequences,

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addiction weakens the most vulnerable individuals and fosters the emergence of delinquent behaviors.

This dual impact—both health-related and social—justifies an in-depth analysis that goes beyond the strict framework of epidemiology to include crime dynamics. Indeed, epidemiological data highlight the magnitude of the phenomenon: in Europe, the annual mortality rate among drug users is estimated at 1.42% [1], a level comparable to that of pandemics such as multidrug-resistant tuberculosis, chronic hepatitis B [2], H1N1 influenza [3], and the 1957–1958 Asian flu [4].

Cohort studies confirm this excess mortality: Degenhardt et al. [5] reported a 14.7-fold higher risk of death compared to the general population, while Mathers et al. [6] observed rates ranging from 0.5% to 2.7% depending on the region. More recently, Sordo et al. [7] confirmed a standardized mortality rate of 14.7 per 1,000 person-years in a cohort of 31,000 users. In France, Lopez et al. (2004) and Brisacier et al. (2015) revealed a very pronounced excess mortality, particularly among women. The World Health Organization estimates that 600,000 annual deaths are attributable to drugs, 80% of which are due to opioids [8], while Peacock et al. [9] evaluated the annual burden of lost years of life (DALYs) at 42.2 million.

These data confirm the severity of addiction as a health problem. However, its effects extend beyond the medical field: chronic consumption often leads to the breakdown of social ties, economic marginalization, and, in many contexts, entry into criminal activity to finance drug use. This phenomenon is particularly evident with certain low-cost, highly addictive substances, such as the so-called drug el Poofa, whose growing diffusion is observed in marginalized communities. The compulsive need to consume thus transforms addiction into a real pathway to crime.

In the scientific literature, several mathematical models and optimal control approaches have been proposed to analyze the dynamics of complex phenomena, particularly in epidemiology [10, 11, 12]. These works demonstrate the relevance of compartmental modeling for representing propagation processes and assessing the impact of various control strategies. However, none of these studies explicitly considers drug consumption as a tipping factor toward criminality.

It is precisely this gap that we aim to fill. We propose a discrete compartmental model explicitly integrating drug consumption as a determinant of crime dynamics. The population is divided into five compartments: individuals susceptible to commit a crime (S), drug-dependent individuals who are not yet criminals (S^D), in particular consumers of el Poofa, active criminals (I), prisoners (P), and rehabilitated individuals (R). The introduction of the S^D compartment makes it possible to represent an intermediate category, reflecting the central role of dependence in the transition to criminal behavior. We assume in particular that the criminalization rate of dependent individuals (α_2) is significantly higher than that of merely susceptible individuals (α_1), reflecting a much greater criminogenic vulnerability among the former.

Furthermore, in order to evaluate the effectiveness of public policies, our model integrates three complementary control levers: the rehabilitation of dependent individuals (u_i), which aims to reduce drug demand and associated crimes; the social reintegration of rehabilitated individuals (v_i), which acts on social stability and the prevention of relapses; and awareness campaigns on the dangers of el Poofa and crime (w_i), which constitute a preventive measure targeting vulnerable populations.

The main objective is therefore to analyze how the expansion of el Poofa consumption influences the criminal equilibrium of a society and how appropriate control strategies can mitigate its impact. To this end, we develop a theoretical framework based on optimal control theory and Pontryagin's maximum principle, from which we establish the necessary conditions for identifying optimal strategies. Numerical simulations carried out in MATLAB complement this analysis and illustrate the behavior of the system with and without intervention. The remainder of the paper is organized as follows. Section 2 presents the model and its fundamental equations; Section 3 describes the optimal control problem and the different strategies considered; Section 4 illustrates the numerical implementation of the model; Section 5 discusses the results obtained; and Section 6 provides a conclusion and opens up perspectives.

2. Model Presentation

We consider a closed population subject to crime dynamics influenced by social, economic, and behavioral factors. One of the most significant elements is the consumption of a local drug, inexpensive and highly addictive, known as el Poofa. This context justifies the development of a discrete compartmental model capable of capturing the differentiated effect of drug consumption on crime.

The population is divided into five compartments denoted by S, S^D, I, P, and R. Compartment S includes individuals considered susceptible, who have not yet committed any criminal act nor developed an addiction. Compartment S^D contains individuals who are already drug users, particularly of el Poofa, and who have a higher probability of transitioning into criminality. Compartment I corresponds to active criminals, compartment P to prisoners temporarily excluded from the free population, and compartment R to rehabilitated individuals, that is, those who have exited the criminal system through reintegration or detoxification programs.

The specificity of our model lies in the explicit introduction of compartment S^D, which is intermediate between the healthy state and the criminal state. This allows for a specific analysis of the impact of drug consumption on the transition to criminal behavior.

The overall evolution of the system is described by the following five equations

$$(1)\qquad \begin{cases} S_{i+1} = S_i - \alpha_1 S_i I_i - \theta S_i S_i^D + \delta_1 R_i \\ S_{i+1}^D = S_i^D - \alpha_2 S_i^D I_i + \theta S_i S_i^D + \delta_2 R_i \\ I_{i+1} = I_i + \alpha_1 S_i I_i + \alpha_2 S_i^D I_i - (\lambda + \beta) I_i \\ P_{i+1} = P_i + \beta I_i - \gamma P_i \\ R_{i+1} = R_i + \gamma P_i - \delta_1 R_i - \delta_2 R_i + \lambda I_i \end{cases}$$

Each equation of this system describes the evolution of a specific compartment of the population. We present and comment on them below in order to clarify the role of each parameter.

The following equation expresses the evolution of the compartment of susceptible individuals

$$(2)\qquad S_{i+1} = S_i - \alpha_1 S_i I_i - \theta S_i S_i^D + \delta_1 R_i$$

Compartment S includes individuals who have not yet developed either criminal behaviors or drug dependence. The term $-\alpha_1 S_i I_i$ represents the direct transition of some individuals into criminality as a result of interactions with active criminals. The second term, $-\theta S_i S_i^D$, models the initiation into drug use through social contact: a susceptible individual interacting with a dependent may in turn become a drug-using susceptible, thereby joining compartment

S^D. Finally, the term $+\delta_1 R_i$ corresponds to the return of rehabilitated individuals who, despite having exited the criminal system, once again become vulnerable.

The following equation represents the evolution of compartment S^D, which groups individuals dependent on drugs (notably el Poofa) but who are not yet criminals

$$S^D_{i+1} = S^D_i - \alpha_2 S^D_i I_i + \theta S_i S^D_i + \delta_2 R_i \tag{3}$$

These individuals can initiate other susceptibles into drug use (term $\theta S_i S^D_i$) and face a high risk of transitioning into criminality (term $\alpha_2 S^D_i I_i$). A portion of the rehabilitated population may also relapse into this state (term $\delta_2 R_i$). In line with the earlier hypothesis, it is assumed that $\alpha_2 \gg \alpha_1$.

The dynamics of active criminals are given by

$$I_{i+1} = I_i + \alpha_1 S_i I_i + \alpha_2 S^D_i I_i - (\lambda + \beta) I_i \tag{4}$$

The criminal population increases through the direct criminalization of susceptibles ($\alpha_1 S_i I_i$) and that of dependent individuals ($\alpha_2 S^D_i I_i$). The losses come from direct rehabilitation (λI_i) or incarceration (βI_i).

The evolution of the prison compartment is described by

$$P_{i+1} = P_i + \beta I_i - \gamma P_i \tag{5}$$

Incarcerated individuals come from the criminal compartment (term βI_i) and may be released at rate γ.

Finally, the dynamics of rehabilitated individuals are given by:

$$R_{i+1} = R_i + \gamma P_i - \delta_1 R_i - \delta_2 R_i + \lambda I_i \tag{6}$$

The reintegrated population increases through rehabilitated prisoners (γP_i) and directly rehabilitated criminals (λI_i). However, relapse remains possible in two forms: a return to the susceptible state ($\delta_1 R_i$) or a relapse into dependence ($\delta_2 R_i$).

Table 1 defines the main parameters of the model and their meaning.

Parameter	**Meaning**
α_1	Rate of transition of susceptible individuals into criminality
α_2	Rate of transition of dependent individuals into criminality
θ	Rate of initiation into drug use among susceptibles
β	Incarceration rate of criminals
γ	Release rate of prisoners
λ	Direct rehabilitation rate of criminals
δ_1	Relapse rate of rehabilitated individuals into the initial vulnerable state
δ_2	Relapse rate of rehabilitated individuals into dependence

TABLE 1. Definitions of the model parameters

Finally, Figure 1 schematically represents the flows between compartments and the associated parameters.

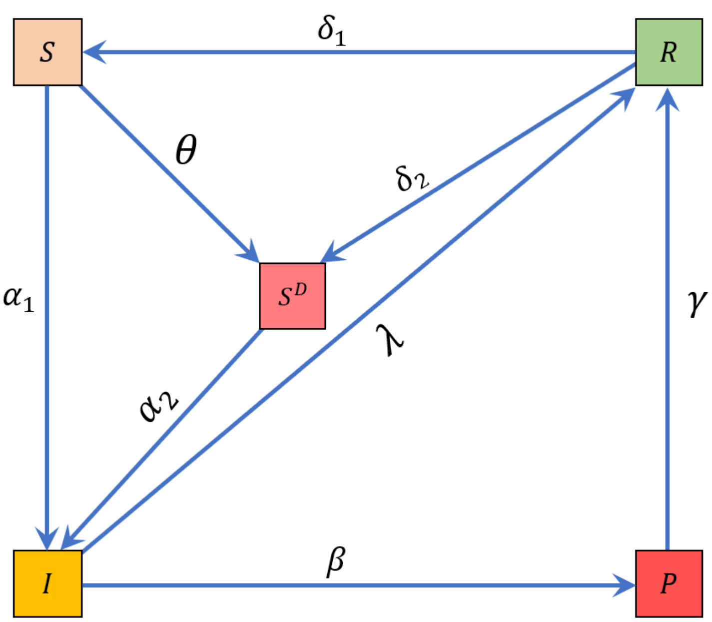

FIGURE 1. Schematic representation of the transitions between the model compartments, with associated parameters.

3. Optimal Control Problem

Optimal control theory provides a relevant framework for the regulation of dynamic systems, as has been shown in epidemiology [13, 14, 15, 16, 17, 18]. Here, we adapt this methodology to the criminological context in order to study the impact of different intervention strategies on the dynamics associated with the consumption of el Poofa.

3.1. Presentation of the controls.

The State's fight against el Poofa, a highly addictive substance threatening social stability, has led to the arrest of numerous dealers [19]. This repressive action, while effective in reducing supply, has resulted in a sharp increase in prices and a scarcity of the drug on the market [19]. Confronted with their dependence and the economic impossibility of maintaining consumption, many users have turned to criminal activity [20], particularly through theft of mobile phones and valuable items intended to finance drug purchases [20]. This phenomenon illustrates a displacement of the problem: repression of trafficking, although effective on supply, has paradoxically intensified acquisitive criminal behaviors among dependent users [20].

To control the evolution of crime rates, we propose an integrated approach based on three complementary control strategies, each targeting a specific stage of the deviant cycle.

The first, denoted u_i, corresponds to the rehabilitation of dependent individuals. It aims to address the root of the problem by providing detoxification programs, withdrawal support, and medical and psychological care. Mathematically, u_i reduces the size of compartment S_i^D through the term $-u_iS_i^D$ and increases the rehabilitated population via $+u_iS_i^D$.

The second strategy, denoted v_i, concerns the social reintegration of rehabilitated individuals. It consists of supporting individuals who have overcome their dependence in their reintegration through professional training, employment assistance, and psychosocial support. The objective is to prevent relapses and strengthen their social stability. This control acts by reducing the outflows from R_i toward deviant states ($-\delta_2 v_i R_i$) and by reinforcing their contribution to society through $+\delta_2 v_i R_i$ in S_i^D.

The third strategy, denoted w_i, corresponds to awareness campaigns on the impact of el Poofa and crime. It aims to inform the population, particularly vulnerable groups, about the consequences of consuming el Poofa and engaging in criminal behavior. From a mathematical perspective, w_i results in a reduction of the S_i compartment ($-w_iS_i$) and an indirect reinforcement of rehabilitation efforts through $+w_iS_i$ in R_i.

These three levers form a coherent control system that operates at different levels: treatment of dependence (u_i), prevention of relapse (v_i), and primary prevention (w_i). This comprehensive approach aims to break the cycle dependence $\rightarrow$ criminality $\rightarrow$ incarceration $\rightarrow$ recidivism by addressing the root causes of the phenomenon rather than its mere manifestations.

The corresponding controlled dynamic system is given by

$$\begin{cases} S_{i+1} = S_i - \alpha_1 S_i I_i - \theta S_i S_i^D + \delta_1 R_i - w_i S_i, \\ S_{i+1}^D = S_i^D - \alpha_2 S_i^D I_i + \theta S_i S_i^D + \delta_2 v_i R_i - u_i S_i^D, \\ I_{i+1} = I_i + \alpha_1 S_i I_i + \alpha_2 S_i^D I_i - (\lambda + \beta) I_i, \\ P_{i+1} = P_i + \beta I_i - \gamma P_i, \\ R_{i+1} = R_i + \gamma P_i - \delta_1 R_i - \delta_2 v_i R_i + \lambda I_i + w_i S_i + u_i S_i^D. \end{cases} \tag{7}$$

This system makes it possible to simulate the evolution of the different sub-populations and to evaluate the impact of control strategies on the crime dynamics associated with the consumption of el Poofa.

3.2. Objective Function. The objective is to define a cost function that simultaneously accounts for the reduction of active criminals, the increase of rehabilitated individuals, and the costs associated with implementing the controls u_i, v_i, and w_i over the considered time horizon.

For an initial state $(S_0, S_0^D, I_0, P_0, R_0)$, we define the following optimization criterion:

$$J(u,v,w) = \alpha_I I_N - \alpha_R R_N + \sum_{i=0}^{N-1} \left[\alpha_I I_i - \alpha_R R_i + \frac{A}{2} u_i^2 + \frac{B}{2} v_i^2 + \frac{C}{2} w_i^2 \right], \tag{8}$$

where $\alpha_I > 0$ and $\alpha_R > 0$ are the weights associated with active criminals and rehabilitated individuals, respectively, while $A > 0$, $B > 0$, and $C > 0$ represent the weights related to the costs of the controls. The control variables are $u = (u_0, \ldots, u_{N-1})$, $v = (v_0, \ldots, v_{N-1})$, and $w = (w_0, \ldots, w_{N-1})$. Finally, N denotes the final time of the strategy.

The problem therefore consists of determining the optimal controls (u^*, v^*, w^*) that minimize (8) subject to the dynamic constraints of the controlled system (7):

$$J(u^*, v^*, w^*) = \min\left\{J(u,v,w) \mid (u,v,w) \in \mathscr{U} \times \mathscr{V} \times \mathscr{W}\right\}, \tag{9}$$

where the sets of admissible controls are defined as follows:

$$\mathscr{U} = \left\{u \mid u_{\min} \leq u_i \leq u_{\max},\ i = 0, \ldots, N-1\right\}, \tag{10}$$

$$\mathscr{V} = \left\{v \mid v_{\min} \leq v_i \leq v_{\max},\ i = 0, \ldots, N-1\right\}, \tag{11}$$

$$\mathscr{W} = \left\{w \mid w_{\min} \leq w_i \leq w_{\max},\ i = 0, \ldots, N-1\right\}, \tag{12}$$

with the following bounds

$$0 < u_{\min} < u_{\max} \leq 1, \quad 0 < v_{\min} < v_{\max} \leq 1, \quad 0 < w_{\min} < w_{\max} \leq 1. \tag{13}$$

3.3. Sufficient Condition.

Theorem 1. *There exists an optimal control $(u^*, v^*, w^*) \in \mathscr{U} \times \mathscr{V} \times \mathscr{W}$ such that*

$$J(u^*, v^*, w^*) = \min_{(u,v,w) \in \mathscr{U} \times \mathscr{V} \times \mathscr{W}} J(u,v,w), \tag{14}$$

subject to system (7) and the initial conditions.

Proof: Since the system parameters are bounded and the time horizon is finite, the trajectories of the states S, S^D, I, P, R remain uniformly bounded for any admissible triplet of controls $(u,v,w) \in \mathscr{U} \times \mathscr{V} \times \mathscr{W}$. In particular, the cost function $J(u,v,w)$, defined by (8), remains bounded over $\mathscr{U} \times \mathscr{V} \times \mathscr{W}$.

It follows that

$$\inf_{(u,v,w) \in \mathscr{U} \times \mathscr{V} \times \mathscr{W}} J(u,v,w)$$

is finite. Hence, there exists a sequence $(u^n, v^n, w^n)_{n \geq 1} \subset \mathscr{U} \times \mathscr{V} \times \mathscr{W}$ such that

$$\lim_{n\to+\infty} J(u^n,v^n,w^n) = \inf_{(u,v,w)\in\mathscr{U}\times\mathscr{V}\times\mathscr{W}} J(u,v,w). \tag{15}$$

The corresponding sequences of states $S^n, S^{D,n}, I^n, P^n, R^n$ are also uniformly bounded. By compactness, there exists a convergent subsequence $(u^n,v^n,w^n)\to(u^*,v^*,w^*)$ and limits $(S^*,S^{D,*},I^*,P^*,R^*)$ such that

$$\begin{cases} S^n \to S^*, \\ S^{D,n} \to S^{D,*}, \\ I^n \to I^*, \\ P^n \to P^*, \\ R^n \to R^*. \end{cases} \tag{16}$$

Thus, $(u^*,v^*,w^*)\in\mathscr{U}\times\mathscr{V}\times\mathscr{W}$ constitutes an optimal control for the studied system, with corresponding states $(S^*,S^{D,*},I^*,P^*,R^*)$. This completes the proof.

3.4. Necessary Condition. The sufficient condition for the existence of an optimal control for this problem follows from Theorem 1 presented in [15]. Moreover, by applying Pontryagin's maximum principle [10, 11, 21, 22, 23, 12], we derive the necessary conditions related to our optimal control problem. To this end, we define the Hamiltonian as follows:

$$\begin{aligned} \mathscr{H}(\Omega) &= \alpha_I I_i - \alpha_R R_i + \frac{A}{2}u_i^2 + \frac{B}{2}v_i^2 + \frac{C}{2}w_i^2 \\ &+ \zeta_{1,i+1}\left[S_i - \alpha_1 S_i I_i - \theta S_i S_i^D + \delta_1 R_i - w_i S_i\right] \\ &+ \zeta_{2,i+1}\left[S_i^D - \alpha_2 S_i^D I_i + \theta S_i S_i^D + \delta_2 v_i R_i - u_i S_i^D\right] \\ &+ \zeta_{3,i+1}\left[I_i + \alpha_1 S_i I_i + \alpha_2 S_i^D I_i - \lambda I_i - \beta I_i\right] \\ &+ \zeta_{4,i+1}\left[P_i + \beta I_i - \gamma P_i\right] \\ &+ \zeta_{5,i+1}\left[R_i + \gamma P_i - \delta_1 R_i - \delta_2 v_i R_i + \lambda I_i + w_i S_i + u_i S_i^D\right]. \end{aligned} \tag{17}$$

Theorem 2. *Given the optimal controls u^*, v^*, w^* and the solutions $S^*, S^{D,*}, I^*, P^*, R^*$, there exist $\zeta_{k,i}$, $i = 0,\dots,N$, $k = 1,2,3,4,5$ (the adjoint variables) satisfying the following equations:*

(18)
$$\Delta\zeta_{1,i} = -\zeta_{1,i+1}\left(1-\alpha_1 I_i - \theta S_i^D - w_i\right) - \zeta_{2,i+1}\theta S_i^D - \zeta_{3,i+1}\alpha_1 I_i - \zeta_{5,i+1} w_i,$$

(19)
$$\Delta\zeta_{2,i} = \zeta_{1,i+1}\theta S_i - \zeta_{2,i+1}\left(1-\alpha_2 I_i + \theta S_i - u_i\right) - \zeta_{3,i+1}\alpha_2 I_i - \zeta_{5,i+1} u_i,$$

(20)
$$\Delta\zeta_{3,i} = -\alpha_I + \zeta_{1,i+1}\alpha_1 S_i + \zeta_{2,i+1}\alpha_2 S_i^D - \zeta_{3,i+1}\left(1+\alpha_1 S_i + \alpha_2 S_i^D - \lambda - \beta\right) - \zeta_{4,i+1}\beta - \zeta_{5,i+1}\lambda,$$

(21)
$$\Delta\zeta_{4,i} = -\zeta_{4,i+1}(1-\gamma) - \zeta_{5,i+1}\gamma,$$

(22)
$$\Delta\zeta_{5,i} = \alpha_R - \zeta_{1,i+1}\delta_1 - \zeta_{2,i+1}\delta_2 v_i - \zeta_{5,i+1}(1-\delta v_i).$$

With the following transversality conditions: $\zeta_{1,N}=0$, $\zeta_{2,N}=0$, $\zeta_{3,N}=\alpha_I$, $\zeta_{4,N}=0$, *and* $\zeta_{5,N}=-\alpha_R$. *Moreover, the optimal controls can be expressed as follows:*

$$u_i^* = \min\left\{\max\left\{\frac{S_i^D(\zeta_{2,i+1}-\zeta_{5,i+1})}{A}, u^{\min}\right\}, u^{\max}\right\}, \quad i=0,\dots,N-1, \tag{23}$$

$$v_i^* = \min\left\{\max\left\{\frac{\delta_2 R_i(\zeta_{5,i+1}-\zeta_{2,i+1})}{B}, v^{\min}\right\}, v^{\max}\right\}, \quad i=0,\dots,N-1, \tag{24}$$

$$w_i^* = \min\left\{\max\left\{\frac{S_i(\zeta_{1,i+1}-\zeta_{5,i+1})}{C}, w^{\min}\right\}, w^{\max}\right\}, \quad i=0,\dots,N-1. \tag{25}$$

Proof: Using Pontryagin's maximum principle [21] and denoting $S^*, S^{D,*}, I^*, P^*, R^*$ together with the optimal controls u^*, v^*, w^*, we obtain the following adjoint equations:

$$\Delta\zeta_{1,i} = -\frac{\partial\mathscr{H}}{\partial S_i} = -\zeta_{1,i+1}\left(1-\alpha_1 I_i - \theta S_i^D - w_i\right) - \zeta_{2,i+1}\theta S_i^D - \zeta_{3,i+1}\alpha_1 I_i - \zeta_{5,i+1} w_i, \tag{26}$$

$$\Delta\zeta_{2,i} = -\frac{\partial\mathscr{H}}{\partial S_i^D} = \zeta_{1,i+1}\theta S_i - \zeta_{2,i+1}\left(1-\alpha_2 I_i + \theta S_i - u_i\right) - \zeta_{3,i+1}\alpha_2 I_i - \zeta_{5,i+1} u_i, \tag{27}$$

$$\Delta\zeta_{3,i} = -\frac{\partial\mathscr{H}}{\partial I_i} = -\alpha_I + \zeta_{1,i+1}\alpha_1 S_i + \zeta_{2,i+1}\alpha_2 S_i^D - \zeta_{3,i+1}\left(1+\alpha_1 S_i + \alpha_2 S_i^D - \lambda - \beta\right)$$

$$- \zeta_{4,i+1}\beta - \zeta_{5,i+1}\lambda, \tag{28}$$

$$\Delta\zeta_{4,i} = -\frac{\partial \mathscr{H}}{\partial P_i} = -\zeta_{4,i+1}(1-\gamma) - \zeta_{5,i+1}\gamma, \tag{29}$$

$$\Delta\zeta_{5,i} = -\frac{\partial \mathscr{H}}{\partial R_i} = \alpha_R - \zeta_{1,i+1}\delta_1 - \zeta_{2,i+1}\delta_2 v_i - \zeta_{5,i+1}(1-\delta v_i). \tag{30}$$

with: $\zeta_{1,N} = 0, \quad \zeta_{2,N} = 0, \quad \zeta_{3,N} = \alpha_I, \quad \zeta_{4,N} = 0, \quad \zeta_{5,N} = -\alpha_R.$

To obtain the optimality conditions, we consider the variation with respect to the controls u_i, v_i, w_i and set the partial derivatives of the Hamiltonian to zero. This leads to the following relations

$$\frac{\partial \mathscr{H}}{\partial u_i} = Au_i + S_i^D(\zeta_{5,i+1} - \zeta_{2,i+1}) = 0, \tag{37}$$

$$\frac{\partial \mathscr{H}}{\partial v_i} = Bv_i + \delta_2 R_i(\zeta_{2,i+1} - \zeta_{5,i+1}) = 0, \tag{38}$$

$$\frac{\partial \mathscr{H}}{\partial w_i} = Cw_i + S_i(\zeta_{5,i+1} - \zeta_{1,i+1}) = 0. \tag{39}$$

Taking into account the bounds imposed by the sets of admissible controls, the expressions of the optimal controls take the following form

$$u_i^* = \max\left\{\min\left\{\frac{S_i^D(\zeta_{2,i+1} - \zeta_{5,i+1})}{A},\, u_{\max}\right\},\, u_{\min}\right\}, \quad i = 0,\ldots,N-1, \tag{40}$$

$$v_i^* = \max\left\{\min\left\{\frac{\delta_2 R_i(\zeta_{5,i+1} - \zeta_{2,i+1})}{B},\, v_{\max}\right\},\, v_{\min}\right\}, \quad i = 0,\ldots,N-1, \tag{41}$$

$$w_i^* = \max\left\{\min\left\{\frac{S_i(\zeta_{1,i+1} - \zeta_{5,i+1})}{C},\, w_{\max}\right\},\, w_{\min}\right\}, \quad i = 0,\ldots,N-1. \tag{42}$$

4. Numerical Simulation and Discussion

In this section, we validate the theoretical model through numerical simulations carried out using MATLAB. The system of equations describes the evolution of the different interacting sub-populations: S, S^D, I, P, and R. These equations capture the mechanisms of transitions between states such as drug initiation, entry into criminality, incarceration, reintegration, and relapse. They are studied both in the absence and in the presence of the three controls u, v, and w, whose values are taken within the interval $[0.1, 0.5]$. The parameters used are: $\alpha_1 = 8\times 10^{-6}$, $\alpha_2 = 1\times 10^{-6}$, $\theta = 5\times 10^{-6}$, $\beta = 9\times 10^{-5}$, $\gamma = 4\times 10^{-4}$, $\lambda = 1\times 10^{-4}$, $\delta_1 = 5\times 10^{-4}$,

$\delta_2 = 5 \times 10^{-4}$. Simulations were performed with the initial conditions $S(0) = 1000,\ S^D(0) = 10,\ I(0) = 10,\ P(0) = 5,\ R(0) = 0$.

Figure 2 illustrates the evolution of the susceptible population S. In both scenarios, S decreases to reach similar values (34 with control versus 39 without control). The difference therefore lies not in the final level but in the speed of decline: under control, the decrease is much faster. This rapidity reflects the fact that some individuals are diverted early towards rehabilitation through awareness campaigns, which considerably reduces the time during which they could feed into classes S^D and I. In other words, it is the initial dynamics, more than the final state, that prevents the explosion in the number of active criminals.

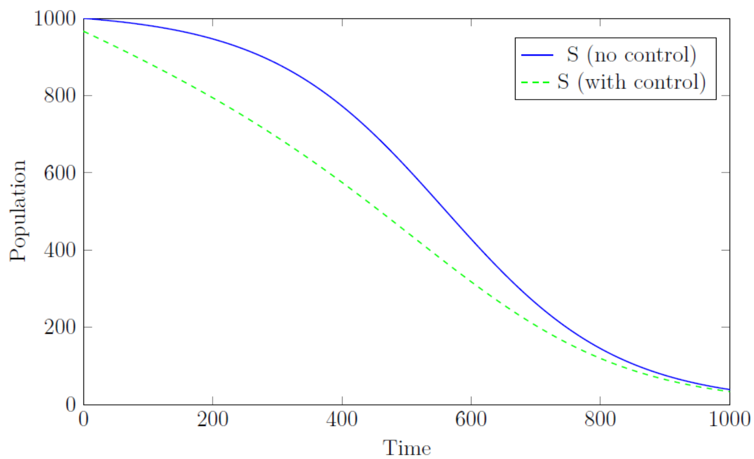

FIGURE 2. Evolution of S with and without control

Figure 3 shows the dynamics of drug-dependent individuals S^D. Without intervention, their number increases sharply, from 10 to about 121, reflecting the spread of consumption and the spillover effect from S. When controls are applied, the trajectory is reversed: the population decreases slightly to around 3. This evolution is explained not only by the effect of u, which promotes the rehabilitation of dependents, but also by the prior decrease in S, which limits the inflow into S^D. The reduction of this compartment weakens a major channel feeding criminality and helps to contain the growth of I.

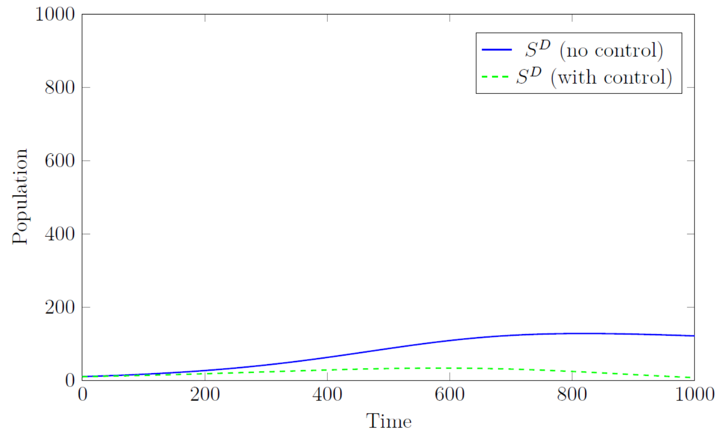

FIGURE 3. Evolution of S^D with and without control

Figure 4 highlights the evolution of active criminals I. Without control, their number grows rapidly to exceed 800, a direct consequence of the increase in S^D and the absence of preventive mechanisms. When controls are introduced, the trajectory changes significantly: growth is curbed and the final level stabilizes around 214. This difference is explained by the combined effect of u, which reduces the transition of dependents to I, and v, which promotes reintegration and limits relapses. In addition, the rapid decrease in S under control restricts the feeding of the criminal reservoir, thereby preventing an explosion in the I population. Altogether, this shows that criminality can be contained by an integrated approach that acts upstream on dependence and downstream on reintegration.

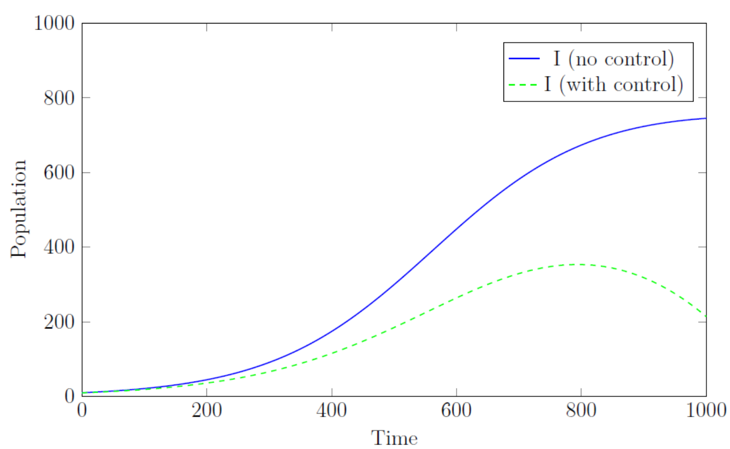

FIGURE 4. Evolution of I with and without control

The evolution of prisoners P, shown in Figure 5, also illustrates the effectiveness of control policies. In the absence of intervention, the prison population rises from 5 to 94, reflecting an approach essentially based on repression. With control, this number is limited to 47. The difference lies in the fact that the controls redirect part of the criminals toward reintegration and rehabilitation pathways, thereby reducing reliance on the prison system. This results in a decline in the role of incarceration as the sole social response to the criminal phenomenon.

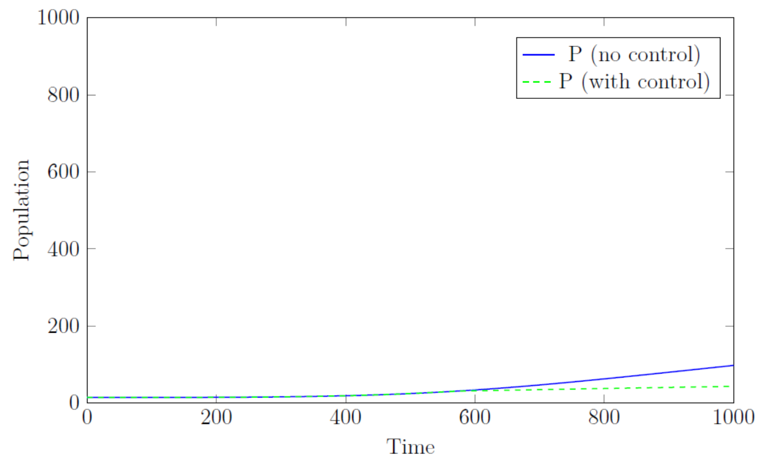

FIGURE 5. Evolution of P with and without control

Figure 6 concerns the population of rehabilitated individuals R. Without control, this compartment remains marginal and does not exceed about thirty individuals, indicating that the system, left on its own, produces very few positive outcomes. With controls, the trajectory is radically different: the rehabilitated population reaches approximately 702, reflecting the combined effectiveness of awareness, rehabilitation, and reintegration. This progression represents a genuine reversal of dynamics: instead of a cycle that traps individuals in dependence, criminality, and incarceration, there is a massive shift toward social reintegration, ensuring a sustainable stabilization of the system.

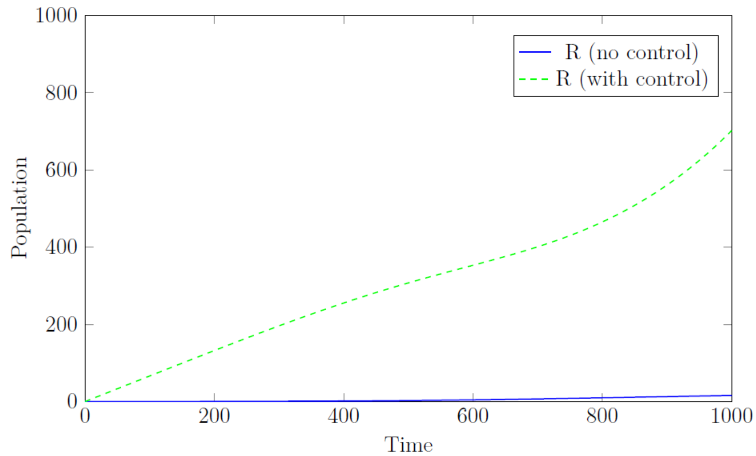

FIGURE 6. Evolution of R with and without control

Ultimately, the joint analysis of the five compartments highlights that the combination of the three controls constitutes a decisive lever to curb the spread of drugs and contain the resulting criminality. The simulations clearly reveal a dual movement: on the one hand, a significant reduction in dependents and active criminals, as well as a decline in the role of the prison population; on the other hand, a massive increase in rehabilitated individuals, who in the long run become a dominant compartment. This shift reveals a profound transformation of the system: instead of a model centered on repression, which sustains recidivism and increases the burden on prisons, an emerging dynamic appears in which prevention, rehabilitation, and reintegration are combined to foster positive trajectories. Altogether, this leads to more sustainable social stabilization, where criminality is contained not by coercion but by reducing the structural causes that fuel it.

5. CONCLUSION

In this article, we proposed a mathematical framework to analyze the influence of drug consumption on crime dynamics and to evaluate the effectiveness of different control strategies. Drug-related criminality remains a major challenge for social stability and public safety, and its complexity requires an integrated approach combining prevention, treatment, and reintegration.

The model highlights the role of the S^D compartment, representing dependent individuals, as a gateway to criminality. The introduction of three forms of control, namely the rehabilitation of dependents, the reintegration of rehabilitated individuals, and awareness campaigns targeting vulnerable populations, shows that the system's dynamics can be profoundly modified. The simulations indicate that a purely repressive policy tends to increase the burden on the prison system, whereas a balanced strategy promoting rehabilitation and prevention reduces criminality and increases the number of rehabilitated individuals.

This work confirms the need for a multidimensional approach to address criminality associated with drug consumption. It also opens perspectives for future research that could integrate socioeconomic factors, validate the model with real data, and explore continuous or spatio-temporal formulations, in order to enhance its role as a decision-support tool for the development of more effective public policies.

Conflict of Interests

The authors declare that there is no conflict of interests.

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