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THE FEAR AND TOXIN EFFECT ON SOKOL-HOWELL PREY-PREDATOR MODEL INVOLVING HUNTING COOPERATION WITH ANTI-PREDATOR BEHAVIOUR AND DISEASE IN TOP PREDATOR

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Abstract: This study examines the complex dynamic behavior of a multi-trophic epidemiological ecology model, encompassing prey, predators, susceptible and infected apex predators. When a predator consumes, preys according to Holling type-IV functional response, while the apex predator utilizes a functional response of the Sokol-Howl type. Among other environmental challenges, the model considers the toxic effect on all species, with toxicity levels decreasing from prey to apex predator. Cooperative hunting among predators and apex predators, as well as prey-predator behavior, enhances the model's dynamics. Additionally, the fear effect addresses the psychological impact, driving prey to fear predators and predators to fear apex predators. We conduct a comprehensive qualitative analysis, including the presence and limitations of solutions, the local and global stability of equilibrium points. Numerical analysis demonstrate how toxicity, fear, and cooperative hunting synergistically influence species survival and extinction, revealing the stability of diseased, three-trophic food webs under environmental stresses.

Keywords: Sokol-Howell; fear; hunting cooperation; anti-predator; toxin; disease.
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1. INTRODUCTION

The mathematical model of predator-prey interactions is a cornerstone of theoretical ecology,

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providing insights into the complex mechanisms that govern biodiversity and ecological stability. The pioneering work of Lotka and Volterra provided the initial framework for these elements [1, 2], which May later expanded to explore the interplay of complexity in microsystems [3]. In recent years, this field has become increasingly prominent, yet also remarkably elusive, encompassing non-participatory functional responses, behavioral adaptations, and micro-constraints [4, 5].

Particular functional responses are pivotal in the long-term dynamics of ecological systems. Classical Holling - IV functional response responses have been extensively studied [6], and more recent variations, such as the Sokol-Howl functional response, have garnered increasing attention for their ability to provide a realistic physiological representation of predation under varying conditions [7]. Furthermore, we often improve integratively through hunting cooperation, and collective movement can completely transform feeding networks [8, 9].

In addition to direct consumption, there are indirect predator effects, specifically the fear effect, and a main factor in prey population dynamics [10-12]. Predator-induced fear leads to significant physiological and behavioral shifts in prey, such as modified foraging efficiency and reduced reproductive output, which may ultimately lead to population declines even in the absence of direct killing [13, 14]. These dynamics are often compounded by the fear, as the effect of a low population leads to reduced growth rate, thus critical extinction thresholds [15, 16].

The novelty of epidemiological ecology has led to the understanding that natural adaptations are not designed to combat disease. The presence of disease in the predator population introduces a new layer of complexity, often leading to a destabilization of the equilibrium or the emergence of oscillatory patterns [17]. Moreover, the inclusion of self-defense measures, where prey defends itself or uses refuge, shifts the nonlinearity towards the system [18, 19]. In many ecosystems, species survival is also threatened by toxins, whether environmental pollutants or simply defensive chemicals, increasing mortality rates and disrupting the balance of prey-prey interactions [20-22].

Although many diverse environmental factors, such as hunting cooperation, fear effect, and toxins [23-26], have been extensively studied, there is a scarcity of studies that simultaneously the Sokol-Howell and Holling-IV functional response with hunting cooperation, self-defense behaviors, toxins, and disease in top predators [27, 28]. This research aims to bridge this gap by developing and analyzing a comprehensive dynamic system.

The proposed dynamical system is rigorously analyzed to ensure its biological and mathematical consistency. We establish the boundedness of the solutions to ensure the model's dissipativeness, followed by a detailed qualitative analysis to determine the existence and stability conditions for all possible equilibrium points. Both local and global stability analyses are conducted to predict the long-term behavior of the populations under the combined influence of Sokol-Howell and Holling type-IV functional responses, hunting cooperation, toxins, and anti-predator behavior. Finally, comprehensive numerical simulations are performed using MATLAB to validate the theoretical findings induced by the interplay of these ecological factors.

2. Mathematical Model

A research model of (prey, predator, and susceptible and infected top predator) with Holling type IV functional reactor for predator and Sokol-Howel functional reaction for susceptible top predator is put forward in this work. At time T, which of the following best describes the total population density $x_1(T), x_2(T), x_s(T)$ and $x_I(T)$ in that order.

$$\left.\begin{aligned}
\frac{dx_1}{dT} &= a_1 x_1\left(1-\frac{x_1}{k_1}\right)\left(\frac{1}{1+f_1x_2}\right)-\frac{b_1x_1x_2}{c_1+x_1^2}-\rho_1x_1x_2^2-\delta_1x_1^3-d_1x_1,\\
\frac{dx_2}{dT} &= a_2x_2\left(1-\frac{x_2}{k_2}\right)\left(\frac{1}{1+f_2x_s}\right)+\frac{g_1x_1x_2}{c_1+x_1^2}-\eta x_1x_2+e_1\rho_1x_1x_2^2-\rho_2x_2x_s^2-\delta_2x_2^2-d_2x_2-\frac{b_2x_2x_s}{c_2+\gamma_1x_2^2},\\
\frac{dx_s}{dT} &= \frac{g_2x_2x_s}{c_2+\gamma_1x_2^2}+e_2\rho_2x_2x_s^2-\delta_3x_s-\beta x_sx_I-d_3x_s,\\
\frac{dx_I}{dT} &= \beta x_sx_I-d_4x_I.
\end{aligned}\right\}\quad(1)$$

With the following initial condition $x_1(T)\geq 0$, $x_2(T)\geq 0$, $x_s(T)\geq 0$ and $x_I(T)\geq 0$. Even though table 1 shows the biological importance of system (1) attributes:

Table 1 Biological significance of system parameters.

Parameters	**Biological meaning**
$a_n > 0, n = 1,2$	The intrinsic growth rates of the prey and the predator in order
$k_n > 0, n = 1,2$	The carrying capacity of the prey and the predator accordingly
$f_n > 0, n = 1,2$	The fear range of the prey species from predator and predator species from Susceptible top predator in order
$b_v > 0, v = 1,2$	The attack rates of the prey and the predator in order
$c_1 > 0$	The interference constant of Holling type IV function response of the prey and the predator accordingly
$\rho_v > 0, v = 1,2$	The hunting cooperation in prey and predator in order
$g_v > 0, g_v \leq b_v$	The rates at which food is converted into energy differ for both the predator and the apex predator

η	The anti-predator rate of the prey
$\gamma_1 > 0$	The inverse measure of inhibitory effect
$c_2 > 0$	The interference constant between the susceptible top predator and the predator's Sokol-Howell function response
$0 < e_v < 1\,, v = 1,2$	The conversion rate by attack the prey and predator in order.
$\delta_v > 0, v = 1,2,3$	The toxin rate of the prey, predator and Susceptible top predator in order
$\beta > 0$	The disease transmission rate
$d_v > 0\,, v = 1,2$	The death rates of prey and predator by external sources in order
$d_3 > 0$	The natural death rate of susceptible top predator
$d_4 > 0$	The death rate of infected top predator by disease

3. BOUNDEDNESS

The boundedness of system (1) has been proved in the next theorem.

Theorem (1): Every solution of system (1) that begins in R_+^4 is a uniform boundary.

Proof: Assume $(x_1(\mathrm{t}), x_2(\mathrm{t}), x_s(t), x_I(t))$ represent any solution of the system (1) with non-negative initial condition $(x_1(0), x_2(0), x_s(0), x_I(0)) \in R_+^4$.

Let that $A(t) = x_1(\mathrm{t}) + x_2(\mathrm{t}) + x_s(t) + x_I(t)$ time derivative of A(t) along system (1) solution is then taken into consideration, and the result is

$$\frac{dA}{dt} = a_1x_1\left(1-\frac{x_1}{k_1}\right)\left(\frac{1}{1+f_1x_2}\right) - (b_1-g_1)\frac{x_1x_2}{c_1+x_1^2} - \delta_1x_1^3 - d_1x_1 + a_2x_2\left(1-\frac{x_2}{k_2}\right)\left(\frac{1}{1+f_2x_s}\right) - \eta x_1x_2 - (1-e_1)\rho_1x_1x_2^2 - \delta_2x_2^2 - d_2x_2 - (b_2-g_2)\frac{x_2x_s}{c_2+\gamma_1x_2^2} - (1-e_2)\rho_2x_2x_s^2 - \delta_3x_s - \beta x_sx_I - d_3x_s + \beta x_sx_I - d_4x_s.$$

Now, from the biological point of view, always $b_i > g_i$ and $0 < e_i < 1$, when $i = 1,2$ it is obtained that.

$$\frac{dA}{dt} \le a_1x_1\left(1-\frac{x_1}{k_1}\right) - \delta_1x_1^3 - d_1x_1 + a_2x_2\left(1-\frac{x_2}{k_2}\right) - \eta x_1x_2 - \delta_2x_2^2 - d_2x_2 - (\delta_3+d_3)x_s - \beta x_sx_I + \beta x_sx_I - d_4x_s,$$

$$\frac{dA}{dt} = a_1x_1\left(1-\frac{x_1}{k_1}\right) - \delta_1x_1^3 - d_1x_1 + a_2x_2\left(1-\frac{x_2}{k_2}\right) - \eta x_1x_2 - \delta_2x_2^2 - d_2x_2 - (\delta_3+d_3)x_s - d_4x_s,$$

$$\frac{dA}{dt} \le a_1x_1\left(1-\frac{x_1}{k_1}\right) + a_2x_2\left(1-\frac{x_2}{k_2}\right) - \omega A,$$

where $\omega = \min\{d_1, d_2, \delta_3 + d_3, d_4\}$.

Now since the logistic function with relation to x_1 is $h_1(x_1) = a_1x_1\left(1-\frac{x_1}{k_1}\right)$ its limited above by $\frac{a_1k_1}{4}$.

Also, since the logistic function with relation to x_2 is $h_2(x_2) = a_2x_2\left(1-\frac{x_2}{k_2}\right)$ its limited above by $\frac{a_2k_2}{4}$, it's observed that

$$\frac{dA}{dt} + \omega A \leq L, \text{ where } L = \left(\frac{a_1k_1}{4} + \frac{a_2k_2}{4}\right).$$

Subsequently, by resolving the aforementioned differential inequality, it is derived that

$$\mathrm{A(t)} \leq \frac{L}{\omega} + \left(A_0 - \frac{L}{\omega}\right)e^{-\omega t}.$$

Then $\lim_{t\to\infty} \mathrm{A(t)} \leq \frac{L}{\omega}$.

So, $0 \leq \mathrm{A(t)} \leq \frac{L}{\omega}$,

hence the solution is uniformly bounded.

4. Existence of Equilibrium Points (EPs)

All possible (EPs) of system (1) are covered in this section. It is noteworthy that system (1) comprises eleven (EPs):

- The trivial (EP) $M_0 = (0,0,0,0)$ is always existed.
- The (EP) $M_1 = (\bar{x}_1, 0, 0, 0)$ exists if there is a positive solution to the following equation.

$$-\delta_1 x_1^2 - \frac{a_1}{k_1}x_1 + (a_1 - d_1) = 0, \tag{2}$$

  in according with Descartes' rule, Eq. (2) possesses only one positive root, depending upon the fulfillment of the specified condition.

$$a_1 > d_1. \tag{3}$$

- The (EP) $M_2 = (0, \bar{\bar{x}}_2, 0, 0)$ where $\bar{\bar{x}}_2 = \frac{k_2(a_2-d_2)}{\delta_2k_2+a_2}$ exists if the following condition hold:

$$a_2 > d_2. \tag{4}$$

- The (EP) $M_3 = (\check{x}_1, \check{x}_2, 0, 0)$, exists if there is a positive solution to the following equation.

$$a_1\left(1-\frac{x_1}{k_1}\right)\left(\frac{1}{1+f_1x_2}\right) - \frac{b_1x_2}{c_1+x_1^2} - \rho_1x_2^2 - \delta_1x_1^2 - d_1 = 0, \tag{5}$$

$$a_2\left(1-\frac{x_2}{k_2}\right) + \frac{g_1x_1}{c_1+x_1^2} - \eta x_1 + e_1\rho_1x_1x_2 - \delta_2x_2 - d_2 = 0. \tag{6}$$

  From Eq. (6) we have

$$x_2 = \frac{g_1x_1 + (a_2 - \eta x_1 - d_2)(c_1 + x_1^2)}{\left(\frac{a_2}{k_2} + \delta_2 - e_1\rho_1x_1\right)(c_1 + x_1^2)}, \tag{7}$$

Now, we get the following equation by putting Eq. (7) into Eq. (5)

$$\breve{\alpha}_1 x_1^{13} + \breve{\alpha}_2 x_1^{12} + \breve{\alpha}_3 x_1^{11} + \breve{\alpha}_4 x_1^{10} + \breve{\alpha}_5 x_1^{9} + \breve{\alpha}_6 x_1^{8} + \breve{\alpha}_7 x_1^{7} + \breve{\alpha}_8 x_1^{6} + \breve{\alpha}_9 x_1^{5} + \breve{\alpha}_{10} x_1^{4} + \breve{\alpha}_{11} x_1^{3} + \breve{\alpha}_{12} x_1^{2} + \breve{\alpha}_{13} x_1 + \breve{\alpha}_{14} = 0, \quad (8)$$

where

$\breve{\alpha}_1 = e_1^2\rho_1^2 f_1\delta_1\eta > 0,$

$\breve{\alpha}_2 = -e_1\rho_1 f_1\delta_1(2\check{\tau}_2\eta + e_2\rho_2\check{\tau}_1),$

$\breve{\alpha}_3 = \eta\big(\rho_1 f_1\eta^2 + e_1\rho_1(\eta + e_1\rho_1\check{\tau}_{17})\big) + \delta_2 f_1[\check{\tau}_2^2\eta + 2e_1\rho_1(e_1\rho_1 c_1\eta + \check{\tau}_2\check{\tau}_1)] - e_1^2\rho_1^2\check{\tau}_3 f_1\delta_1,$

$\breve{\alpha}_4 = 2\check{\tau}_2 e_1\rho_1\check{\tau}_3 f_1\delta_1 - \eta^2(3\rho_1 f_1\check{\tau}_1 + \check{\tau}_2) - 2e_1\rho_1\eta(\check{\tau}_1 + 2\check{\tau}_2 c_2 f_1\delta_1 + \check{\tau}_2\check{\tau}_{17}) - \check{\tau}_1((\check{\tau}_2^2 + 2e_1^2\rho_1^2 c_1)f_1\delta_1 - e_1^2\rho_1^2(c_1 f_1\delta_1 + \check{\tau}_{17})),$

$\breve{\alpha}_5 = \rho_1 f_1(\eta^3 c_1 + \check{\tau}_{14}) + 2\check{\tau}_1\check{\tau}_2\eta + e_1\rho_1\big[c_1\eta^2 + (\check{\tau}_{15}\eta^2 + \check{\tau}_{17}) + 2\check{\tau}_2\check{\tau}_1\big(2c_1 f_1\delta_1 + (c_1 f_1\delta_1 + \check{\tau}_{17}))\big] + c_1(2\check{\tau}_2^2 + e_1^2\rho_1^2 c_1)f_1\delta_1\eta - (\check{\tau}_2^2 + 2e_1^2\rho_1^2 c_1)(\check{\tau}_3 f_1\delta_1 - \check{\tau}_{17}\eta) - e_1^2\rho_1^2(\check{\tau}_3\check{\tau}_{17} - \check{\tau}_{16}\eta),$

$\breve{\alpha}_6 = e_1\rho_1[\check{\tau}_6 + 2\check{\tau}_2\check{\tau}_3(2c_1 f_1\delta_1 + \check{\tau}_{17})] - \rho_1 f_1(3\check{\tau}_1 c_1\eta^2 + \check{\tau}_{13}) - \check{\tau}_2 c_1\eta^2 - \check{\tau}_2(\check{\tau}_{15}\eta^2 + \check{\tau}_7) - 2e_1\rho_1[\check{\tau}_1(c_1\eta + \check{\tau}_{15}\eta) + \check{\tau}_2 c_1^2 f_1\delta_1\eta + \check{\tau}_2\check{\tau}_{16}\eta + 2\check{\tau}_2 c_2\check{\tau}_{17}\eta] - (\check{\tau}_2^2 + 2e_1^2\rho_1^2 c_1)\check{\tau}_1(c_1 f_1\delta_1 + \check{\tau}_{17}) - c_1(2\check{\tau}_2^2 + e_1^2\rho_1^2 c_1)\check{\tau}_1 f_1\delta_1 - e_1^2\rho_1^2\check{\tau}_1(\check{\tau}_{17}c_1 + \check{\tau}_{16}),$

$\breve{\alpha}_7 = c_1(\rho_1 f_1\check{\tau}_{14} + 2\check{\tau}_1\check{\tau}_2\eta + \check{\tau}_2^2 c_1 f_1\delta_1\eta) + e_1\rho_1\big[c_1(\check{\tau}_{15}\eta^2 + \check{\tau}_7) + (\check{\tau}_{17}\check{\tau}_{15} + \check{\tau}_5) + 2\check{\tau}_1\check{\tau}_2\big(c_1^2 f_1\delta_1 + 2c_2\,(c_1 f_1\delta_1 + \check{\tau}_{17}) + (\check{\tau}_{17}c_1 + \check{\tau}_{16})\big)\big] - \rho_1 f_1\check{\tau}_{12} - \check{\tau}_2\check{\tau}_6 - c_1(2\check{\tau}_2^2 + e_1^2\rho_1^2 c_1)(\check{\tau}_3 f_1\delta_1 - \check{\tau}_{17}\eta) - (\check{\tau}_2^2 + 2e_1^2\rho_1^2 c_1)(\check{\tau}_3\check{\tau}_{17} - \check{\tau}_{16}\eta) - e_1^2\rho_1^2\check{\tau}_3\check{\tau}_{16},$

$\breve{\alpha}_8 = e_1\rho_1[(\check{\tau}_6\check{\tau}_{15} + \check{\tau}_4) + 2\check{\tau}_2\check{\tau}_3\check{\tau}_{16} + c_1(2\check{\tau}_2(\check{\tau}_3 f_1\delta_1 - \check{\tau}_{17}\eta)c_1 + 4\check{\tau}_2(\check{\tau}_3\check{\tau}_{17} - \check{\tau}_{16}\eta) + \check{\tau}_6)] - \rho_1 f_1(\check{\tau}_{13}c_1 + \check{\tau}_{11}) - c_1[\check{\tau}_2(\check{\tau}_{15}\eta^2 + \check{\tau}_7) + e_1\rho_1\check{\tau}_1(2\check{\tau}_{15}\eta + e_1\rho_1\check{\tau}_{16}) + \check{\tau}_1\check{\tau}_2^2 c_1 f_1\delta_1 + (2\check{\tau}_2^2 + e_1^2\rho_1^2 c_1)\check{\tau}_1(c_1 f_1\delta_1 + \check{\tau}_{17})] - \check{\tau}_2(\check{\tau}_7\check{\tau}_{15} + \check{\tau}_5) - (\check{\tau}_2^2 + 2e_1^2\rho_1^2 c_1)\check{\tau}_1(\check{\tau}_{17}c_1 + \check{\tau}_{16}),$

$\breve{\alpha}_9 = 2\check{\tau}_1(\check{\tau}_2(c_1 f_1\delta_1 + \check{\tau}_{17}) + \check{\tau}_{15}\check{\tau}_2 c_1\eta) + e_1\rho_1\left[c_1\left((\check{\tau}_7\check{\tau}_{15} + \check{\tau}_5) + \check{\tau}_1^2 c_1 + 4\check{\tau}_2\check{\tau}_1(\check{\tau}_{17}c_1 + \check{\tau}_{16}) + 2\check{\tau}_2\check{\tau}_1\check{\tau}_{16}\right) + \check{\tau}_5\check{\tau}_{15}\right] + \rho_1 f_1\check{\tau}_{16} - c_1\big[\rho_1 f_1\check{\tau}_{12} + \check{\tau}_2\big(\check{\tau}_6 + c_1\check{\tau}_2(\check{\tau}_3 f_1\delta_1 - \check{\tau}_{17}\eta)\big) + (2\check{\tau}_2^2 + e_1^2 e_1^2 c_1)(\check{\tau}_3\check{\tau}_{17} - \check{\tau}_{16}\eta)\big] - \check{\tau}_2(\check{\tau}_6\check{\tau}_{15} + \check{\tau}_4) - (\check{\tau}_2^2 + 2e_1^2 e_1^2 c_1)\check{\tau}_3\check{\tau}_{16},$

$\breve{\alpha}_{10} = e_1\rho_1\left[c_1\left((\check{\tau}_6\check{\tau}_{15} + \check{\tau}_4) + 2\check{\tau}_2\big((\check{\tau}_3\check{\tau}_{17} - \check{\tau}_{16}\eta)c_1 + 2\check{\tau}_3\check{\tau}_{16}\big)\right) + \check{\tau}_4\check{\tau}_{15}\right] - \rho_1 f_1(\check{\tau}_{11}c_1 + \check{\tau}_9) - c_1\left[(\check{\tau}_7\check{\tau}_{15} + \check{\tau}_5)\check{\tau}_2 + \check{\tau}_1\left(\check{\tau}_2^2 c_1(c_1 f_1\delta_1 + \check{\tau}_{17}) + (2\check{\tau}_2^2 + e_1^2\rho_1^2 c_1)(\check{\tau}_{17}c_1 + \check{\tau}_{16}) + \check{\tau}_{16}(\check{\tau}_2^2 + 2e_1^2\rho_1^2 c_1)\right)\right] - \check{\tau}_2(\check{\tau}_{15}\check{\tau}_5 + \check{\tau}_1^2 c_1^2),$

$\breve{\alpha}_{11} = e_1\rho_1 c_1[(\check{\tau}_5\check{\tau}_{15} + \check{\tau}_1^2 c_1^2) + \check{\tau}_1(\check{\tau}_1\check{\tau}_{15}c_1 + 4\check{\tau}_2^2\check{\tau}_{16}) + 2\check{\tau}_1(\check{\tau}_{17}c_1 + \check{\tau}_{16})\check{\tau}_2 c_1] + \frac{a_1 c_1}{k_1} - \rho_1 f_1(\check{\tau}_{10}c_1 + \check{\tau}_8) - \check{\tau}_2\check{\tau}_4\check{\tau}_{15} - c_1\big[\check{\tau}_2\big((\check{\tau}_6\check{\tau}_{15} + \check{\tau}_4) + \check{\tau}_2 c_1(\check{\tau}_3\check{\tau}_{17} - \check{\tau}_{16}\eta)\big) + (2\check{\tau}_2^2 + e_1^2 e_1^2 c_1)\check{\tau}_3\check{\tau}_{16}\big],$

$\check{\alpha}_{12} = e_1\rho_1 c_1(\check{\tau}_4\check{\tau}_{15} + 2\check{\tau}_2\check{\tau}_3\check{\tau}_{16}c_1) + \delta_1 c_1 + d_1 - c_1[\rho_1 f_1(\check{\tau}_9 + \check{\tau}_1^3 c_1^2) + \check{\tau}_2(\check{\tau}_5\check{\tau}_{15} + \check{\tau}_1^2 c_1^2)] -$
$\check{\tau}_1 c_1^2\left[\check{\tau}_2\left(\check{\tau}_1\check{\tau}_{15} + \check{\tau}_2(\check{\tau}_{17}c_1 + \check{\tau}_{16})\right) + (2\check{\tau}_2^2 + e_1^2\rho_1^2 c_1)\check{\tau}_{16}\right] - a_1,$

$\check{\alpha}_{13} = c_1[e_1\rho_1 c_1^2\check{\tau}_1(\check{\tau}_1\check{\tau}_{15} + 2\check{\tau}_2\check{\tau}_{16}) - (\rho_1 f_1\check{\tau}_8 + \check{\tau}_2\check{\tau}_4\check{\tau}_{15} + \check{\tau}_2^2\check{\tau}_3\check{\tau}_{16}c_1)],$

$\check{\alpha}_{14} = c_1[(a_1 + d_1) - c_1^2\check{\tau}_1(\rho_1 f_1\check{\tau}_1^2 c_1 + \check{\tau}_1\check{\tau}_2\check{\tau}_{15} + \check{\tau}_2^2\check{\tau}_{16})],$

$\check{\tau}_1 = a_2 - d_2,\ \check{\tau}_2 = \frac{a_2}{k_2} + \delta_2,\ \ \check{\tau}_3 = g_1 - c_1\eta,\ \check{\tau}_4 = 2\check{\tau}_1\check{\tau}_2 c_1,\ \check{\tau}_5 = 2\check{\tau}_1^2 c_1 + \check{\tau}_3^2,\ \check{\tau}_6 = 2\check{\tau}_1(\check{\tau}_3 - c_1\eta),$

$\check{\tau}_7 = \check{\tau}_1^2 - 2\check{\tau}_3\eta,\quad \check{\tau}_8 = \check{\tau}_1 c_1(\check{\tau}_4 + \check{\tau}_1\check{\tau}_3 c_1),\ \check{\tau}_9 = (\check{\tau}_1 c_1\check{\tau}_5 + \check{\tau}_3\check{\tau}_4 + \check{\tau}_1^3 c_1^2),\qquad \check{\tau}_{10} = \check{\tau}_1\check{\tau}_6 c_1 + \check{\tau}_3\check{\tau}_5 +$
$\check{\tau}_1\check{\tau}_4 - \check{\tau}_1^2 c_1^2\eta,\qquad \check{\tau}_{11} = \check{\tau}_1\check{\tau}_7 + \check{\tau}_3\check{\tau}_6 + \check{\tau}_1\check{\tau}_5 - \check{\tau}_4\eta,\qquad \check{\tau}_{12} = \check{\tau}_3\check{\tau}_7 + \check{\tau}_1\check{\tau}_6 - 2\check{\tau}_1^2 c_1\eta - \check{\tau}_5\eta,$
$\check{\tau}_{13} = \check{\tau}_1 c_1\eta^2 - \eta(2\check{\tau}_1\check{\tau}_3 + \check{\tau}_6) + \check{\tau}_1\check{\tau}_7,\qquad \check{\tau}_{14} = \eta(-\tau_3\eta + 2\tau_1^2 + \check{\tau}_7),\quad \check{\tau}_{15} = b_1 f_1 + \rho_1 c_1,$
$\check{\tau}_{16} = d_1 f_1 - b_1,\ \check{\tau}_{17} = f_1(c_1\delta_1 + d_1),$

By Desecrate, the rule has a single positive solution, namely $\check{x}_1$ if the next set of conditions with Eq. (4) hold.

$$\eta < Min\left\{\frac{g_1}{c_1}, \frac{\check{\tau}_3}{c_1}, \frac{\check{\tau}_1^2}{2\check{\tau}_3}, \frac{\check{\tau}_3\check{\tau}_7 + \check{\tau}_1\check{\tau}_6}{2\check{\tau}_1^2 c_1 + \check{\tau}_5}, \frac{\check{\tau}_1 c_1\eta^2 + \check{\tau}_1\check{\tau}_7}{2\check{\tau}_1\check{\tau}_3 + \check{\tau}_6}, \frac{2\check{\tau}_1^2 + \check{\tau}_7}{\check{\tau}_3}, \frac{\check{\tau}_3 f_1\delta_1}{\check{\tau}_{17}}, \frac{\check{\tau}_3\check{\tau}_{17}}{\check{\tau}_{16}}\right\}, \tag{8.1}$$

$$\tau_1 > Max\left\{\frac{\check{\tau}_1^2 c_1^2\eta}{\check{\tau}_6 c_1 + \check{\tau}_4 + \check{\tau}_3\check{\tau}_5}, \frac{\check{\tau}_4\eta}{(\check{\tau}_7 + \check{\tau}_5) + \check{\tau}_3\check{\tau}_6}\right\}, \tag{8.2}$$

$$d_1 f_1 > b_1, \tag{8.3}$$

$$\eta\left(\rho_1 f_1\eta^2 + e_1\rho_1(\eta + e_1\rho_1\check{\tau}_{17})\right) + \delta_2 f_1[\check{\tau}_2^2\eta + 2e_1\rho_1(e_1\rho_1 c_1\eta + \check{\tau}_2\check{\tau}_1)] < e_1^2\rho_1^2\check{\tau}_3 f_1\delta_1, \tag{8.4}$$

$$2\check{\tau}_2 e_1\rho_1\check{\tau}_3 f_1\delta_1 < \eta^2(3\rho_1 f_1\check{\tau}_1 + \check{\tau}_2) + 2e_1\rho_1\eta(\check{\tau}_1 + 2\check{\tau}_2 c_2 f_1\delta_1 + \check{\tau}_2\check{\tau}_{17}) + \check{\tau}_1((\check{\tau}_2^2 + 2e_1^2\rho_1^2 c_1)f_1\delta_1 + e_1^2\rho_1^2(c_1 f_1\delta_1 + \check{\tau}_{17})), \tag{8.5}$$

$$\rho_1 f_1(\eta^3 c_1 + \check{\tau}_{14}) + 2\check{\tau}_1\check{\tau}_2\eta + e_1\rho_1\left[c_1\eta^2 + (\check{\tau}_{15}\eta^2 + \check{\tau}_{17}) + 2\check{\tau}_2\check{\tau}_1\left(2c_1 f_1\delta_1 + (c_1 f_1\delta_1 + \check{\tau}_{17})\right)\right] + c_1(2\check{\tau}_2^2 + e_1^2\rho_1^2 c_1)f_1\delta_1\eta < (\check{\tau}_2^2 + 2e_1^2\rho_1^2 c_1)(\check{\tau}_3 f_1\delta_1 - \check{\tau}_{17}\eta) + e_1^2\rho_1^2(\check{\tau}_3\check{\tau}_{17} - \check{\tau}_{16}\eta), \tag{8.6}$$

$$e_1\rho_1[\check{\tau}_6 + 2\check{\tau}_2\check{\tau}_3(2c_1 f_1\delta_1 + \check{\tau}_{17})] < \rho_1 f_1(3\check{\tau}_1 c_1\eta^2 + \check{\tau}_{13}) + \check{\tau}_2 c_1\eta^2 + \check{\tau}_2(\check{\tau}_{15}\eta^2 + \check{\tau}_7) + 2e_1\rho_1[\check{\tau}_1(c_1\eta + \check{\tau}_{15}\eta) + \check{\tau}_2 c_1^2 f_1\delta_1\eta + \check{\tau}_2\check{\tau}_{16}\eta + 2\check{\tau}_2 c_2\check{\tau}_{17}\eta] + (\check{\tau}_2^2 + 2e_1^2\rho_1^2 c_1)\check{\tau}_1(c_1 f_1\delta_1 + \check{\tau}_{17}) + c_1(2\check{\tau}_2^2 + e_1^2\rho_1^2 c_1)\check{\tau}_1 f_1\delta_1 + e_1^2\rho_1^2\check{\tau}_1(\check{\tau}_{17}c_1 + \check{\tau}_{16}) \tag{8.7}$$

$$c_1(\rho_1 f_1\check{\tau}_{14} + 2\check{\tau}_1\check{\tau}_2\eta + \check{\tau}_2^2 c_1 f_1\delta_1\eta) + e_1\rho_1\left[c_1(\check{\tau}_{15}\eta^2 + \check{\tau}_7) + (\check{\tau}_{17}\check{\tau}_{15} + \check{\tau}_5) + 2\check{\tau}_1\check{\tau}_2\left(c_1^2 f_1\delta_1 + 2c_2\,(c_1 f_1\delta_1 + \check{\tau}_{17}) + (\check{\tau}_{17}c_1 + \check{\tau}_{16})\right)\right] < \rho_1 f_1\check{\tau}_{12} + \check{\tau}_2\check{\tau}_6 + c_1(2\check{\tau}_2^2 + e_1^2\rho_1^2 c_1)(\check{\tau}_3 f_1\delta_1 - \check{\tau}_{17}\eta) + (\check{\tau}_2^2 + 2e_1^2\rho_1^2 c_1)(\check{\tau}_3\check{\tau}_{17} - \check{\tau}_{16}\eta) + e_1^2\rho_1^2\check{\tau}_3\check{\tau}_{16}, \tag{8.8}$$

$$e_1\rho_1[(\check{\tau}_6\check{\tau}_{15} + \check{\tau}_4) + 2\check{\tau}_2\check{\tau}_3\check{\tau}_{16} + c_1(2\check{\tau}_2(\check{\tau}_3 f_1\delta_1 - \check{\tau}_{17}\eta)c_1 + 4\check{\tau}_2(\check{\tau}_3\check{\tau}_{17} - \check{\tau}_{16}\eta) + \check{\tau}_6)] < \rho_1 f_1(\check{\tau}_{13}c_1 + \check{\tau}_{11}) + c_1[\check{\tau}_2(\check{\tau}_{15}\eta^2 + \check{\tau}_7) + e_1\rho_1\check{\tau}_1(2\check{\tau}_{15}\eta + e_1\rho_1\check{\tau}_{16}) + \check{\tau}_1\check{\tau}_2^2 c_1 f_1\delta_1 + (2\check{\tau}_2^2 + e_1^2\rho_1^2 c_1)\check{\tau}_1(c_1 f_1\delta_1 + \check{\tau}_{17})] + \check{\tau}_2(\check{\tau}_7\check{\tau}_{15} + \check{\tau}_5) + (\check{\tau}_2^2 + 2e_1^2\rho_1^2 c_1)\check{\tau}_1(\check{\tau}_{17}c_1 + \check{\tau}_{16}), \tag{8.9}$$

$2\check{\tau}_1(\check{\tau}_2(c_1f_1\delta_1+\check{\tau}_{17})+\check{\tau}_{15}\check{\tau}_2c_1\eta)+e_1\rho_1\Big[c_1\Big((\check{\tau}_7\check{\tau}_{15}+\check{\tau}_5)+\check{\tau}_1^2c_1+4\check{\tau}_2\check{\tau}_1(\check{\tau}_{17}c_1+\check{\tau}_{16})+$

$2\check{\tau}_2\check{\tau}_1\check{\tau}_{16}\Big)+\check{\tau}_5\check{\tau}_{15}\Big]+\rho_1f_1\check{\tau}_{16}<c_1\big[\rho_1f_1\check{\tau}_{12}+\check{\tau}_2\big(\check{\tau}_6+c_1\check{\tau}_2(\check{\tau}_3f_1\delta_1-\check{\tau}_{17}\eta)\big)+(2\check{\tau}_2^2+$

$e_1^2e_1^2c_1)(\check{\tau}_3\check{\tau}_{17}-\check{\tau}_{16}\eta)\big]+\check{\tau}_2(\check{\tau}_6\check{\tau}_{15}+\check{\tau}_4)+(\check{\tau}_2^2+2e_1^2e_1^2c_1)\check{\tau}_3\check{\tau}_{16},$ (8.10)

$e_1\rho_1\Big[c_1\Big((\check{\tau}_6\check{\tau}_{15}+\check{\tau}_4)+2\check{\tau}_2\big((\check{\tau}_3\check{\tau}_{17}-\check{\tau}_{16}\eta)c_1+2\check{\tau}_3\check{\tau}_{16}\big)\Big)+\check{\tau}_4\check{\tau}_{15}\Big]<\rho_1f_1(\check{\tau}_{11}c_1+\check{\tau}_9)+$

$c_1\Big[(\check{\tau}_7\check{\tau}_{15}+\check{\tau}_5)\check{\tau}_2+\check{\tau}_1\Big(\check{\tau}_2^2c_1(c_1f_1\delta_1+\check{\tau}_{17})+(2\check{\tau}_2^2+\ e_1^2\rho_1^2c_1)(\check{\tau}_{17}c_1+\check{\tau}_{16})+\check{\tau}_{16}(\check{\tau}_2^2+$

$2e_1^2\rho_1^2c_1)\Big)\Big]+\check{\tau}_2(\check{\tau}_{15}\check{\tau}_5+\check{\tau}_1^2c_1^2),$ (8.11)

$e_1\rho_1c_1[(\check{\tau}_5\check{\tau}_{15}+\check{\tau}_1^2c_1^2)+\check{\tau}_1(\check{\tau}_1\check{\tau}_{15}c_1+4\check{\tau}_2^2\check{\tau}_{16})+2\check{\tau}_1(\check{\tau}_{17}c_1+\check{\tau}_{16})\check{\tau}_2c_1]+\frac{a_1c_1}{k_1}<\rho_1f_1(\check{\tau}_{10}c_1+$

$\check{\tau}_8)+\check{\tau}_2\check{\tau}_4\check{\tau}_{15}+c_1\big[\check{\tau}_2\big((\check{\tau}_6\check{\tau}_{15}+\check{\tau}_4)+\check{\tau}_2c_1(\check{\tau}_3\check{\tau}_{17}-\check{\tau}_{16}\eta)\big)+(2\check{\tau}_2^2+e_1^2e_1^2c_1)\check{\tau}_3\check{\tau}_{16}\big],$ (8.12)

$e_1\rho_1c_1(\check{\tau}_4\check{\tau}_{15}+2\check{\tau}_2\check{\tau}_3\check{\tau}_{16}c_1)+\delta_1c_1+d_1<c_1[\rho_1f_1(\check{\tau}_9+\check{\tau}_1^3c_1^2)+\check{\tau}_2(\check{\tau}_5\check{\tau}_{15}+\check{\tau}_1^2c_1^2)]+$

$\check{\tau}_1c_1^2[\check{\tau}_2(\check{\tau}_1\check{\tau}_{15}+\check{\tau}_2(\check{\tau}_{17}c_1+\check{\tau}_{16}))+(2\check{\tau}_2^2+e_1^2\rho_1^2c_1)\check{\tau}_{16}]+a_1,$ (8.13)

$e_1\rho_1c_1^2\check{\tau}_1(\check{\tau}_1\check{\tau}_{15}+2\check{\tau}_2\check{\tau}_{16})<\rho_1f_1\check{\tau}_8+\check{\tau}_2\check{\tau}_4\check{\tau}_{15}+\check{\tau}_2^2\check{\tau}_3\check{\tau}_{16}c_1,$ (8.14)

$a_1+d_1<c_1^2\check{\tau}_1(\rho_1f_1\check{\tau}_1^2c_1+\check{\tau}_1\check{\tau}_2\check{\tau}_{15}+\check{\tau}_2^2\check{\tau}_{16}),$ (8.15)

So, the (EP) $M_3=(\check{x}_1,\check{x}_2,0,0)$, where $\check{x}_2=x_2(\check{x}_1)$, exists if in addition to conditions (4) the following conditions hold.

$a_2-d_2>\eta\check{x}_1,$ (8.16)

$\frac{a_2}{k_2}+\delta_2>e_1\rho_1\check{x}_1,$ (8.17)

- The (EPs) $M_4=(0,\hat{x}_2,\hat{x}_s,0)$ and $M_5=\big(0,\hat{\hat{x}}_2,\hat{\hat{x}}_s,0\big)$ in the event that and only in the event that the equations that follow have a positive answer

$a_2\left(1-\frac{x_2}{k_2}\right)\left(\frac{1}{1+f_2x_s}\right)-\rho_2x_s^2-\delta_2x_2-d_2-\frac{b_2x_s}{c_2+\gamma_1x_2^2}=0,$ (9.1)

$\frac{g_2x_2}{c_2+\gamma_1x_2^2}+e_2\rho_2x_2x_s-(\delta_3+d_3)=0,$ (9.2)

From Eq. (9.2) we get:

$x_s=\frac{(c_2+\gamma_1x_2^2)(\delta_3+d_3)-g_2x_2}{e_2\rho_2x_2(c_2+\gamma_1x_2^2)},$ (9.3)

We get the following equation by putting Eq. (9.3) into Eq. (9.1):

$\hat{\alpha}_1x_2^{10}+\hat{\alpha}_2x_2^9+\hat{\alpha}_3x_2^8+\hat{\alpha}_4x_2^7+\hat{\alpha}_5x_2^6+\hat{\alpha}_6x_2^5+\hat{\alpha}_7x_2^4+\hat{\alpha}_8x_2^3+\hat{\alpha}_9x_2^2+\hat{\alpha}_{10}x_2+\hat{\alpha}_{11}=0,$ (9.4)

where

$\hat{\alpha}_1=-\gamma_1^3\left(e_2^3\rho_2^3\delta_2+\frac{\hat{a}_2}{K_2}\right)<0,$

$\hat{\alpha}_2 = \gamma_1^3(\hat{a}_2 - e_2^2\rho_2^2(e_2\rho_2 d_2 - \delta_2 f_2 \hat{\tau}_1)),$

$\hat{\alpha}_3 = -\gamma_2^2\left(3c_2\left(\frac{\hat{a}_2}{k_2} + e_2^3 e_2^3 \delta_2\right) - e_2^2\rho_2^2(\delta_2 f_2 g_2 + d_2\hat{\tau}_1\gamma_1)\right),$

$\hat{\alpha}_4 = \gamma_1^2[3c_2\hat{a}_2 - e_2\rho_2^2(\hat{\tau}_1^2\gamma_1 + e_2(3e_2\rho_2 c_2 d_2 + 3\delta_2 f_2\hat{\tau}_1 c_2 + d_2 f_2 g_2))],$

$\hat{\alpha}_5 = -\gamma_1\left[3c_2\left(\frac{\hat{a}_2}{k_2} + e_2^3\rho_2^3\delta_2 c_2\right) + \rho_2\hat{\tau}_1\gamma_1(f_2\hat{\tau}_1^2\gamma_1 + b_2 e_2^2\rho_2^2) - e_2\rho_2^2(2g_2\gamma_1\hat{\tau}_1 - f_2 e_2 c_2(2\delta_2 g_2 + 3d_2\hat{\tau}_1\gamma_1))\right],$

$\hat{\alpha}_6 = \gamma_1[3\hat{a}_2 c_2^2 + \rho_2 g_2(3f_2\hat{\tau}_1^2\gamma_1 + b_2 e_2^2\rho_2) - e_2\rho_2^2(\gamma_1(3\hat{\tau}_1^2 c_2 + b_2 e_2\hat{\tau}_1 + g_2^2) + f_2 e_2 c_2(3\delta_2\hat{\tau}_1 c_2 + 2d_2 g_2) + 3e_2^2\rho_2 c_2^2 d_2)],$

$\hat{\alpha}_7 = c_2 e_2\rho_2^2[4\hat{\tau}_1\gamma_1 g_2 + f_2 c_2 e_2(\delta_2 g_2 + 3d_2\hat{\tau}_1\gamma_1)] - [\frac{\hat{a}_2 c_2^3}{k_2} + 3\rho_2 f_2\hat{\tau}_1\gamma_1(\hat{\tau}_1^2\gamma_1 c_2 + 3g_2^2) + e_2^2\rho_2^2(e_2\rho_2 c_2^3\delta_2 + 2\hat{\tau}_1\gamma_1(b_2 c_2 + b_2 g_2))],$

$\hat{\alpha}_8 = \hat{a}_2 c_2^3 + \rho_2 f_2 g_2(4\hat{\tau}_1^2\gamma_1 + 2\hat{\tau}_1^2 c_2\gamma_1 + g_2^2) + e_2^2\rho_2^2 g_2 c_2(d_2 f_2 c_2 + b_2) - [e_2\rho_2^2\gamma_1(3\hat{\tau}_1^2 c_2^3 + g_2^2) + e_2^2\rho_2^2(e_2\rho_2 c_2^3 d_2 + \delta_2 f_2\hat{\tau}_1 c_2^3 + 2b_2\hat{\tau}_1(c_2\gamma_1 + g_2^2))],$

$\hat{\alpha}_9 = e_2\rho_2^2\hat{\tau}_1 c_2^2(2g_2 + d_2 f_2 e_2 c_2) - 3\rho_2 f_2\hat{\tau}_1 c_2(\hat{\tau}_1^2\gamma_1 c_2 + \rho_2 g_2^2) - b_2 e_2^2\rho_2^2\hat{\tau}_1 c_2(c_2 + 2g_2),$

$\hat{\alpha}_{10} = \hat{\tau}_1 c_2^2[3\rho_2 f_2\hat{\tau}_1 g_2 - e_2\rho_2^2(c_2\hat{\tau}_1 + b_2 e_2)],$

$\hat{\alpha}_{11} = -\rho_2 f_2\hat{\tau}_1^3 c_1^3,\ \hat{\tau}_1 = \delta_3 + f_3$ and $\hat{a}_2 = e_2^3\rho_2^3 a_2.$

The Descartes rule states that Eq. (9.4) either does not have any roots or has two positive roots $\{\hat{x}_2, \hat{\hat{x}}_2\}$ depending on the following conditions:

$$\hat{a}_2 < e_2^2\rho_2^2(e_2\rho_2 d_2 + \delta_2 f_2\hat{\tau}_1), \tag{9.5}$$

$$3c_2\left(\frac{\hat{a}_2}{k_2} + e_2^3 e_2^3\delta_2\right) > e_2^2\rho_2^2(\delta_2 f_2 g_2 + d_2\hat{\tau}_1\gamma_1), \tag{9.6}$$

$$3c_2\hat{a}_2 < e_2\rho_2^2[\hat{\tau}_1^2\gamma_1 + e_2(3e_2\rho_2 c_2 d_2 + 3\delta_2 f_2\hat{\tau}_1 c_2 + d_2 f_2 g_2)], \tag{9.7}$$

$$3c_2\left(\frac{\hat{a}_2}{k_2} + e_2^3\rho_2^3\delta_2 c_2\right) + \rho_2\hat{\tau}_1\gamma_1(f_2\hat{\tau}_1^2\gamma_1 + b_2 e_2^2\rho_2) > e_2\rho_2^2[2g_2\gamma_1\hat{\tau}_1 + f_2 e_2 c_2(2\delta_2 g_2 + 3d_2\hat{\tau}_1\gamma_1)], \tag{9.8}$$

$$3\hat{a}_2 c_2^2 + \rho_2 g_2(3f_2\hat{\tau}_1^2\gamma_1 + b_2 e_2^2\rho_2) < e_2\rho_2^2[\gamma_1(3\hat{\tau}_1^2 c_2 + b_2 e_2\hat{\tau}_1 + g_2^2) + f_2 e_2 c_2(3\delta_2\hat{\tau}_1 c_2 + 2d_2 g_2) + 3e_2^2\rho_2 c_2^2 d_2], \tag{9.9}$$

$$c_2 e_2\rho_2^2[4\hat{\tau}_1\gamma_1 g_2 + f_2 c_2 e_2(\delta_2 g_2 + 3d_2\hat{\tau}_1\gamma_1)] < \frac{\hat{a}_2 c_2^3}{k_2} + 3\rho_2 f_2\hat{\tau}_1\gamma_1(\hat{\tau}_1^2\gamma_1 c_2 + 3g_2^2) + +e_2^2\rho_2^2(e_2\rho_2 c_2^3\delta_2 + 2\hat{\tau}_1\gamma_1(b_2 c_2 + b_2 g_2)), \tag{9.10}$$

$$\hat{a}_2 c_2^3 + \rho_2 f_2 g_2(4\hat{\tau}_1^2\gamma_1 + 2\hat{\tau}_1^2 c_2\gamma_1 + g_2^2) + e_2^2\rho_2^2 g_2 c_2(d_2 f_2 c_2 + b_2) < e_2\rho_2^2\gamma_1(3\hat{\tau}_1^2 c_2^3 + g_2^2) + e_2^2\rho_2^2(e_2\rho_2 c_2^3 d_2 + \delta_2 f_2\hat{\tau}_1 c_2^3 + 2b_2\hat{\tau}_1(c_2\gamma_1 + g_2^2)), \tag{9.11}$$

$$e_2\rho_2^2\hat{\tau}_1 c_2^2(2g_2 + d_2 f_2 e_2 c_2) < 3\rho_2 f_2\hat{\tau}_1 c_2(\hat{\tau}_1^2\gamma_1 c_2 + \rho_2 g_2^2) + b_2 e_2^2\rho_2^2\hat{\tau}_1 c_2(c_2 + 2g_2), \tag{9.12}$$

$$e_2\rho_2^2(c_2\hat{\tau}_1 + b_2 e_2) < 3\rho_2 f_2\hat{\tau}_1 g_2. \tag{9.13}$$

And same way with $\hat{\hat{x}}_2$ we get similar result so $M_4 = (0, \hat{x}_2, \hat{x}_s, 0)$ and $M_5 = \left(0, \hat{\hat{x}}_2, \hat{\hat{x}}_s, 0\right)$ where

$\hat{x}_s = x_s(\hat{x}_2)$ and $\hat{\hat{x}}_s = x_s(\hat{\hat{x}}_2)$ exist proved that

$$g_2\hat{x}_2 < (c_2 + \gamma_1\hat{x}_2^2)(\delta_3 + d_3). \tag{9.14}$$

- The (EPs) $M_6 = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_s, 0)$ and $M_7 = (\tilde{\tilde{x}}_1, \tilde{\tilde{x}}_2, \tilde{\tilde{x}}_s, 0)$ in the event that and only in the event that the equations that follow have a positive answer

$$a_1\left(1 - \frac{x_1}{k_1}\right)\left(\frac{1}{1+f_1x_2}\right) - \frac{b_1x_2}{c_1+x_1^2} - \rho_1x_2^2 - \delta_1x_1^2 - d_1 = 0, \tag{10.1}$$

$$a_2\left(1 - \frac{x_2}{k_2}\right)\left(\frac{1}{1+f_2x_s}\right) + \frac{g_1x_1}{c_1+x_1^2} - \eta x_1 + e_1\rho_1x_1x_2 - \rho_2x_s^2 - \delta_2x_2 - d_2 - \frac{b_2x_s}{c_2+\gamma_1x_2^2} = 0, \tag{10.2}$$

$$\frac{g_2x_2}{c_2+\gamma_1x_2^2} + e_2\rho_2x_2x_s - (\delta_3 + d_3) = 0, \tag{10.3}$$

From Eq. (10.3) we get:

$$x_s = \frac{(c_2+\gamma_1x_2^2)(\delta_3+d_3) - g_2x_2}{e_2\rho_2(c_2+\gamma_1x_2^2)x_2}, \tag{10.4}$$

by substituting Eq. (10.4) in Eq. (10.2), after multiple Eq. (10.2) by $(1 + f_2x_s)$ we obtain the following equations:

$$L_1(x_1, x_2) = a_1\left(1 - \frac{x_1}{k_1}\right)\left(\frac{1}{1+f_1x_2}\right) - \frac{b_1x_2}{c_1+x_1^2} - \rho_1x_2^2 - \delta_1x_1^2 - d_1 = 0, \tag{10.5}$$

$$\begin{aligned} L_2(x_1, x_2) = {} & a_2\left(1 - \frac{x_2}{k_2}\right) + \frac{g_1x_1}{c_1+x_1^2}\left(1 + f_2\frac{(c_2+\gamma_1x_2^2)(\delta_3+d_3) - g_2x_2}{e_2\rho_2(c_2+\gamma_1x_2^2)x_2}\right) - (\eta x_1 - e_1\rho_1x_1x_2 + \delta_2x_2 + \\ & d_2)\left(1 + f_2\frac{(c_2+\gamma_1x_2^2)(\delta_3+d_3) - g_2x_2}{e_2\rho_2(c_2+\gamma_1x_2^2)x_2}\right) - \left(\rho_2\left(\frac{(c_2+\gamma_1x_2^2)(\delta_3+d_3) - g_2x_2}{e_2\rho_2(c_2+\gamma_1x_2^2)x_2}\right)^2 + \right. \\ & \left. \frac{b_2}{c_2+\gamma_1x_2^2}\left(\frac{(c_2+\gamma_1x_2^2)(\delta_3+d_3) - g_2x_2}{e_2\rho_2(c_2+\gamma_1x_2^2)x_2}\right)\right)\left(1 + f_2\frac{(c_2+\gamma_1x_2^2)(\delta_3+d_3) - g_2x_2}{e_2\rho_2(c_2+\gamma_1x_2^2)x_2}\right) = 0. \end{aligned} \tag{10.6}$$

Subsequent to Eq. (10.5) we notice that, when $x_1 \to 0$ then $x_2 \to \tilde{x}_2$ where $\tilde{x}_2$ is a positive root of the equation

$$\tilde{\alpha}_1x_2^3 + \tilde{\alpha}_2x_2^2 + \tilde{\alpha}_3x_2 + \tilde{\alpha}_4 = 0, \tag{10.7}$$

where

$$\tilde{\alpha}_1 = -\rho_1f_1,$$

$$\tilde{\alpha}_2 = -(b_1f_1 + \rho_1),$$

$$\tilde{\alpha}_3 = -(b_1 + d_1f_1),$$

$$\tilde{\alpha}_4 = a_1 - d_1.$$

In accordance with Descartes' rule and condition (3), Eq. (10.7) possesses only one positive root say $\{\tilde{x}_2\}$.

More over from Eq. (10.5), $\frac{dx_1}{dx_2} = -\left(\frac{\frac{dL_1}{dx_2}}{\frac{dL_1}{dx_1}}\right)$, so $\frac{dx_1}{dx_2} < 0$, if following set condition hold:

$\frac{dL_1}{dx_1} > 0,\ \frac{dL_1}{dx_2} > 0$ or $\frac{dL_1}{dx_1} < 0,\ \frac{dL_1}{dx_2} < 0.$

Now, from Eq. (10.6) we notice that, when $x_1 \to 0$, then $x_2 \to \tilde{x}_2'$ or $x_2 \to \tilde{\tilde{x}}_2'$ where $\tilde{x}_2'$ and $\tilde{\tilde{x}}_2'$ are a positive root of the Eq. (9.4).

The Descartes rule states that Eq. (9.4) either does not have any roots or has two positive roots $\{\tilde{x}_2', \tilde{\tilde{x}}_2'\}$ depending on the condition (9.5-9.13) holds.

More over from Eq. (10.6), $\frac{dx_1}{dx_2} = -\left(\frac{\frac{dL_2}{dx_2}}{\frac{dL_2}{dx_1}}\right)$, so $\frac{dx_1}{dx_2} > 0$, if following set condition hold:

$\frac{dL_2}{dx_1} < 0,\ \frac{dL_2}{dx_2} > 0$ or $\frac{dL_2}{dx_1} > 0,\ \frac{dL_2}{dx_2} < 0.$

The two isoclines (10.5) and (10.6) intersect at positive point $(\tilde{x}_1, \tilde{x}_2)$ in $\text{Int.}R_+^2$ of x_1x_2-space. So, the (EPs) $M_6 = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_s, 0)$ and $M_7 = (\tilde{\tilde{x}}_1, \tilde{\tilde{x}}_2, \tilde{\tilde{x}}_s, 0)$ exists where $\tilde{x}_s = x_s(\tilde{x}_2)$ and $\tilde{\tilde{x}}_s = x_s(\tilde{\tilde{x}}_2)$ if in addition to above conditions with

$$\tilde{\tilde{x}}_2 < \tilde{x}_2, \tag{10.8}$$

$$g_2\tilde{x}_2 < (c_2 + \gamma_1\tilde{x}_2^2)(\delta_3 + d_3). \tag{10.9}$$

- The (EP) $M_8 = (0, \check{x}_2, \check{x}_s, \check{x}_I)$ in the event that and only in the event that the equations that follow have a positive answer

$$a_2\left(1 - \frac{x_2}{k_2}\right)\left(\frac{1}{1+f_2x_s}\right) - \rho_2x_s^2 - \delta_2x_2 - d_2 - \frac{b_2x_s}{c_2+\gamma_1x_2^2} = 0, \tag{11.1}$$

$$\frac{g_2x_2}{c_2+\gamma_1x_2^2} + e_2\rho_2x_2x_s - \beta x_I - (\delta_3 + d_3) = 0, \tag{11.2}$$

$$\beta x_s - d_4 = 0. \tag{11.3}$$

From Eq. (11.3) we get: $\check{x}_s = \frac{d_4}{\beta}$, (11.4)

We get the following equation by putting Eq. (11.4) into Eq. (11.1):

$$\check{\alpha}_1x_2^3 + \check{\alpha}_2x_2^2 + \check{\alpha}_3x_2 + \check{\alpha}_4 = 0, \tag{11.5}$$

where

$$\check{\alpha}_1 = -\gamma_1\left(\frac{\check{\tau}_1a_2}{k_2} + \delta_2\right),$$

$$\check{\alpha}_2 = \gamma_1(\check{\tau}_1a_2 - \check{\tau}_2 - d_2),$$

$$\check{\alpha}_3 = -c_2\left(\frac{a_2\check{\tau}_1}{k_2} + \delta_2\right),$$

$\breve{\alpha}_4 = a_2 c_2 \breve{\tau}_1 - [c_2(\breve{\tau}_2 + d_2) + \breve{\tau}_3]$,

$\breve{\tau}_1 = \frac{1}{1+f_2\left(\frac{d_4}{\beta}\right)}$, $\quad \breve{\tau}_2 = \rho_2 \left(\frac{d_4}{\beta}\right)^2$, $\quad \breve{\tau}_3 = b_2 \left(\frac{d_4}{\beta}\right)$,

in accordance with Descartes' rule, Eq. (11.5) possesses only one positive root say $\{\breve{x}_2\}$, if the following conditions hold

$$\breve{\tau}_1 a_2 < \breve{\tau}_2 + d_2, \tag{11.6}$$

$$a_2 c_2 \breve{\tau}_1 > c_2(\breve{\tau}_2 + d_2) + \breve{\tau}_3. \tag{11.7}$$

Now substituting Eq. (11.4) and $\breve{x}_2$ in Eq. (11.2), we obtain:

$\breve{x}_I = \frac{\breve{\tau}_4 + \breve{\tau}_5 - (\delta_3 + d_3)}{\beta}$, where

$\breve{\tau}_4 = \frac{g_2 \breve{x}_2}{c_2 + \gamma_1 \breve{x}_2^2}, \breve{\tau}_5 = e_2 \rho_2 \breve{x}_2 \left(\frac{d_4}{\beta}\right)$,

so, the (EPs) $M_8 = (0, \breve{x}_2, \breve{x}_s, \breve{x}_I)$ where $\breve{x}_I = x_I(\breve{x}_2, \breve{x}_s)$ exist if in addition to conditions(11.6-11.7) the next condition holds:

$$\breve{\tau}_4 + \breve{\tau}_5 > \delta_3 + d_3, \tag{11.8}$$

- The (EP) $M_9 = (\dot{x}_1, \dot{x}_2, \dot{x}_s, \dot{x}_I), M_{10} = (\dot{x}_1, \ddot{x}_2, \dot{x}_s, \ddot{x}_I)$ and $M_{11} = (\dot{x}_1, \dddot{x}_2, \dot{x}_s, \dddot{x}_2)$ in the event that and only in the event that the equations that follow have a positive answer

$$a_1 \left(1 - \frac{x_1}{k_1}\right)\left(\frac{1}{1+f_1 x_2}\right) - \frac{b_1 x_2}{c_1 + x_1^2} - \rho_1 x_2^2 - \delta_1 x_1^2 - d_1 = 0, \tag{12.1}$$

$$a_2 \left(1 - \frac{x_2}{k_2}\right)\left(\frac{1}{1+f_2 x_s}\right) + \frac{g_1 x_1}{c_1 + x_1^2} - \eta x_1 + e_1 \rho_1 x_1 x_2 - \rho_2 x_s^2 - \delta_2 x_2 - d_2 - \frac{b_2 x_s}{c_2 + \gamma_1 x_2^2} = 0, \tag{12.2}$$

$$\frac{g_2 x_2}{c_2 + \gamma_1 x_2^2} + e_2 \rho_2 x_2 x_s - \beta x_I - (\delta_3 + d_3) = 0, \tag{12.3}$$

$$\beta x_s - d_4 = 0. \tag{12.4}$$

From Eq. (12.4) we get:

$$\dot{x}_s = \frac{d_4}{\beta}, \tag{12.5}$$

We get the following equation by putting Eq. (12.5) into Eq. (12.2):

$$P_1(x_1, x_2) = a_1 \left(1 - \frac{x_1}{k_1}\right)\left(\frac{1}{1+f_1 x_2}\right) - \frac{b_1 x_2}{c_1 + x_1^2} - \rho_1 x_2^2 - \delta_1 x_1^2 - d_1 = 0, \tag{12.6}$$

$$P_2(x_1, x_2) = \dot{\tau}_1 \left(1 - \frac{x_2}{k_2}\right) + \frac{g_1 x_1}{c_1 + x_1^2} - \eta x_1 + e_1 \rho_1 x_1 x_2 - \dot{\tau}_2 - \delta_2 x_2 - d_2 - \frac{\dot{\tau}_3}{c_2 + \gamma_1 x_2^2} = 0, \tag{12.7}$$

where $\dot{\tau}_1 = \frac{a_2}{1+f_2\left(\frac{d_4}{\beta}\right)}$, $\dot{\tau}_2 = \rho_2 \left(\frac{d_4}{\beta}\right)^2$ and $\dot{\tau}_3 = \frac{b_2 d_4}{\beta}$.

Subsequent to Eq. (12.6) we notice that, when $x_1 \to 0$ then $x_2 \to \dot{x}_2$ where $\dot{x}_2$ is a positive unique root of the equation

$$\dot{\alpha}_1 x_2^3 + \dot{\alpha}_2 x_2^2 + \dot{\alpha}_3 x_2 + \dot{\alpha}_4 = 0, \tag{12.8}$$

where

$$\dot{\alpha}_1 = -\rho_1 f_1,$$

$$\dot{\alpha}_2 = -\left(\frac{b_1 f_1}{c_1} + \rho_2\right),$$

$$\dot{\alpha}_3 = -\left(\frac{b_1}{c_1} + d_1 f_1\right),$$

$$\dot{\alpha}_4 = a_1 - d_1,$$

in accordance with Descartes' rule with condition (3), Eq. (12.8) possesses only one positive root say $\{\dot{x}_2\}$.

More over from Eq. (12.6), $\frac{dx_1}{dx_2} = -\left(\frac{\frac{dP_1}{dx_2}}{\frac{dP_1}{dx_1}}\right)$, so $\frac{dx_1}{dx_2} > 0,$ if following set of conditions hold:

$\frac{dP_1}{dx_1} > 0,$ $\frac{dP_1}{dx_2} < 0$ or $\frac{dP_1}{dx_1} < 0,$ $\frac{dP_1}{dx_2} > 0.$

Now, from Eq. (12.7) we notice that, when $x_1 \to 0,$ then $x_2 \to \dot{x}_2',$ $x_2 \to \ddot{x}_2'$ and $x_2 \to \dddot{x}_2'$ where $\dot{x}_2',$ $\ddot{x}_2'$ and $\dddot{x}_2'$ are a positive root of the equation.

$$\dot{\alpha}_1' x_2^3 + \dot{\alpha}_2' x_2^2 + \dot{\alpha}_3' x_2 + \dot{\alpha}_4' = 0, \tag{12.9}$$

where

$$\dot{\alpha}_1' = -\gamma_1 \left(\frac{\dot{\tau}_1}{k_2} + \delta_2\right),$$

$$\dot{\alpha}_2' = \gamma_1(\dot{\tau}_1 - \dot{\tau}_2 - d_2),$$

$$\dot{\alpha}_3' = -c_2 \left(\frac{\dot{\tau}_1}{k_2} + \delta_2\right),$$

$$\dot{\alpha}_4' = c_2\dot{\tau}_1 - c_2(\dot{\tau}_2 + d_2) - \dot{\tau}_3,$$

in accordance with Descartes' rule, Eq. (12.9) possesses one positive root or three positive roots say $\{\dot{x}_2', \ddot{x}_2', \dddot{x}_2'\}$, if the flowing conditions hold.

$$\dot{\tau}_1 > \dot{\tau}_2 + d_2, \tag{12.10}$$

$$c_2\dot{\tau}_1 > c_2(\dot{\tau}_2 + d_2) + \dot{\tau}_3. \tag{12.11}$$

So, from Eq. (12.7), $\frac{dx_1}{dx_2} = -\left(\frac{\frac{dP_2}{dx_2}}{\frac{dP_2}{dx_1}}\right)$, so $\frac{dx_1}{dx_2} < 0,$ if following set condition holds:

$\frac{dP_2}{dx_1} < 0,$ $\frac{dP_2}{dx_2} < 0$ or $\frac{dP_2}{dx_1} > 0,$ $\frac{dP_2}{dx_2} > 0.$

The two isoclines (12.6) and (12.7) intersect at positive point $(\dot{x}_1, \dot{x}_2')$ or $(\dot{x}_1, \ddot{x}_2')$ or $(\dot{x}_1, \dddot{x}_2')$ in $\text{Int.} R_+^2$ of x_1x_2-space.

Now substituting Eq. (12.5) and $\dot{x}_2$ in Eq. (12.3), we obtain:

$$\dot{x}_I = \frac{1}{\beta}\left[\frac{g_2\dot{x}_2}{c_2+\gamma_1\dot{x}_2^2} + e_2\rho_2\dot{x}_2\left(\frac{d_4}{\beta}\right) - (\delta_3 + d_3)\right], \tag{12.12}$$

So, the (EPs) M_9, M_{10} and M_{11} exists where $\dot{x}_I = x_I(\dot{x}_2, \dot{x}_s)$ if in addition to the above condition with the next conditions hold:

$$\frac{g_2\dot{x}_2}{c_2+\gamma_1\dot{x}_2^2} + e_2\rho_2\dot{x}_2\left(\frac{d_4}{\beta}\right) > \delta_3 + d_3, \tag{12.13}$$

$$\dot{x}_2 < \dot{x}_2'. \tag{12.14}$$

5. LOCAL STABILITY ANALYSIS (LSA)

Through the process of constructing the Jacobian matrix, the localized stability area (LSA) surrounding each of the equilibrium sites is addressed in this section. (JM), $J(x_1, x_2, x_s, x_I)$ after which the eigenvalues for the system (1) were discovered at all of them. Be aware that we made use of the symbols. $\lambda_{lx_1}, \lambda_{lx_2}, \lambda_{lx_s}$ and λ_{lx_I} to represent the eigenvalues of (JM) J_l; $l = 0, \dots, 11$ where the matrix of Jacobian $J(x_1, x_2, x_s, x_I)$ of the system (1) for all of them may be stated as follows: that describes the dynamics along the x_1-direction, the x_2-direction, the x_s-direction and the x_I-direction correspondingly

$$J = [n_{ij}]_{4\times4}, \tag{13.1}$$

$$n_{11} = a_1\left(1 - \frac{2x_1}{k_1}\right)\left(\frac{1}{1+f_1x_2}\right) - \frac{b_1x_2(c_1 - x_1^2)}{(c_1+x_1^2)^2} - \rho_1x_2^2 - 3\delta_1x_1^2 - d_1,$$

$$n_{12} = a_1x_1\left(1 - \frac{x_1}{k_1}\right)\left(\frac{-f_1}{(1+f_1x_2)^2}\right) - \frac{b_1x_1}{c_1+x_1^2} - 2\rho_1x_1x_2, \quad n_{13} = 0, \quad n_{14} = 0,$$

$$n_{21} = \frac{g_1x_2(c_1 - x_1^2)}{(c_1+x_1^2)^2} - \eta x_2 + e_1\rho_1x_2^2,$$

$$n_{22} = a_2\left(1 - \frac{2x_2}{k_2}\right)\left(\frac{1}{1+f_2x_s}\right) + \frac{g_1x_1}{c_1+x_1^2} - \eta x_1 + 2e_1\rho_1x_1x_2 - \rho_2x_s^2 - 2\delta_2x_2 - d_2 - \frac{b_2x_s(c_2-\gamma_1x_2^2)}{(c_2+\gamma_1x_2^2)^2},$$

$$n_{23} = a_2x_2\left(1 - \frac{x_2}{k_2}\right)\left(\frac{-f_2}{(1+f_2x_s)^2}\right) - 2\rho_2x_2x_s - \frac{b_2x_2}{(c_2+\gamma_1x_2^2)}, \quad n_{24} = 0,$$

$$n_{31} = 0, \quad n_{32} = \frac{g_2x_s(c_2-\gamma_1x_2^2)}{(c_2+\gamma_1x_2^2)^2} + e_2\rho_2x_s^2,$$

$$n_{33} = \frac{g_2x_2}{(c_2+\gamma_1x_2^2)} + 2e_2\rho_2x_2x_s - \beta x_I - (\delta_3 + d_3), \quad n_{34} = -\beta x_s,$$

$n_{41} = 0, \quad n_{42} = 0, \quad n_{43} = \beta x_I, \quad n_{44} = \beta x_s - d_4.$

- It is simplest to prove that the (JM) of system (1) at $M_0 = (0,0,0,0)$ it could be expressed as:

$$J_0 = J(M_0) = \begin{bmatrix} a_1 - d_1 & 0 & 0 & 0 \\ 0 & a_2 - d_2 & 0 & 0 \\ 0 & 0 & -(\delta_3 + d_3) & 0 \\ 0 & 0 & 0 & -d_4 \end{bmatrix}. \tag{14.1}$$

After that, the equation that is distinctive of J_0 is given by:

$$(a_1 - d_1 - \lambda)(a_2 - d_2 - \lambda)(-(\delta_3 + d_3) - \lambda)(-d_4 - \lambda) = 0,$$

so the eigenvalues of J_0 are

$$\lambda_{0x_1} = a_1 - d_1, \quad \lambda_{0x_2} = a_2 - d_2, \quad \lambda_{0x_s} = -(\delta_3 + d_3), \text{ and } \lambda_{0x_I} = -d_4$$

Assuming conditions (3) and (4) are true, the equilibrium point M_0 is (LAS) in the R_+^4, but it is unstable in all other cases.

- The (JM) of system (1) at $M_1 = (\bar{x}_1, 0, 0, 0)$ can be written as

$$J_1 = J(M_1) = \begin{bmatrix} a_1\left(1 - \frac{2\bar{x}_1}{k_1}\right) - 3\delta_1 \bar{x}_1^2 - d_1 & a_1 f_1 \bar{x}_1\left(\frac{\bar{x}_1}{k_1} - 1\right) - \frac{b_1\bar{x}_1}{c_1 + \bar{x}_1^2} & 0 & 0 \\ 0 & a_2 + \frac{g_1\bar{x}_1}{c_1 + \bar{x}_1^2} - \eta\bar{x}_1 - d_2 & 0 & 0 \\ 0 & 0 & -(\delta_3 + d_3) & 0 \\ 0 & 0 & 0 & -d_4 \end{bmatrix}, \tag{15.1}$$

Therefore, the equation that is typical of J_1 comes in by

$$\left(a_1\left(1 - \frac{2\bar{x}_1}{k_1}\right) - 3\delta_1\bar{x}_1^2 - d_1 - \lambda\right)\left(a_1 f_1 \bar{x}_1\left(\frac{\bar{x}_1}{k_1} - 1\right) - \frac{b_1\bar{x}_1}{c_1 + \bar{x}_1^2} - \lambda\right)(-\delta_3 - d_3 - \lambda)(-d_4 - \lambda) = 0,$$

Then, J_1 has the following eigenvalues:

$$\lambda_{1x_1} = a_1\left(1 - \frac{2\bar{x}_1}{k_1}\right) - 3\delta_1\bar{x}_1^2 - d_1,$$

$$\lambda_{1x_2} = a_2 + \frac{g_1\bar{x}_1}{c_1 + \bar{x}_1^2} - \eta\bar{x}_1 - d_2,$$

$$\lambda_{1x_s} = -(\delta_3 + d_3) \quad \text{and} \quad \lambda_{1x_I} = -d_4.$$

If the next subsequent statements are true,

$$a_1 < \frac{2a_1\bar{x}_1}{k_1} + 3\delta_1\bar{x}_1^2 + d_1, \tag{15.2}$$

$$a_2 + \frac{g_1\bar{x}_1}{c_1 + \bar{x}_1^2} < \eta\bar{x}_1 + d_2, \tag{15.3}$$

then M_1 is (LAS) otherwise M_1 is unstable.

- The (JM) of system (1) at $M_2 = (0, \bar{\bar{x}}_2, 0, 0)$ can be written as

$$J_2 = J(M_2) = [m_{ij}]_{4\times 4}, \tag{16.1}$$

where $m_{11}=\frac{a_1}{1+f_1\bar{\bar{x}}_2}-\frac{b_1\bar{\bar{x}}_2}{c_1}-\rho_1\bar{\bar{x}}_2^2-d_1$, $m_{12}=m_{13}=m_{14}=0$,

$m_{21}=\frac{g_1\bar{\bar{x}}_2}{c_1}-\eta\bar{\bar{x}}_2+e_1\rho_1\bar{\bar{x}}_2^2$,

$m_{22}=a_2\left(1-\frac{2\bar{\bar{x}}_2}{k_2}\right)-2\delta_2\bar{\bar{x}}_2-d_2$,

$m_{23}=a_2f_2\bar{\bar{x}}_2\left(\frac{\bar{\bar{x}}_2}{k_2}-1\right)-\frac{b_2\bar{\bar{x}}_2}{c_2+\gamma_1\bar{\bar{x}}_2^2}$, $m_{24}=0$, $m_{31}=m_{32}=0$,

$m_{33}=\frac{g_2\bar{\bar{x}}_2}{(c_2+\gamma_1\bar{\bar{x}}_2^2)}-(\delta_3+d_3)$, $m_{34}=0$, $m_{41}=m_{42}=m_{43}=0$, $m_{44}=-d_4$.

After that, the equation that is distinctive of J_2 is given by

$(m_{11}-\lambda)(m_{22}-\lambda)(m_{33}-\lambda)(m_{44}-\lambda)=0$,

Then, J_2 has the following eigenvalues:

$\lambda_{2x_1}=\frac{a_1}{1+f_1\bar{\bar{x}}_2}-\frac{b_1\bar{\bar{x}}_2}{c_1}-\rho_1\bar{\bar{x}}_2^2-d_1$,

$\lambda_{2x_2}=a_2\left(1-\frac{2\bar{\bar{x}}_2}{k_2}\right)-2\delta_2\bar{\bar{x}}_2-d_2$,

$\lambda_{2x_S}=\frac{g_2\bar{\bar{x}}_2}{(c_2+\gamma_1\bar{\bar{x}}_2^2)}-(\delta_3+d_3)$ and $\lambda_{2x_I}=-d_4$.

If the next subsequent statements are true,

$$\frac{a_1}{1+f_1\bar{\bar{x}}_2}<\frac{b_1\bar{\bar{x}}_2}{c_1}+\rho_1\bar{\bar{x}}_2^2+d_1, \tag{16.2}$$

$$a_2<\frac{2a_2\bar{\bar{x}}_2}{k_2}+2\delta_2\bar{\bar{x}}_2+d_2, \tag{16.3}$$

$$\frac{g_2\bar{\bar{x}}_2}{c_2+\gamma_1\bar{\bar{x}}_2^2}<\delta_3+d_3, \tag{16.4}$$

then M_2 is (LAS) otherwise M_2 is unstable.

- The (JM) of system (2) at $M_3=(\breve{x}_1,\breve{x}_2,0,0)$, can be written as:

$$J_3=J(M_3)=[p_{ij}]_{4\times4}, \tag{17.1}$$

where $p_{11}=a_1\left(1-\frac{2\breve{x}_1}{k_1}\right)\left(\frac{1}{1+f_1\breve{x}_2}\right)-\frac{b_1\breve{x}_2(c_1-\breve{x}_1^2)}{(c_1+\breve{x}_1^2)^2}-\rho_1\breve{x}_2^2-3\delta_1\breve{x}_1^2-d_1$

$p_{12}=a_1\breve{x}_1\left(\frac{\breve{x}_1}{k_1}-1\right)\left(\frac{f_1}{(1+f_1\breve{x}_2)^2}\right)-\frac{b_1\breve{x}_1}{c_1+\breve{x}_1^2}-2\rho_1\breve{x}_1\breve{x}_2$, $n_{13}=n_{14}=0$,

$p_{21}=\frac{g_1\breve{x}_2(c_1-\breve{x}_1^2)}{(c_1+\breve{x}_1^2)^2}-\eta\breve{x}_2+e_1\rho_1\breve{x}_2^2$,

$p_{22}=a_2\left(1-\frac{2\breve{x}_2}{k_2}\right)+\frac{g_1\breve{x}_1}{c_1+\breve{x}_1^2}-\eta\breve{x}_1+2e_1\rho_1\breve{x}_1\breve{x}_2-2\delta_2\breve{x}_2-d_2$,

$p_{23}=a_2f_2\breve{x}_2\left(\frac{\breve{x}_2}{k_2}-1\right)-\frac{b_2\breve{x}_2}{(c_2+\gamma_1\breve{x}_2^2)}$, $p_{24}=p_{31}=p_{32}=0$,

$p_{33} = \frac{g_2 \check{x}_2}{(c_2+\gamma_1 \check{x}_2^2)} - (\delta_3 + d_3), \qquad p_{34} = 0,$

$p_{41} = p_{42} = p_{43} = 0, \qquad p_{44} = -d_4.$

After that, the equation that is distinctive of J_3 is

$\left(\lambda^2 - (p_{11}+p_{22})\lambda + (p_{11}p_{22} - p_{12}p_{21})\right)(p_{33} - \lambda)(p_{44} - \lambda) = 0,$

either $(p_{33} - \lambda) = 0 \Longrightarrow \lambda_{3x_s} = p_{33} = \left(\frac{g_2 \check{x}_2}{c_2+\gamma_1 \check{x}_2^2} - \delta_3 - d_3\right)$ or $(p_{44} - \lambda) = 0$ $\Longrightarrow \lambda_{3x_I} = p_{44} = -d_4 < 0$, or $\left(\lambda^2 - (p_{11}+p_{22})\lambda + (p_{11}p_{22} - p_{12}p_{21})\right) = 0$, that provides

$\lambda_{3x_1} + \lambda_{3x_2} = p_{11} + p_{22}$ and $\lambda_{3x_1} \cdot \lambda_{3x_2} = p_{11}p_{22} - p_{12}p_{21}.$

So, $\lambda_{3x_1}, \lambda_{3x_2}$ and λ_{3x_s} are negative if the following conditions hold:

$$\frac{g_2 \check{x}_2}{c_2+\gamma_1 \check{x}_2^2} < \delta_3 + d_3, \tag{17.2}$$

$$c_1 > \check{x}_1^2, \tag{17.3}$$

$$\check{x}_1 < \frac{k_1}{2}, \tag{17.4}$$

$$a_1\left(1 - \frac{2\check{x}_1}{k_1}\right)\left(\frac{1}{1+f_1\check{x}_2}\right) < \frac{b_1\check{x}_2(c_1 - \check{x}_1^2)}{(c_1+\check{x}_1^2)^2} + \rho_1\check{x}_2^2 + 3\delta_1\check{x}_1^2 + d_1, \tag{17.5}$$

$$\check{x}_2 < \frac{k_2}{2}, \tag{17.6}$$

$$a_2\left(1 - \frac{2\check{x}_2}{k_2}\right) + \frac{g_1\check{x}_1}{c_1+\check{x}_1^2} + 2e_1\rho_1\check{x}_1\check{x}_2 < \eta\check{x}_1 + 2\delta_2\check{x}_2 + d_2, \tag{17.7}$$

$$\frac{g_1\check{x}_2(c_1-\check{x}_1^2)}{(c_1+\check{x}_1^2)^2} + e_1\rho_1\check{x}_2^2 > \eta\check{x}_2, \tag{17.8}$$

then M_3 is (LAS) otherwise M_3 is unstable.

- The (JM) of system (2) at $M_4 = (0, \hat{x}_2, \hat{x}_s, 0)$ can be written as:

$$J_4 = J(M_4) = [q_{ij}]_{4\times 4}, \tag{18.1}$$

where $q_{11} = \frac{a_1}{1+f_1\hat{x}_2} - \frac{b_1\hat{x}_2}{c_1} - \rho_1\hat{x}_2^2 - d_1, \qquad q_{12} = q_{13} = q_{14} = 0,$

$q_{21} = \frac{g_1\hat{x}_2}{c_1} - \eta\hat{x}_2 + e_1\rho_1\hat{x}_2^2,$

$q_{22} = a_2\left(1 - \frac{2\hat{x}_2}{k_2}\right)\left(\frac{1}{1+f_2\hat{x}_s}\right) - \rho_2\hat{x}_s^2 - 2\delta_2\hat{x}_2 - d_2 - \frac{b_2\hat{x}_s(c_2-\gamma_1\hat{x}_2^2)}{(c_2+\gamma_1\hat{x}_2^2)^2},$

$q_{23} = \left(\frac{\hat{x}_2}{k_2} - 1\right)\left(\frac{a_2 f_2\hat{x}_2}{(1+f_2\hat{x}_s)^2}\right) - 2\rho_2\hat{x}_2\hat{x}_s - b_2\frac{\hat{x}_2}{c_2+\gamma_1\hat{x}_2^2}, \quad q_{24} = 0, \; q_{31} = 0,$

$$q_{32} = \frac{g_2\hat{x}_s(c_2-\gamma_1\hat{x}_2^2)}{(c_2+\gamma_1\hat{x}_2^2)^2} + e_2\rho_2\hat{x}_s^2,$$

$$q_{33} = \frac{g_2\hat{x}_2}{(c_2+\gamma_1\hat{x}_2^2)} + 2e_2\rho_2\hat{x}_2\hat{x}_s - (\delta_3+d_3), \qquad q_{34} = -\beta\hat{x}_s,$$

$$q_{41} = q_{42} = q_{43} = 0, \qquad q_{44} = \beta\hat{x}_s - d_4.$$

After that, the equation that is distinctive of J_4 is

$$(q_{11}-\lambda)(q_{44}-\lambda)\big(\lambda^2-(q_{22}+q_{33})\lambda+(p_{22}p_{33}-p_{23}p_{32})\big)=0,$$

either $\left(q_{11}-\lambda\right)=0 \Longrightarrow \lambda_{4x_1} = q_{11} = \frac{a_1}{1+f_1\hat{x}_2} - \frac{b_1\hat{x}_2}{c_1} - \rho_1\hat{x}_2^2 - d_1,$

or $\left(q_{44}-\lambda\right)=0 \Longrightarrow \lambda_{4x_I} = q_{44} = \beta\hat{x}_s - d_4 < 0,$

or $\left(\lambda^2-(q_{22}+q_{33})\lambda+(q_{22}q_{33}-q_{23}q_{32})\right)=0$, that provides

$\lambda_{4x_2}+\lambda_{4x_s} = q_{22}+q_{33}$ and $\lambda_{4x_2}\cdot\lambda_{4x_s} = q_{22}q_{33}-q_{23}q_{32}$.

So, $\lambda_{4x_1}, \lambda_{4x_2}, \lambda_{4x_s}$ and λ_{4x_I} are negative if the following conditions hold:

$$\frac{a_1}{1+f_1\hat{x}_2} < \frac{b_1\hat{x}_2}{c_1} + \rho_1\hat{x}_2^2 + d_1, \tag{18.2}$$

$$\beta\hat{x}_s < d_4, \tag{18.3}$$

$$c_2 > \gamma_1\hat{x}_2^2, \tag{18.4}$$

$$\hat{x}_2 < \frac{k_2}{2}, \tag{18.5}$$

$$a_2\left(1-\frac{2\hat{x}_2}{k_2}\right)\left(\frac{1}{1+f_2\hat{x}_s}\right) < \rho_2\hat{x}_s^2 + 2\delta_2\hat{x}_2 + d_2 + \frac{b_2\hat{x}_s(c_2-\gamma_1\hat{x}_2^2)}{(c_2+\gamma_1\hat{x}_2^2)^2}, \tag{18.6}$$

$$\frac{g_2\hat{x}_2}{(c_2+\gamma_1\hat{x}_2^2)} + 2e_2\rho_2\hat{x}_2\hat{x}_s < \delta_3 + d_3, \tag{18.7}$$

then M_4 is (LAS) otherwise M_4 is unstable, similarly for the equilibrium point M_5.

- The (JM) of system (2) at $M_6 = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_s, 0)$ can be written as:

$$J_6 = J(M_6) = [r_{ij}]_{4\times4}, \tag{19.1}$$

where $r_{11} = a_1\left(1-\frac{2\tilde{x}_1}{k_1}\right)\left(\frac{1}{1+f_1\tilde{x}_2}\right) - \frac{b_1\tilde{x}_2(c_1-\tilde{x}_1^2)}{(c_1+\tilde{x}_1^2)^2} - \rho_1\tilde{x}_2^2 - 3\delta_1\tilde{x}_1^2 - d_1,$

$$r_{12} = \left(\frac{\tilde{x}_1}{k_1}-1\right)\left(\frac{a_1f_1\tilde{x}_1}{1+f_1\tilde{x}_2}\right) - \frac{b_1\tilde{x}_1}{c_1+\tilde{x}_1^2} - 2\rho_1\tilde{x}_1\tilde{x}_2, \qquad r_{13} = r_{14} = 0,$$

$$r_{21} = \frac{g_1\tilde{x}_2(c_1-\tilde{x}_1^2)}{(c_1+\tilde{x}_1^2)^2} - \eta\tilde{x}_2 + e_1\rho_1\tilde{x}_2^2,$$

$$r_{22} = \left(1-\frac{2\tilde{x}_2}{k_2}\right)\left(\frac{a_2}{1+f_2\tilde{x}_s}\right) + \frac{g_1\tilde{x}_1}{c_1+\tilde{x}_1^2} - \eta\tilde{x}_1 + 2e_1\rho_1\tilde{x}_1\tilde{x}_2 - \rho_2\tilde{x}_s^2 - 2\delta_2\tilde{x}_2 - d_2 - \frac{b_2\tilde{x}_s(c_2-\gamma_1\tilde{x}_2^2)}{(c_2+\gamma_1\tilde{x}_2^2)^2},$$

$$r_{23} = \left(\frac{\tilde{x}_2}{k_2}-1\right)\left(\frac{a_2f_2\tilde{x}_2}{(1+f_2\tilde{x}_s)^2}\right) - 2\rho_2\tilde{x}_2\tilde{x}_s - \frac{b_2\tilde{x}_2}{(c_2+\gamma_1\tilde{x}_2^2)}, \qquad r_{24} = r_{31} = 0,$$

$r_{32} = \frac{g_2 \tilde{x}_s (c_2 - \gamma_1 \tilde{x}_2^2)}{(c_2 + \gamma_1 \tilde{x}_2^2)^2} + e_2 \rho_2 \tilde{x}_S^2,$

$r_{33} = \frac{g_2 \tilde{x}_2}{(c_2 + \gamma_1 \tilde{x}_2^2)} + 2 e_2 \rho_2 \tilde{x}_2 \tilde{x}_s - (\delta_3 + d_3), \quad r_{34} = -\beta \tilde{x}_s,$

$r_{41} = r_{42} = r_{43} = 0, \quad r_{44} = \beta \tilde{x}_s - d_4.$

After that, the equation that is distinctive of J_6 is

$(r_{44} - \lambda)(\lambda^3 + \tilde{\varphi}_1 \lambda^2 + \tilde{\varphi}_2 \lambda + \tilde{\varphi}_3) = 0,$

either $(r_{44} - \lambda) = 0 \Longrightarrow \lambda_{6x_I} = r_{44} = \beta \tilde{x}_s - d_4 < 0.$

Or $(\lambda^3 + \tilde{\varphi}_1 \lambda^2 + \tilde{\varphi}_2 \lambda + \tilde{\varphi}_3) = 0,$ (19.2)

where $\tilde{\varphi}_1 = -(r_{11} + r_{22} + r_{33}),$

$\tilde{\varphi}_2 = r_{11}(r_{22} + r_{33}) + r_{22} r_{33} - r_{23} r_{32} - r_{12} r_{21},$

$\tilde{\varphi}_3 = -r_{11}(r_{22} r_{33} - r_{23} r_{32}) + r_{12} r_{21} r_{23}.$

All of the eigenvalues, which are the roots of the equation, are considered to be valid according to the Routh-Hurwitz criterion Eq.(19.2), have negative real portions if and only if the following conditions are met: $\tilde{\varphi}_1 > 0,\ \tilde{\varphi}_3 > 0$ and $\tilde{\Delta} = (\tilde{\varphi}_1 \tilde{\varphi}_2 - \tilde{\varphi}_3)\tilde{\varphi}_3 > 0.$

Clearly, we have $\tilde{\varphi}_1 > 0,\ \tilde{\varphi}_3 > 0$ and

$\tilde{\Delta} = \left[(r_{22} + r_{33})(r_{23} r_{32} - r_{11}^2) + (r_{11} + r_{22})\left(r_{12} r_{21} - (r_{22}^2 + r_{33}^2)\right) - 2 r_{11} r_{22} r_{33}\right] > 0,$

provided that

$c_1 > \tilde{x}_1^2,$ (19.3)

$\tilde{x}_1 < \frac{k_1}{2},$ (19.4)

$\left(1 - \frac{2\tilde{x}_1}{k_1}\right)\left(\frac{a_1}{1 + f_1 \tilde{x}_2}\right) < \frac{b_1 \tilde{x}_2 (c_1 - \tilde{x}_1^2)}{(c_1 + \tilde{x}_1^2)^2} + \rho_1 \tilde{x}_2^2 + 3\delta_1 \tilde{x}_1^2 + d_1,$ (19.5)

$\frac{g_1 \tilde{x}_2 (c_1 - \tilde{x}_1^2)}{(c_1 + \tilde{x}_1^2)^2} + e_1 \rho_1 \tilde{x}_2^2 > \eta \tilde{x}_2,$ (19.6)

$\tilde{x}_2 < \frac{k_2}{2},$ (19.7)

$c_2 > \gamma_1 \tilde{x}_2^2,$ (19.8)

$\left(1 - \frac{2\tilde{x}_2}{k_2}\right)\left(\frac{a_2}{1 + f_2 \tilde{x}_s}\right) + \frac{g_1 \tilde{x}_1}{c_1 + \tilde{x}_1^2} + 2 e_1 \rho_1 \tilde{x}_1 \tilde{x}_2 < \eta \tilde{x}_1 + \rho_2 \tilde{x}_s^2 + 2\delta_2 \tilde{x}_2 + d_2 + \frac{b_2 \tilde{x}_s (c_2 - \gamma_1 \tilde{x}_2^2)}{(c_2 + \gamma_1 \tilde{x}_2^2)^2},$ (19.9)

$\frac{g_2 \tilde{x}_2}{(c_2 + \gamma_1 \tilde{x}_2^2)} + 2 e_2 \rho_2 \tilde{x}_2 \tilde{x}_s < \delta_3 + d_3,$ (19.10)

$\beta \tilde{x}_s < d_4,$ (19.11)

then M_6 is (LAS) otherwise M_6 is unstable, similarly for the equilibrium point M_7.

- The (JM) of system (1) at $M_8 = (0, \breve{x}_2, \breve{x}_s, \breve{x}_I)$ can be written as:

$$J_8 = J(M_8) = [u_{ij}]_{4\times 4}, \tag{20.1}$$

where $u_{11} = \frac{a_1}{1+f_1\breve{x}_2} - \frac{b_1\breve{x}_2}{c_1} - \rho_1\breve{x}_2^2 - d_1$, $\quad u_{12} = u_{13} = u_{14} = 0$,

$$u_{21} = \frac{g_1\breve{x}_2}{c_1} - \eta\breve{x}_2 + e_1\rho_1\breve{x}_2^2,$$

$$u_{22} = \left(1 - \frac{2\breve{x}_2}{k_2}\right)\left(\frac{a_2}{1+f_2\breve{x}_s}\right) - \rho_2\breve{x}_s^2 - 2\delta_2\breve{x}_2 - d_2 - \frac{b_2\breve{x}_s(c_2-\gamma_1\breve{x}_2^2)}{(c_2+\gamma_1\breve{x}_2^2)^2},$$

$$u_{23} = \left(\frac{\breve{x}_2}{k_2} - 1\right)\left(\frac{a_2f_2\breve{x}_2}{(1+f_2\breve{x}_s)^2}\right) - 2\rho_2\breve{x}_2\breve{x}_s - \frac{b_2\breve{x}_2}{(c_2+\gamma_1\breve{x}_2^2)}, \qquad u_{24} = u_{31} = 0,$$

$$u_{32} = \frac{g_2\breve{x}_s(c_2-\gamma_1\breve{x}_2^2)}{(c_2+\gamma_1\breve{x}_2^2)^2} + e_2\rho_2\breve{x}_s^2,$$

$$u_{33} = \frac{g_2\breve{x}_2}{(c_2+\gamma_1\breve{x}_2^2)} + 2e_2\rho_2\breve{x}_2\breve{x}_s - \beta\breve{x}_I - (\delta_3 + d_3), \qquad u_{34} = -\beta\breve{x}_s,$$

$$u_{41} = u_{42} = 0, \quad u_{43} = \beta\breve{x}_I, \qquad u_{44} = \beta\breve{x}_s - d_4.$$

After that, the equation that is distinctive of J_8 is

$$(u_{11} - \lambda)(\lambda^3 + \breve{\varphi}_1\lambda^2 + \breve{\varphi}_2\lambda + \breve{\varphi}_3) = 0,$$

either $(u_{11} - \lambda) = 0 \Longrightarrow \lambda_{8x_1} = u_{11}$.

Or $\quad (\lambda^3 + \breve{\varphi}_1\lambda^2 + \breve{\varphi}_2\lambda + \breve{\varphi}_3) = 0, \qquad (20.2)$

where $\breve{\varphi}_1 = -(u_{22} + u_{33} + u_{44})$,

$$\breve{\varphi}_2 = u_{22}(u_{33} + u_{44}) + u_{33}u_{44} - u_{34}u_{43} - u_{23}u_{32},$$

$$\breve{\varphi}_3 = -(u_{22}(u_{33}u_{44} - u_{34}u_{43}) - u_{23}u_{32}u_{44}).$$

All of the eigenvalues, which are the roots of the equation, are considered to be valid according to the Routh-Hurwitz criterion Eq.(20.2), have negative real portions if and only if the following conditions are met: $\breve{\varphi}_1 > 0$, $\breve{\varphi}_3 > 0$ and $\breve{\Delta} = (\breve{\varphi}_1\breve{\varphi}_2 - \breve{\varphi}_3)\breve{\varphi}_3 > 0$.

Clearly, we have $\breve{\varphi}_1 > 0$, $\breve{\varphi}_3 > 0$ and

$\breve{\Delta} = (u_{33} + u_{44})(u_{34}u_{43} - u_{22}^2) + (u_{22} + u_{33})(u_{23}u_{32} - u_{44}^2) - u_{33}^2(u_{22} + u_{44}) - 2u_{22}u_{33}u_{44}$,

provided that

$$\frac{a_1}{1+f_1\breve{x}_2} < \frac{b_1\breve{x}_2}{c_1} + \rho_1\breve{x}_2^2 + d_1, \tag{20.3}$$

$$c_2 > \gamma_1\breve{x}_2^2, \tag{20.4}$$

$$\breve{x}_2 < \frac{k_2}{2}, \tag{20.5}$$

$$\left(1 - \frac{2\breve{x}_2}{k_2}\right)\left(\frac{a_2}{1+f_2\breve{x}_s}\right) < \rho_2\breve{x}_s^2 + 2\delta_2\breve{x}_2 + d_2 + \frac{b_2\breve{x}_s(c_2-\gamma_1\breve{x}_2^2)}{(c_2+\gamma_1\breve{x}_2^2)^2}, \tag{20.6}$$

$$\frac{g_2\breve{x}_2}{(c_2+\gamma_1\breve{x}_2^2)} + 2e_2\rho_2\breve{x}_2\breve{x}_s < \beta\breve{x}_I + (\delta_3 + d_3), \tag{20.7}$$

$$\beta \breve{x}_s < d_4, \tag{20.8}$$

then M_8 is (LAS) otherwise M_8 is unstable.

- The (JM) of system (1) at $M_9 = (\dot{x}_1, \dot{x}_2, \dot{x}_s, \dot{x}_I)$ may be expressed as:

$$J_9 = J(M_9) = [z_{ij}]_{4\times 4}, \tag{21.1}$$

where

$$z_{11} = a_1\left(1 - \frac{2\dot{x}_1}{k_1}\right)\left(\frac{1}{1+f_1\dot{x}_2}\right) - \frac{b_1\dot{x}_2(c_1 - \dot{x}_1^2)}{(c_1+\dot{x}_1^2)^2} - \rho_1\dot{x}_2^2 - 3\delta_1\dot{x}_1^2 - d_1,$$

$$z_{12} = a_1\dot{x}_1\left(1 - \frac{\dot{x}_1}{k_1}\right)\left(\frac{-f_1}{(1+f_1\dot{x}_1)^2}\right) - \frac{b_1\dot{x}_1}{c_1+\dot{x}_1^2} - 2\rho_1\dot{x}_1\dot{x}_2, \quad z_{13} = z_{14} = 0,$$

$$z_{21} = \frac{g_1\dot{x}_2(c_1-\dot{x}_1^2)}{(c_1+\dot{x}_1^2)^2} - \eta\dot{x}_2 + e_1\rho_1\dot{x}_2^2,$$

$$z_{22} = a_2\left(1 - \frac{2\dot{x}_2}{k_2}\right)\left(\frac{1}{1+f_2\dot{x}_s}\right) + \frac{g_1\dot{x}_1}{c_1+\dot{x}_1^2} - \eta\dot{x}_1 + 2e_1\rho_1\dot{x}_1\dot{x}_2 - \rho_2\dot{x}_s^2 - 2\delta_2\dot{x}_2 - d_2 - \frac{b_2\dot{x}_s(c_2-\gamma_1\dot{x}_2^2)}{(c_2+\gamma_1\dot{x}_2^2)^2},$$

$$z_{23} = a_2\dot{x}_2\left(1 - \frac{\dot{x}_2}{k_2}\right)\left(\frac{-f_2}{(1+f_2\dot{x}_s)^2}\right) - 2\rho_2\dot{x}_2\dot{x}_s - \frac{b_2\dot{x}_2}{(c_2+\gamma_1\dot{x}_2^2)}, \qquad z_{24} = z_{31} = 0\ ,$$

$$z_{32} = \frac{b_2\dot{x}_s(c_2-\gamma_1\dot{x}_2^2)}{(c_2+\gamma_1\dot{x}_2^2)^2} + 2e_2\rho_2\dot{x}_s^2\ ,$$

$$z_{33} = \frac{g_2\dot{x}_2}{(c_2+\gamma_1\dot{x}_2^2)} + 2e_2\rho_2\dot{x}_2\dot{x}_s - \beta\dot{x}_I - (\delta_3 + d_3), \qquad z_{34} = -\beta\dot{x}_s$$

$$z_{41} = z_{42} = 0, \quad z_{43} = \beta\dot{x}_I, \quad z_{44} = \beta\dot{x}_s - d_4.$$

A characteristic equation of M_9 may be expressed as follows:

$$\lambda^4 + \dot{\varphi}_1\lambda^3 + \dot{\varphi}_2\lambda^2 + \dot{\varphi}_3\lambda + \dot{\varphi}_4 = 0, \tag{21.2}$$

where $\dot{\varphi}_1 = -(z_{11} + z_{22} + z_{33} + z_{44})$,

$$\dot{\varphi}_2 = z_{33}z_{44} + (z_{11} + z_{22})(z_{33} + z_{44}) + z_{11}z_{22} - z_{34}z_{43} - z_{23}z_{32} - z_{12}z_{21},$$

$$\dot{\varphi}_3 = (z_{11} + z_{22})(z_{34}z_{43} - z_{33}z_{44}) + z_{23}z_{32}(z_{11} + z_{44}) + (z_{11} + z_{44})(z_{12}z_{21} - z_{11}z_{22}),$$

$$\dot{\varphi}_4 = (z_{11}z_{22} - z_{12}z_{21})(z_{33}z_{44} - z_{34}z_{43}) - z_{11}z_{23}z_{32}z_{44}.$$

All of the eigenvalues, which are the roots of the equation, are considered to be valid according to the Routh-Hurwitz criterion Eq.(21.2), have negative real portions if and only if the following conditions are met: $\dot{\varphi}_1 > 0$, $\dot{\varphi}_3 > 0, \dot{\varphi}_4 > 0$ and $\dot{\Delta} = (\dot{\varphi}_1\dot{\varphi}_2 - \dot{\varphi}_3)\dot{\varphi}_3 - \dot{\varphi}_1^2\dot{\varphi}_4 > 0$.

Clearly, we have $\dot{\varphi}_1 > 0$, $\dot{\varphi}_3 > 0$, $\dot{\varphi}_4 > 0$ and

$$\dot{\Delta} = (z_{33}z_{44} - z_{34}z_{43})\{(z_{11} + z_{22})^2[(z_{11}z_{22} - z_{12}z_{21}) + (z_{33} + z_{44})(z_{11} + z_{22} + z_{33} + z_{44})] + (z_{11} + z_{22})[(z_{33} + z_{44})(z_{33}z_{44} - z_{34}z_{43}) - (z_{22} + z_{33})z_{23}z_{32}] - (z_{33} + z_{44})[z_{23}z_{32}(z_{11} + z_{44}) - (z_{33} + z_{44})(z_{11}z_{22} - z_{12}z_{21})]]\} - (z_{11} + z_{22} + z_{33} + z_{44})\{(z_{11} + z_{22})(z_{33} + z_{44})[z_{11}z_{23}z_{32} + z_{23}z_{32}z_{44} - (z_{33} + z_{44})(z_{11}z_{22} - z_{12}z_{21})] + (z_{11} + z_{22} + z_{33} +$$

$z_{44})[(z_{11}z_{22}-z_{12}z_{21})(z_{33}z_{44}-z_{34}z_{43})-z_{11}z_{23}z_{32}z_{44}]\}-(z_{11}+z_{22})(z_{11}z_{22}-z_{12}z_{21})\{z_{23}z_{32}(z_{11}+z_{44})-(z_{11}z_{22}-z_{12}z_{21})(z_{33}+z_{44})\}+z_{23}z_{32}(z_{22}+z_{33})\{z_{23}z_{32}(z_{11}+z_{44})-(z_{33}+z_{44})(z_{11}z_{22}-z_{12}z_{21})\}$,

provided that

$$c_1 > \dot{x}_1^2, \tag{21.3}$$

$$\dot{x}_1 < \frac{k_1}{2}, \tag{21.4}$$

$$\left(1-\frac{2\dot{x}_1}{k_1}\right)\left(\frac{a_1}{1+f_1\dot{x}_2}\right) < \frac{b_1\dot{x}_2\left(c_1-\dot{x}_1^2\right)}{\left(c_1+\dot{x}_1^2\right)^2} + \rho_1\dot{x}_2^2 + 3\delta_1\dot{x}_1^2 + d_1, \tag{21.5}$$

$$\frac{g_1\dot{x}_2(c_1-\dot{x}_1^2)}{(c_1+\dot{x}_1^2)^2} + e_1\rho_1\dot{x}_2^2 > \eta\dot{x}_2, \tag{21.6}$$

$$\frac{a_1f_1\dot{x}_1^2}{k_1(1+f_1\dot{x}_2)^2} < \frac{a_1f_1\dot{x}_1}{(1+f_1\dot{x}_2)^2} + \frac{b_1\dot{x}_1}{c_1+\dot{x}_1^2} + 2\rho_1\dot{x}_1\dot{x}_2, \tag{21.7}$$

$$c_2 > \gamma_1\dot{x}_2^2, \tag{21.8}$$

$$\dot{x}_2 < \frac{k_2}{2}, \tag{21.9}$$

$$\left(1-\frac{2\dot{x}_2}{k_2}\right)\left(\frac{a_2}{1+f_2\dot{x}_s}\right) + \frac{g_1\dot{x}_1}{c_1+\dot{x}_1^2} + 2e_1\rho_1\dot{x}_1\dot{x}_2 < \eta\dot{x}_1 + \rho_2\dot{x}_s^2 + 2\delta_2\dot{x}_2 + d_2 + \frac{b_2\dot{x}_s(c_2-\gamma_1\dot{x}_2^2)}{\left(c_2+\gamma_1\dot{x}_2^2\right)^2}, \tag{21.10}$$

$$\frac{g_2\dot{x}_2}{(c_2+\gamma_1\dot{x}_2^2)} + 2e_2\rho_2\dot{x}_2\dot{x}_s < \beta\dot{x}_I + (\delta_3+d_3), \tag{21.11}$$

$$\beta\dot{x}_s < d_4, \tag{21.12}$$

$$(z_{11}+z_{22})(z_{33}+z_{44})[z_{11}z_{23}z_{32}+z_{23}z_{32}z_{44}-(z_{33}+z_{44})(z_{11}z_{22}-z_{12}z_{21})] < (z_{11}+z_{22}+z_{33}+z_{44})[(z_{12}z_{21}-z_{11}z_{22})(z_{34}z_{43}-z_{33}z_{44})+z_{11}z_{23}z_{32}z_{44}]. \tag{21.13}$$

There for, under previous condition, all of the eigen values have a negative real part. This mean M_9 is (LAS) otherwise are unstable. similarly for the equilibrium point M_{10} and M_{11}.

6. GLOBAL STABILITY ANALYSIS (GSA)

As shown in the following theorems, the (GAS) for the (EPs) on the (LAS) of system (1) have been subjected to an analytical investigation by the use of the Lyapunov technique that is applicable for the situation.

Theorem (2): system (1) is (GAS) at the point $M_0=(0,0,0,0)$ in the sub area $\vartheta_0 \subset R_+^4$. With the further conditions are applicable:

$$\frac{a_1k_1}{4} + \frac{a_2k_2}{4} < \delta_1x_1^3 + d_1x_1 + \eta x_1x_2 + \delta_2x_2^2 + d_2x_2 + (\delta_3+d_3)x_s + d_4x_I, \tag{22.1}$$

Proof: The following function should be considered. $S_0(x_1,x_2,x_s,x_I) = x_1+x_2+x_s+x_I$.

Clearly, the function $S_0: R_+^4 \to R$ is positive definite in C^1.

The use of specific algebraic computations and differentiation of S_0 with respect to time t, and by find sup of logistic growth functions led to the following result:

$$\frac{dS_0}{dt} \le \frac{a_1 k_1}{4} - (b_1 - g_1)\frac{x_1 x_2}{c_1 + x_1^2} - \delta_1 x_1^3 - d_1 x_1 + \frac{a_2 k_2}{4} - \eta x_1 x_2 + (e_1 - 1)\rho_1 x_1 x_2^2 - \delta_2 x_2^2 - d_2 x_2 - (b_2 - g_2)\frac{x_2 x_s}{c_2 + \gamma_1 x_2^2} + (e_2 - 1)\rho_2 x_2 x_s^2 - (\delta_3 + d_3)x_s - d_4 x_I.$$

Now, due to the biological facts $b_i > g_i$ and $0 < e_i < 1$ for $i = 1,2$ we get

$$\frac{dS_0}{dt} < \frac{a_1 k_1}{4} - \delta_1 x_1^3 - d_1 x_1 + \frac{a_2 k_2}{4} - \eta x_1 x_2 - \delta_2 x_2^2 - d_2 x_2 - (\delta_3 + d_3)x_s - d_4 x_I.$$

hence $\frac{dS_0}{dt} < 0$ in the region ϑ_0, by conditions (22.1) S_0 is a Lyapunov function. So, M_0 is a (GAS) in $\vartheta_0 \subset R_+^4$.

Theorem (3): system (1) is (GAS) at the point $M_1 = (\overline{x}_1, 0, 0, 0)$ in the sub area $\vartheta_1 \subset R_+^4$. With the further condition are applicable:

$$\frac{b_1 \bar{x}_1 x_2}{c_1 + x_1^2} + \frac{a_2 k_2}{4} + \rho_1 \bar{x}_1 x_2^2 < a_1 \frac{(x_1 - \bar{x}_1)^2}{k_1} + \delta_1 (x_1 + \bar{x}_1)(x_1 - \bar{x}_1)^2 + \eta x_1 x_2 + \delta_2 x_2^2 + d_2 x_2 + (\delta_3 + d_3)x_s + d_4 x_I, \tag{23.1}$$

Proof: The following function should be considered.

$$S_1(x_1, x_2, x_s, x_I) = (x_1 - \bar{x}_1 - \bar{x}_1 \ln\frac{x_1}{\bar{x}_1}) + x_2 + x_s + x_I.$$

Clearly, the function $S_1: R_+^4 \to R$ is positive definite in C^1.

The use of specific algebraic computations and differentiation of S_1 with respect to time t, and by find sup of logistic growth function for predator led to the following result:

$$\frac{dS_1}{dt} \le \frac{-a_1 (x_1 - \bar{x}_1)^2}{k_1} - (b_1 - g_1)\frac{x_1 x_2}{c_1 + x_1^2} + \frac{b_1 \bar{x}_1 x_2}{c_1 + x_1^2} + (e_1 - 1)\rho_1 x_1 x_2^2 + \rho_1 \bar{x}_1 x_2^2 - \delta_1 (x_1 - \bar{x}_1)^2 (x_1 + \bar{x}_1) + \frac{a_2 k_2}{4} - \eta x_1 x_2 + (e_2 - 1)\rho_2 x_2 x_s^2 - \delta_2 x_2^2 - (b_2 - g_2)\frac{x_2 x_s}{c_2 + \gamma_1 x_2^2} - d_2 x_2 - (\delta_3 + d_3)x_s - d_4 x_I.$$

Now, due to the biological facts $b_i > g_i$ and $0 < e_i < 1$ for $i = 1,2$ we obtain

$$\frac{dS_1}{dt} \le \frac{-a_1 (x_1 - \bar{x}_1)^2}{k_1} + \frac{b_1 \bar{x}_1 x_2}{c_1 + x_1^2} + \rho_1 \bar{x}_1 x_2^2 - \delta_1 (x_1 + \bar{x}_1)(x_1 - \bar{x}_1)^2 + \frac{a_2 k_2}{4} - \eta x_1 x_2 - \delta_2 x_2^2 - d_2 x_2 - (\delta_3 + d_3)x_s - d_4 x_I,$$

hence $\frac{dS_1}{dt} < 0$ in the region ϑ_1, by conditions (23.1) S_1 is a Lyapunov function. so, M_1 is a (GAS) in the region $\vartheta_1 \subset R_+^4$.

Theorem (4): system (1) is (GAS) at the point $M_2 = (0, \bar{\bar{x}}_2, 0, 0)$ in the sub area $\vartheta_2 \subset R_+^4$. With the further conditions are applicable:

$$\frac{a_1 k_1}{4} + \rho_2 \bar{\bar{x}}_2 x_s^2 + \frac{b_2 \bar{\bar{x}}_2 x_s}{c_2+\gamma_1 x_2^2} < (\delta_1 {x_1}^3 + d_1 x_1) + \frac{a_2(x_2-\bar{\bar{x}}_2)^2}{k_2} + \frac{g_1 \bar{\bar{x}}_2 x_1}{c_1+x_1^2} + e_1\rho_1 x_1 x_2 \bar{\bar{x}}_2 + \delta_2(x_2-\bar{\bar{x}}_2)^2 + (\delta_3+d_3)x_s + d_4 x_I, \tag{24.1}$$

Proof: The following function should be considered.

$$S_2(x_1, x_2, x_s, x_I) = x_1 + (x_2 - \bar{\bar{x}}_2 - \bar{\bar{x}}_2 \ln\frac{x_2}{\bar{\bar{x}}_2}) + x_s + x_I.$$

Clearly, the function $S_2 \colon \mathrm{R}_+^4 \to \mathrm{R}$ is positive definite in C^1.

The use of specific algebraic computations and differentiation of S_2 with respect to time t, and by find sup of logistic growth function for predator led to the following result:

$$\frac{dS_2}{dt} \le \frac{a_1 k_1}{4} - (b_1 - g_1)\frac{x_1 x_2}{c_1+x_1^2} + (e_1-1)\rho_1 x_1 x_2^2 - (\delta_1 {x_1}^3 + d_1 x_1) - \frac{a_2(x_2-\bar{\bar{x}}_2)^2}{k_2} - \frac{g_1 \bar{\bar{x}}_2 x_1}{c_1+x_1^2} - e_1\rho_1 x_1 x_2 \bar{\bar{x}}_2 + (e_2-1)\rho_2 x_2 x_s^2 + \rho_2 \bar{\bar{x}}_2 x_s^2 - \delta_2(x_1-\bar{\bar{x}}_2)^2 - (b_2-g_2)\frac{x_2 x_s}{c_2+\gamma_1 x_2^2} + \frac{b_2 \bar{\bar{x}}_2 x_s}{c_2+\gamma_1 x_2^2} - (\delta_3+d_3)x_s - d_4 x_I.$$

Now, due to the biological facts $b_i > g_i$ and $0 < e_i < 1$ for $i = 1,2$ we get

$$\frac{dS_2}{dt} \le \frac{a_1 k_1}{4} - (\delta_1 {x_1}^2 + d_1)x_1 - a_2\frac{(x_2-\bar{\bar{x}}_2)^2}{k_2} - g_1\frac{\bar{\bar{x}}_2 x_1}{c_1+x_1^2} - e_1\rho_1 x_1 \bar{\bar{x}}_2 x_2 + \rho_2 \bar{\bar{x}}_2 x_s^2 - \delta_2(x_2-\bar{\bar{x}}_2)^2 + \frac{b_2 \bar{\bar{x}}_2 x_s}{c_2+\gamma_1 x_2^2} - (\delta_3+d_3)x_s - d_4 x_I,$$

hence $\frac{dS_2}{dt} < 0$ in the region ϑ_2, by conditions (24.1) S_2 is a Lyapunov function. so, M_2 is a (GAS) in the region $\vartheta_2 \subset \mathrm{R}_+^4$.

Theorem (5): system (1) is (GAS) at the point $M_3 = (\check{x}_1, \check{x}_2, 0, 0)$ in the sub area $\vartheta_3 \subset R_+^4$. With the further conditions are applicable:

$$\check{Z}_1 < \check{Z}_2, \tag{25.1}$$

$$\check{Z}_1 = b_1\frac{\check{x}_1 x_2}{c_1+x_1^2} + \rho_1 \check{x}_1 x_2^2 + \rho_1 x_1 \check{x}_2^2 + b_1\frac{x_1 \check{x}_2}{c_1+\check{x}_1^2} + \rho_2 \check{x}_2 x_s^2 + b_2\frac{\check{x}_2 x_s}{c_2+\gamma_1 x_2^2},$$

$$\check{Z}_2 = \frac{a_1(x_1-\check{x}_1)^2}{k_1} + \delta_1(x_1-\check{x}_1)^2(x_1+\check{x}_1) + \frac{a_2(x_2-\check{x}_2)^2}{k_2} + g_1\frac{x_1\check{x}_2}{c_1+x_1^2} + e_1\rho_1 x_1 \check{x}_1 x_2 + e_1\rho_1 \check{x}_1 \check{x}_2 x_2 + g_1\frac{\check{x}_1 x_2}{c_1+\check{x}_1^2} + \delta_2(x_2-\check{x}_2)^2 + (\delta_3+d_3)x_s + d_4 x_I.$$

Proof: The following function should be considered.

$$S_3(x_1, x_2, x_s, x_I) = (x_1 - \check{x}_1 - \check{x}_1 \ln\frac{x_1}{\check{x}_1}) + (x_2 - \check{x}_2 - \check{x}_2 \ln\frac{x_2}{\check{x}_2}) + x_s + x_I.$$

Clearly, the function $S_3 \colon \mathrm{R}_+^4 \to \mathrm{R}$ is positive definite in C^1.

The use of specific algebraic computations and differentiation of S_3 with respect to time t, and by find sup of logistic growth function for predator led to the following result:

$$\frac{dS_3}{dt} \le \frac{-a_1(x_1-\breve{x}_1)^2}{k_1} - (b_1-g_1)\frac{x_1x_2}{c_1+x_1^2} - (b_1-g_1)\frac{\breve{x}_1\breve{x}_2}{c_1+\breve{x}_1^2} + b_1\frac{\breve{x}_1x_2}{c_1+x_1^2} + \rho_1\breve{x}_1x_2^2 + \rho_1x_1\breve{x}_2^2 - \delta_1(x_1 - \breve{x}_1)^2(x_1+\breve{x}_1) + b_1\frac{x_1\breve{x}_2}{c_1+\breve{x}_1^2} - \frac{a_2(x_2-\breve{x}_2)^2}{k_2} - g_1\frac{x_1\breve{x}_2}{c_1+x_1^2} + (e_1-1)\rho_1x_1x_2^2 + (e_1-1)\rho_1\breve{x}_1\breve{x}_2^2 - e_1\rho_1x_1\breve{x}_1x_2 - e_1\rho_1\breve{x}_1\breve{x}_2x_2 - g_1\frac{\breve{x}_1x_2}{c_1+\breve{x}_1^2} + (e_2-1)\rho_2x_2x_s^2 + \rho_2\breve{x}_2x_s^2 - \delta_2(x_2-\breve{x}_2)^2 + b_2\frac{\breve{x}_2x_s}{c_2+\gamma_1x_2^2} - (b_2-g_2)\frac{x_2x_s}{c_2+\gamma_1x_2^2} - (\delta_3+d_3)x_s - d_4x_I.$$

Now, due to the biological facts $b_i > g_i$ and $0 < e_i < 1$ for $i = 1,2$ we get

$$\frac{dS_3}{dt} \le \frac{-a_1(x_1-\breve{x}_1)^2}{k_1} + b_1\frac{\breve{x}_1x_2}{c_1+x_1^2} + \rho_1\breve{x}_1x_2^2 + \rho_1\breve{x}_2^2x_1 - \delta_1(x_1+\breve{x}_1)(x_1-\breve{x}_1)^2 + b_1\frac{x_1\breve{x}_2}{c_1+\breve{x}_1^2} - \frac{a_2(x_2-\breve{x}_2)^2}{k_2} - g_1\frac{x_1\breve{x}_2}{c_1+x_1^2} - e_1\rho_1x_1\breve{x}_1x_2 - e_1\rho_1\breve{x}_1\breve{x}_2x_2 - g_1\frac{\breve{x}_1x_2}{c_1+\breve{x}_1^2} + \rho_2\breve{x}_2x_s^2 - \delta_2(x_2-\breve{x}_2)^2 + b_2\frac{\breve{x}_2x_s}{c_2+\gamma_1x_2^2} - (\delta_3+d_3)x_s - d_4x_I,$$

$$= \breve{Z}_1 - \breve{Z}_2$$

hence $\frac{dS_3}{dt} < 0$ in the region ϑ_3, by conditions (25.1) S_3 is a Lyapunov function. so, M_3 is a (GSA) in the region $\vartheta_3 \subset R_+^4$.

Theorem (6): system (1) is (GAS) at the point $M_8 = (0, \breve{x}_2, \breve{x}_s, \breve{x}_I)$ in the sub area $\vartheta_4 \subset R_+^4$. With the further conditions are applicable:

$$a_1x_1\left(1-\frac{x_1}{k_1}\right) + g_2\left(\frac{x_2}{c_2+\gamma_1x_2^2} - \frac{\breve{x}_2}{c_2+\gamma_1\breve{x}_2^2}\right)(x_s-\breve{x}_s) + e_2\rho_2(x_2x_s-\breve{x}_2\breve{x}_s)(x_s-\breve{x}_s) < \frac{a_2(x_2-\breve{x}_2)^2}{k_2} + \eta x_1(x_2-\breve{x}_2) + \rho_2(x_2-\breve{x}_2)(x_s^2-\breve{x}_s^2) + \delta_2(x_2-\breve{x}_2)^2 + b_2\left(\frac{x_s}{c_2+\gamma_1x_2^2} - \frac{\breve{x}_s}{c_2+\gamma_1\breve{x}_2^2}\right)(x_2-\breve{x}_2), \qquad (26.1)$$

Proof: The following function should be considered.

$$S_4(x_1, x_2, x_s, x_I) = x_1 + \left(x_2 - \breve{x}_2 - \breve{x}_2\ln\frac{x_2}{\breve{x}_2}\right) + \left(x_s - \breve{x}_s - \breve{x}_s\ln\frac{x_s}{\breve{x}_s}\right) + \left(x_2 - \breve{x}_I - \breve{x}_I\ln\frac{x_I}{\breve{x}_I}\right).$$

Clearly, the function $S_4: \Omega \subseteq R_+^4 \to R$ is positive definite in C^1.

The use of specific algebraic computations and differentiation of S_4 with respect to time t, and by find sup of logistic growth function for predator led to the following result:

$$\frac{dS_4}{dt} \le a_1\left(1-\frac{x_1}{k_1}\right)x_1 - (b_1-g_1)\frac{x_1x_2}{c_1+x_1^2} + (e_1-1)\rho_1x_1x_2^2 - (\delta_1x_1^2+d_3)x_1 - \frac{a_2(x_2-\breve{x}_2)^2}{k_2} - g_1\frac{x_1\breve{x}_2}{c_1+x_1^2} - \eta x_1(x_2-\breve{x}_2) - e_1\rho_1x_1x_2\breve{x}_2 - \rho_2(x_2-\breve{x}_2)(x_s^2-\breve{x}_s^2) - \delta_2(x_2-\breve{x}_2)^2 - b_2\left(\frac{x_s}{c_2+\gamma_1x_2^2}\frac{\breve{x}_s}{c_2+\gamma_1\breve{x}_2^2}\right)(x_2-\breve{x}_2) + g_2\left(\frac{x_2}{c_2+\gamma_1x_2^2} - \frac{\breve{x}_2}{c_2+\gamma_1\breve{x}_2^2}\right)(x_s-\breve{x}_s) + e_2\rho_2(x_2x_s-\breve{x}_2\breve{x}_s)(x_s-\breve{x}_s).$$

Now, due to the biological facts $b_i > g_i$ and $0 < e_i < 1$ for $i = 1,2$ we get

$$\frac{dS_4}{dt} \le a_1\left(1-\frac{x_1}{k_1}\right)x_1 - \frac{a_2(x_2-\breve{x}_2)^2}{k_2} - \eta x_1(x_2-\breve{x}_2) - \rho_2(x_2-\breve{x}_2)\left(x_s^2-\breve{x}_s^2\right) - \delta_2(x_2-\breve{x}_2)^2 -$$
$$b_2\left(\frac{x_s}{c_2+\gamma_1 x_2^2}-\frac{\breve{x}_s}{c_2+\gamma_1\breve{x}_2^2}\right)(x_2-\breve{x}_2) + g_2\left(\frac{x_2}{c_2+\gamma_1 x_2^2}-\frac{\breve{x}_2}{c_2+\gamma_1\breve{x}_2^2}\right)(x_s-\breve{x}_s) + e_2\rho_2(x_2x_s-\breve{x}_2\breve{x}_s)(x_s-\breve{x}_s),$$

Then by using condition (26.1) hence $\frac{dS_4}{dt}$ is negative semidefinite, and $\frac{dS_4}{dt}=0$ on the set $\{(0,\breve{x}_2,\breve{x}_s,\breve{x}_I)\in\Omega: x_1=0, x_2=\breve{x}_2, x_s=\breve{x}_s, x_I>0\}$ so according to Lyapunov's first theorem, M_8 is a globally stable point. Furthermore, since on this set we have

$$\frac{dx_s}{dT}=\frac{g_2x_2x_s}{c_2+\gamma_1x_2^2}+e_2\rho_2x_2x_s^2-\delta_3x_s-\beta x_sx_I-d_3x_s=0. \qquad (26.2)$$

If and only if $x_I = \breve{x}_I$, then the largest compact invariant set contained in this set is reduced to the endemic equilibrium point M_8. Hence, according to LaSalle's invariant principle, M_8 is an attractive point and therefore it is globally asymptotically stable in Ω.

Furthermore, it is not viable to use the Lyapunov function to examine the global stability of equilibrium point $\{M_4,M_5\},\{M_6,M_7\}$ and $\{M_9,M_{10},M_{11}\}$ that are located in the interior of R_+^4 and have distinct beginning points neighbourhood but the same local stability conditions. Consequently, as opposed to studying it analytically as demonstrated in the previous theorems, we shall investigate it numerically.

7.NUMERICAL SIMULATION

In this section, the behavior of dynamics of system (1) is examined numerically by mathematical methods. The purpose of this study is to verify the analytical results we have obtained and to investigate the impact of varying the given values of all parameters on the dynamical behavior of the system. It has been noted that system (1) has a (GAS) positive equilibrium point, as seen in Figure (2), based on the ensuing set of hypothetical parameters that satisfy the stability criteria of equilibrium point is the positive.

$$\left.\begin{array}{l} a_1=2.5,\ k_1=3,\ f_1=0.1,\ b_1=0.5,\ c_1=6.0,\ \rho_1=0.03,\ \delta_1=0.02,\ d_1=0.01,\ a_2=1.5,\ k_2=2.7, \\ f_2=0.1,\ g_1=0.3,\ b_2=0.4,\ c_2=4.0,\ \gamma_1=0.1;\ \eta=0.05,\ e_1=0.01,\ \rho_2=0.03,\ \delta_2=0.05,\ d_2=0.03, \\ g_2=0.2,\ e_2=0.01,\ \delta_3=0.05,\ \beta=0.01,\ d_3=0.01,\ d_4=0.01. \end{array}\right\} \qquad (27)$$

(a) **(b)**

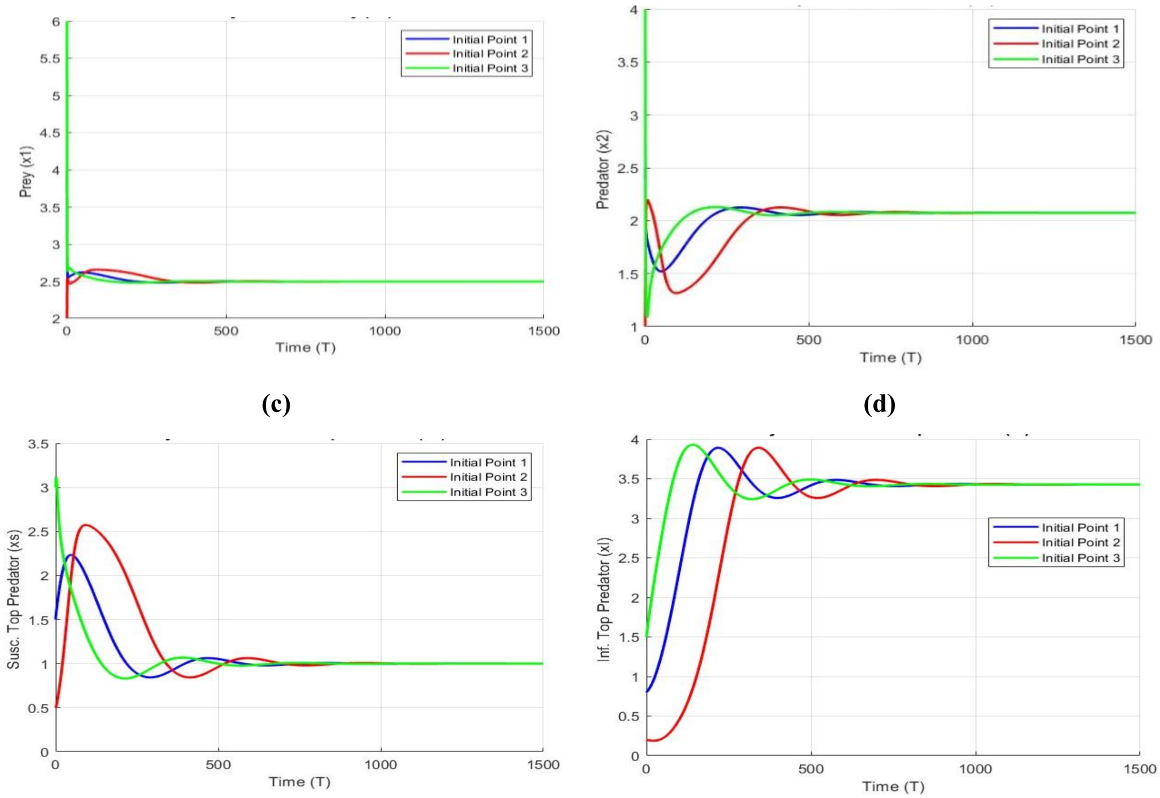


Figure -1 The paths followed by system (1), which began at three different starting points (3.2, 2.8, 1.0, 0.5), (2.0, 1.0, 0.5, 0.2), and (6.0, 4.0, 3.0, 1.5), for the data in (27). (a) the path of x_1 over time, (b) the path of x_2 over time, (c) the path of x_s over time, and (d) the path of x_I over time, approach to $M_9 = (2.495, 2.074, 1, 3.427)$.

As the positive equilibrium point of system (1) is asymptotically approached, the generalized approximation of the solution (GAS) of system (1) is clear in Figure 1. $M_9 = (2.495, 2.074, 1, 3.427)$.

To examine the influence of parameter values on the system's dynamic behavior, we will alter one parameter at a time while maintaining the other parameters as specified in the data (27) with the beginning point (3.2, 2.8, 1.0, 0.5), and the results are shown in Table 2.

Table 2: The dynamic behavior of system (1) at every system parameter

Parameter Range	**Stable (EP)**
$0.00001 \leq a_1 < 0.40461$, $0.40461 \leq a_1 < 5$, $0.00001 \leq a_1 < 0.00724, a_2 = 0.01$	M_8 M_9 M_0
$0.036 \leq k_1 < 5$.	M_9
$0.00001 \leq f_1 < 2.96188$, $2.96188 \leq f_1 < 5$.	M_9 M_8
$0.031 \leq b_1 < 5$.	M_9
$0 \leq c_1 < 5$.	M_9
$0.00001 \leq \rho_1 < 0.38793$,	M_9

$0.38793 \leq \rho_1 < 5.$	M_8
$0.00001 \leq \delta_1 < 5.$	M_9
$0.00001 \leq d_1 < 1.$	M_9
$0.00001 \leq a_2 < 0.10863,$ $0.10863 \leq a_2 < 0.2971,$ $0.2971 \leq a_2 < 0.52196,$ $0.52196 \leq a_2 < 5,$ $0.09772 \leq a_2 < 0.16684, a_1 = 0.1,$ $0.16684 \leq a_2 < 0.37843, a_1 = 0.1.$	M_1 M_3 M_6 M_9 M_2 M_4
$0.02187 \leq k_2 < 1.35428,$ $1.35428 \leq k_2 < 1.53771,$ $1.53771 \leq k_2 < 5.$	M_3 M_6 M_9
$0 < f_2 < 2.64849,$ $2.64849 \leq f_2 < 5.$	M_9 M_6
$0.00001 \leq g_1 < 0.5.$	M_9
$0.21 \leq b_2 < 2.94226,$ $2.94226 \leq b_2 < 5.$	M_9 M_6
$0.00001 \leq c_2 < 0.00021,$ $0.00021 \leq c_2 < 0.00023,$ $0.00023 \leq c_2 < 0.00024.$	M_6 M_9 M_1
$0.00001 \leq \gamma_1 < 0.68506,$ $0.68506 \leq \gamma_1 < 0.74342,$ $0.74342 \leq \gamma_1 < 5.$	M_9 M_6 M_3
$0.00001 \leq \eta < 0.23225,$ $0.23225 \leq \eta < 0.29801,$ $0.29801 \leq \eta < 0.54654,$ $0.54654 \leq \eta < 5.$	M_9 M_6 M_3 M_1
$0 < e_1 < 1.$	M_9
$0.00001 \leq \rho_2 < 0.95367,$ $0.95367 \leq \rho_2 < 5.$	M_9 M_6
$0.00001 \leq \delta_2 < 0.43883,$ $0.43883 \leq \delta_2 < 0.60005,$ $0.60005 \leq \delta_2 < 5.$	M_9 M_6 M_3
$0.00001 \leq d_2 < 0.51602,$ $0.51602 \leq d_2 < 0.69311,$ $0.69311 \leq d_2 < 1.$	M_9 M_6 M_3
$0 \leq g_2 < 0.11347,$ $0.11347 \leq g_2 < 0.12494,$ $0.12494 \leq g_2 < 0.4$	M_3 M_6 M_9
$0 < e_2 < 1.$	M_9
$0.00001 \leq \delta_3 < 0.08601,$ $0.08601 \leq \delta_3 < 0.09429,$ $0.09429 \leq \delta_3 < 5.$	M_9 M_6 M_3

$0.00001 \leq \beta < 0.00295$, $0.00295 \leq \beta < 0.29$, $0.29 \leq \beta \leq 1$.	M_6 M_9 M_3
$0.00001 \leq d_3 < 0.04601$, $0.04601 \leq d_3 < 0.05429$, $0.05429 \leq d_3 < 1$.	M_9 M_6 M_3
$0.00001 \leq d_4 < 0.0289$, $0.0289 \leq d_4 < 0.05429$.	M_9 M_6

The influence of changing the growth rate of prey within the range $0.0001 \leq a_1 < 0.40461$ was investigated, It has been noted that system (1) continues to approach M_8, Nonetheless, extending this parameter to its extremes further $0.40461 \leq a_1 < 5$ the solution approach to M_9, when, pushing this parameter to its limits further $0.00001 \leq a_1 < 0.00724$ and $a_2 = 0.01$ the solution close to M_0, as seen in Figure (2).

(a) **(b)** **(c)**

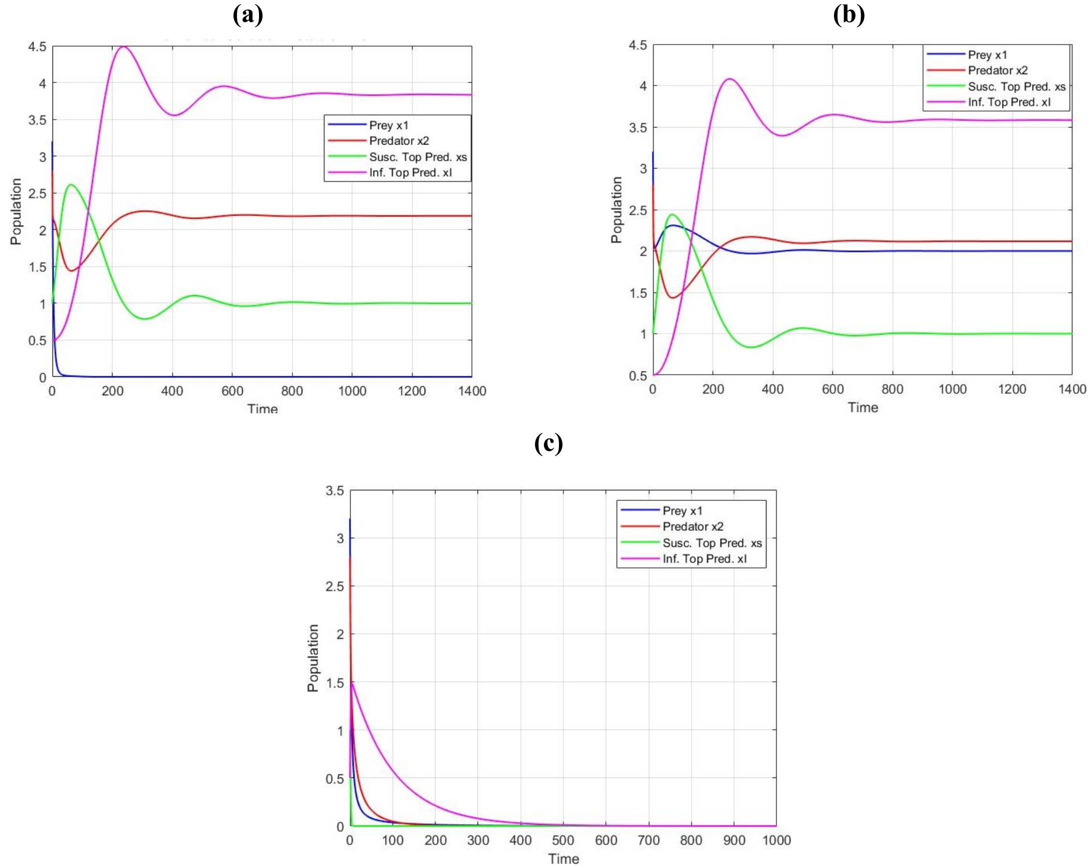
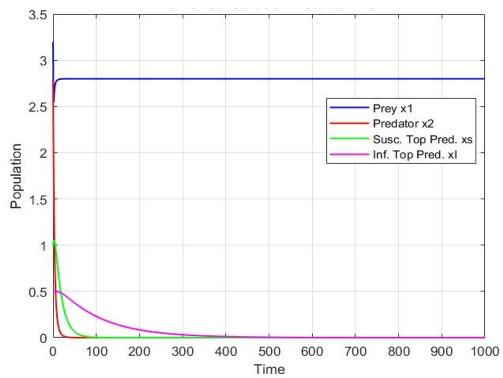
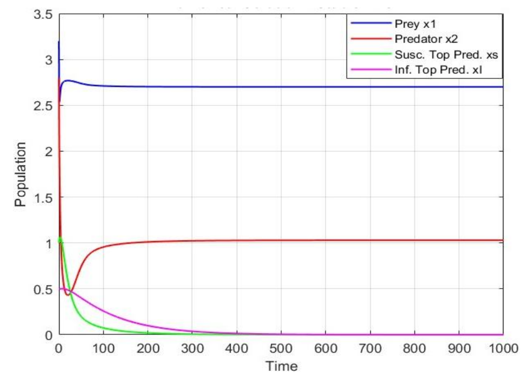


Figure-2 (a) the solution time series for system (1), which approaches to $M_8 = (0, 2.188, 1, 3.834)$ when a_1= 0.2, (b) the solution time series for system (1) which approach to $M_9 = (2, 2.117, 1, 3.582)$ when a_1= 1.2, (c) the solution time series for system (1), which approaches to $M_0 = (0, 0, 0, 0)$ when a_1= 0.002 and a_1= 0.01.

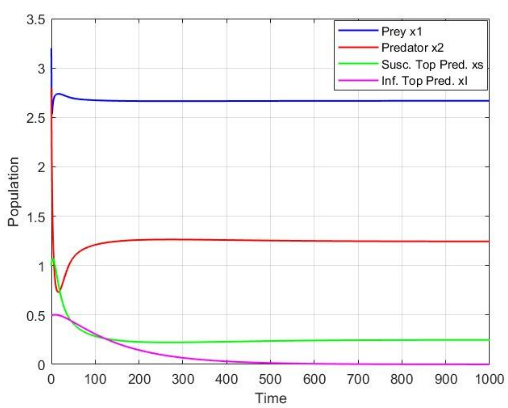
The study examines the predator population's response to changes in the growth rate of the predator population, with a range of $0.00001 \leq a_2 < 0.10863$. It is observed that system (1) is close to M_1, but the system reaches M_3 when the parameter is increased by $0.10863 \leq a_2 < 0.2971$. When the parameter is increased by $0.2971 \leq a_2 < 0.52196$, the system is close to M_6, but the system reaches M_9 when the parameter is increased by $0.52196 \leq a_2 < 5$, also increasing this parameter further $0.09772 \leq a_2 < 0.16684$ when $a_1 = 0.1$, the system reaches M_2, finally upping this parameter even further $0.16684 \leq a_2 < 0.37843$ when $a_1 = 0.1$, the system is close to M_4, as seen in Figure (3).

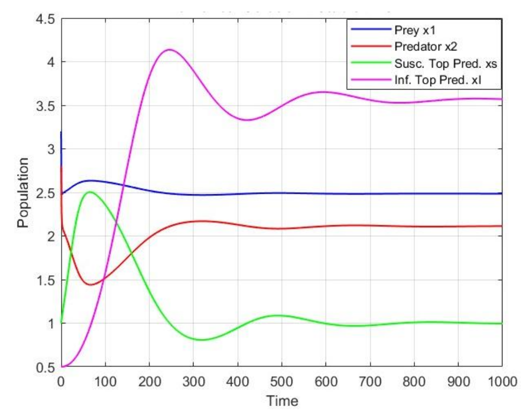
(a) **(b)**



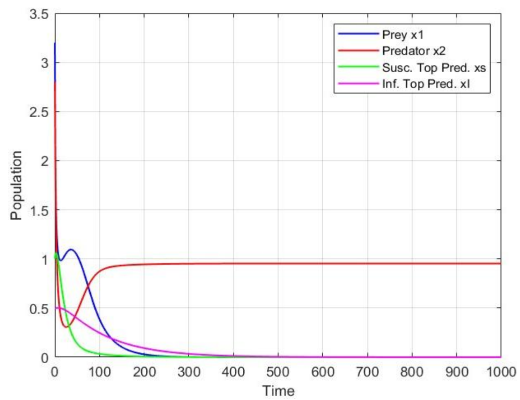


(c) **(d)**





(e) **(f)**



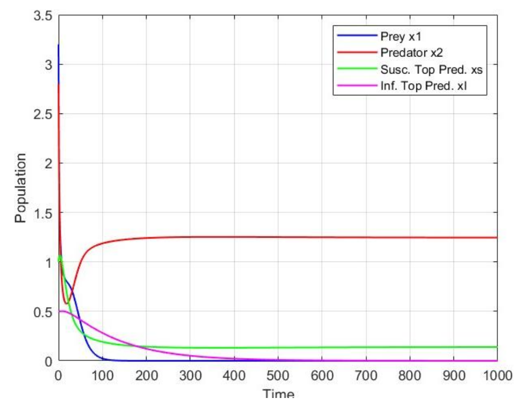
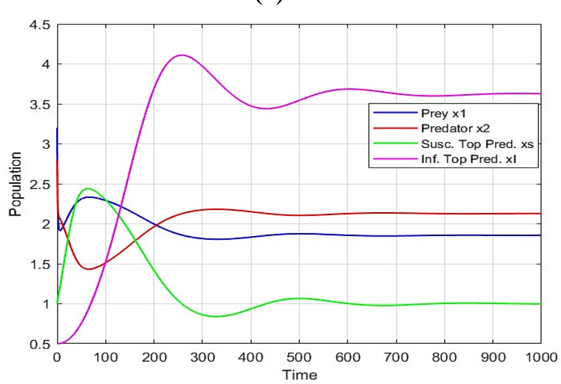


Figure -3 (a) The solution time series for system (1) which approach to $M_1 = (2.8, 0, 0, 0)$ when $a_2 = 0.005$, (b) the solution time series for system (1) which approach to $M_3 = (2.7, 1.029, 0, 0)$ when a_2= 0.25, (c) the solution time series for system (1) which approach to $M_6 = (2.668, 1.245, 0.248, 0)$ when a_2= 0.36, (d) the solution time series for system (1) which approach to $M_9 = (2.485, 2.113, 0.995, 0.569)$ when a_2=1.6, (e) the solution time series for system (1) which approach to $M_2 = (0, 0.953, 0, 0)$ when a_2= 1.2 and a_1= 0.1, (f) the solution time series for system (1) which approach to $M_4 = (0, 1.246, 0.139, 0)$ when a_2= 0.2 and a_1= 0.1.

The impact of switching up the amount of fear of prey inside the range $0.00001 \leq f_1 < 2.96188$, It was found that system (1) is still getting closer to M_9 asymptotically throughout the course of the research, regardless of finding this parameter was raised much further $2.96188 \leq f_1 < 5$, the probability of the infected apex predator becoming extinct increases, and the system is close to M_8 as seen in Figure (4).

(a) **(b)**



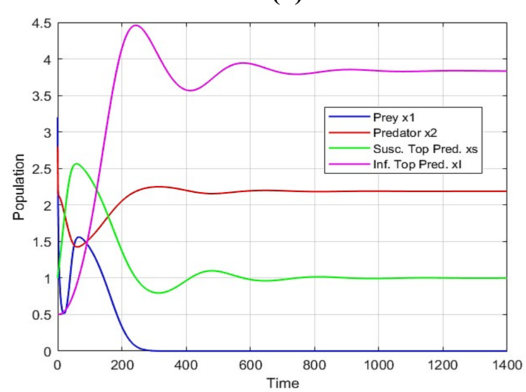


Figure -4 (a) The solution time series for system (1) which approach to $M_9 = (1.854, 2.129, 0.997, 3.627)$ when $f_1 = 0.9$, (b) the solution time series for system (1) which approach to $M_8 = (0, 2.188, 1, 3.834)$ when $f_1 = 3.4$.

The impact of switching up the amount of fear of predator inside the range $0.00001 \leq f_2 < 2.64849$, the investigation revealed that system (1) continued to asymptotically approach the positive equilibrium point defined by M_9, even as this parameter was further raised. $2.64849 \leq$

$f_2 < 5$, augments the probability of extinction for the infected apex predator, hence putting the system is close to M_6, as seen in Figure (5).

(a) **(b)**

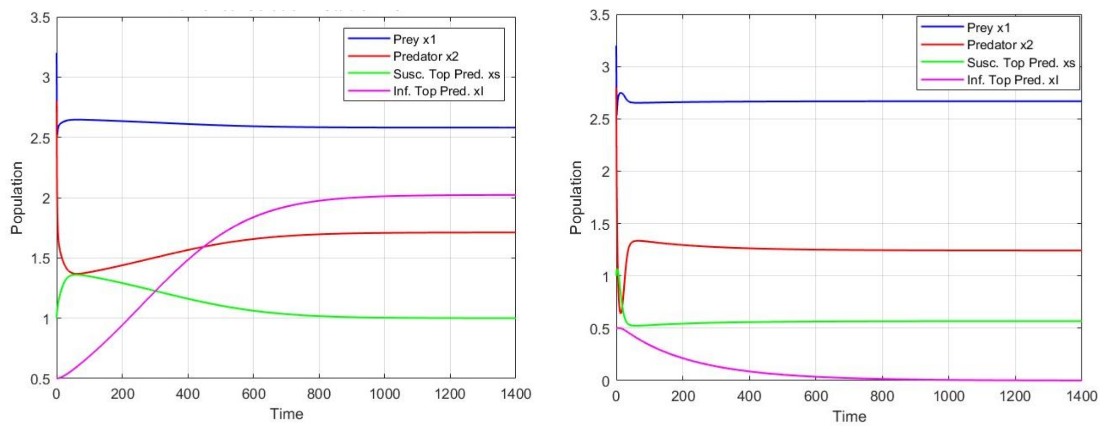


Figure -5 (a) The solution time series for system (1) which approach to $M_9 = (2.581, 1.711, 1, 2.022)$ when $f_2 = 0.8$, (b) the solution time series for system (1) which approach to $M_6 = (2.668, 1.242, 0.567, 0)$ when $f_2 = 4.5$.

When the hunting cooperation in predator the impact of this difference in the range $0.00001 \leq \rho_1 < 0.38793$ was investigated, it has been noted that system (1) continues to approach M_9, regardless of finding this parameter was raised much further $0.38793 \leq \rho_1 < 5$ and the system is close to M_8, as seen in Figure (6).

(a) **(b)**

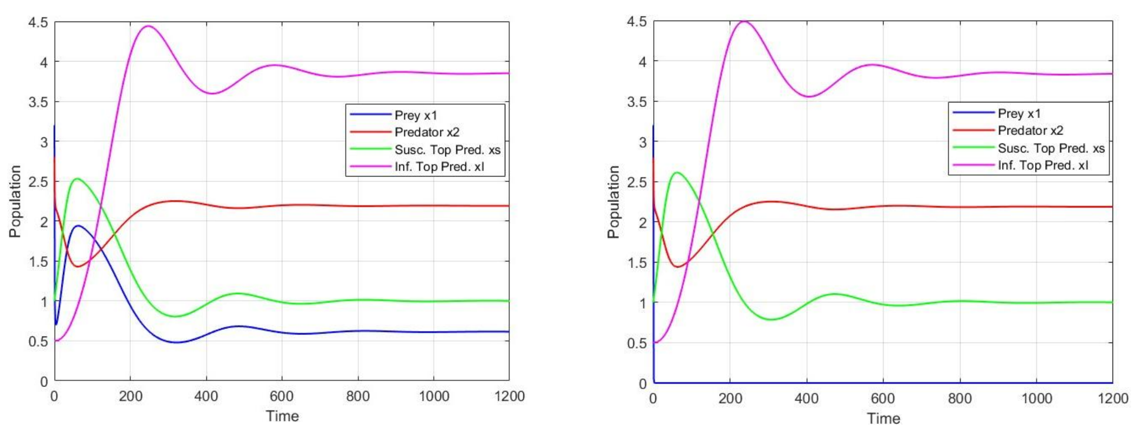


Figure -6 (a) The solution time series for system (1) which approach to $M_9 = (0.616, 2.191, 1, 3.852)$ when $\rho_1 = 0.3$, (b) the solution time series for system (1) which approach to $M_8 = (0, 2.187, 1, 3.838)$ when $\rho_1 = 0.8$.

When the hunting cooperation in susceptible top predator the impact of this difference in the range $0.00001 \leq \rho_2 < 0.95367$ was investigated, it has been noted that system (1) continues to approach M_9, Nonetheless, extending this parameter to its extremes further $0.95367 \leq \rho_2 < 5$

and the system is close to M_6, as seen in Figure (7).

(a) **(b)**

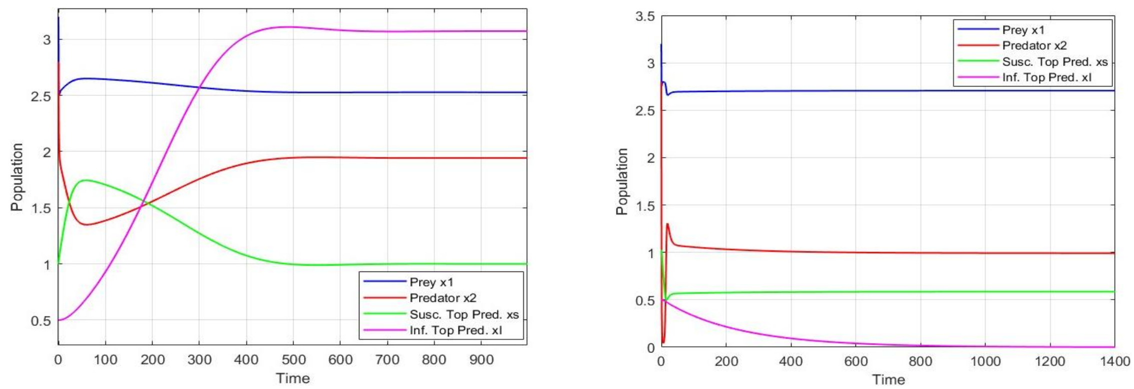
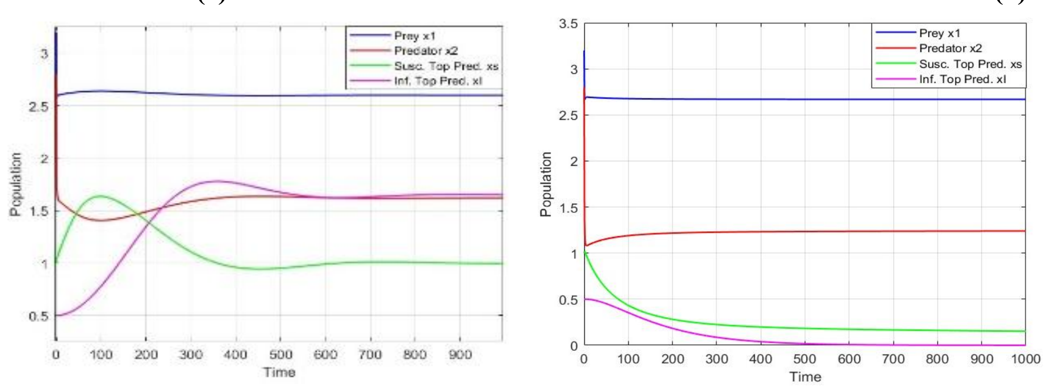


Figure -7 (a) The solution time series for system (1) which approach to $M_9 = (2.528, 1.943, 1, 3.072)$ when ρ_2 = 0.1, (b) the solution time series for system (1) which approach to $M_8 = (2.706, 0.992, 0.586, 0)$ when ρ_2 = 2.

This research investigated the toxin rate of the predator and the effects of variations within the range of $0.00001 \leq \delta_2 < 0.43883$, it has been noted that system (1) continues to approach M_9, the effect of this variation $0.43883 \leq \delta_2 < 0.60005$ was investigated, it has been noted that system (1) continues to approach M_6. Nonetheless, extending this parameter to its extremes further $0.60005 \leq \delta_2 < 5$ the system is close to M_3, as seen in Figure (8).

(a) **(b)**



(c)

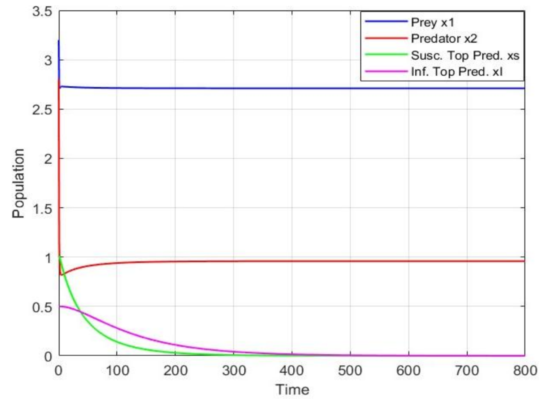
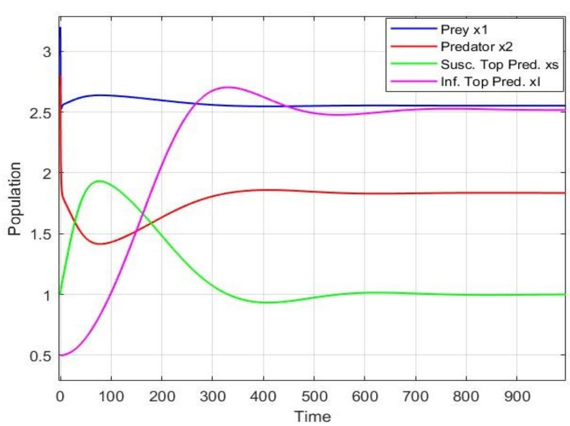
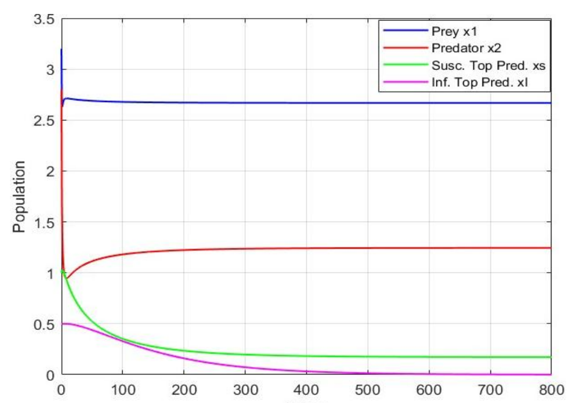


Figure -8 (a)- The solution time series for system (1) which approach to $M_9 = (2.599, 1.621, 1, 1.653)$ when δ_2 = 0.2, (b) the solution time series for system (1) which approach to $M_6 = (2.669, 1.24, 0.153, 0)$ when δ_2 =0.55, (c) the solution time series for system (1) which approach to $M_3 = (2.71, 0.959, 0, 0)$ when δ_2 = 0.9.

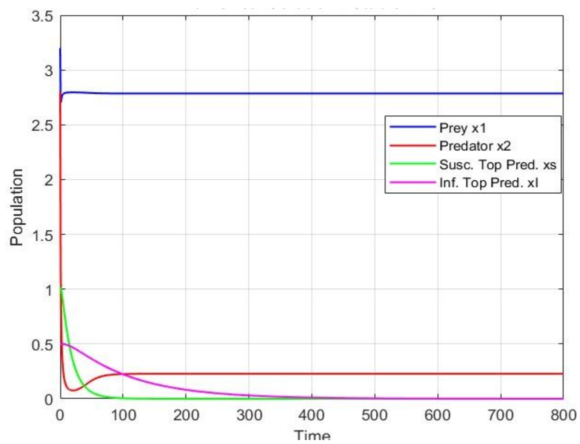
The number of predators in the range changes when the number of food that is protected by predators changes $0.00001 \leq \eta < 0.23225$, it is observed that system (1) is close to M_9, although the system reaches M_6 when the parameter is increased further by $0.23225 \leq \eta < 0.29801$, and M_3 when the parameter is increased by $0.29801 \leq \eta < 0.54654$, finally, the system is close to M_1 when the parameter is increased by $0.54654 \leq \eta < 5$, , as seen in Figure (9).

(a) **(b)**





(c) **(d)**



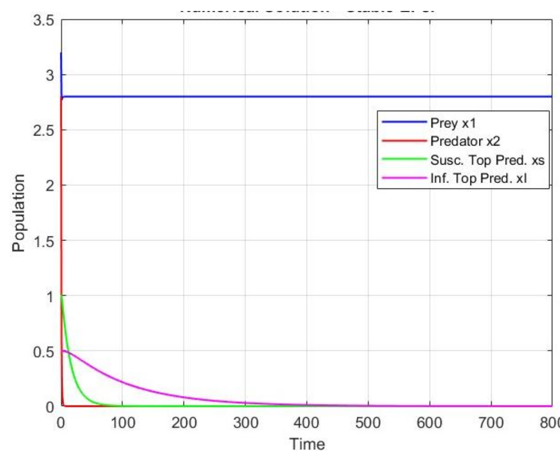
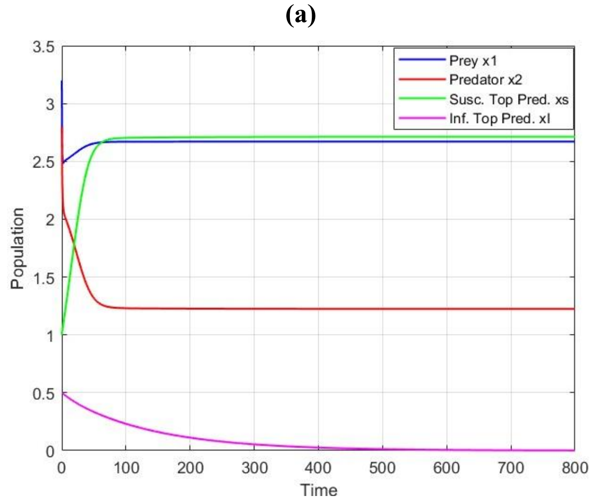


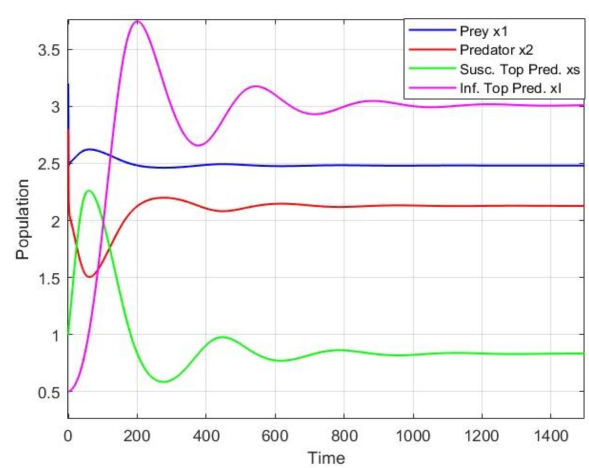
Figure -9 (a) The solution time series for system (1) which approach to $M_1 = (2.553, 1.835, 1, 2.517)$ when $\eta =$ 0.1, (b) the solution time series for system (1) which approach to $M_6 = (2.668, 1.245, 0.172, 0)$ when η= 0.28, (c) the solution time series for system (1) which approach to $M_3 = (2.785, 0.228, 0, 0)$ when η= 0.5, (d) the solution time series for system (1) which approach to $M_1 = (2.8, 0, 0, 0)$ when η=0.9.

The dynamical behavior of system (1) is examined in the range of changes in the disease transmission rate of the top predator $0.00001 \le \beta < 0.00295$, it is observed that system (1) approaches M_6. However, when this parameter is further increased to $0.00295 \le \beta < 0.29$, the system reaches M_9. Finally, when this parameter is increased to $0.29 \le \beta \le 1$,, the system is close to M_3, as seen in Figure (10).

(a)



(b)



(c)

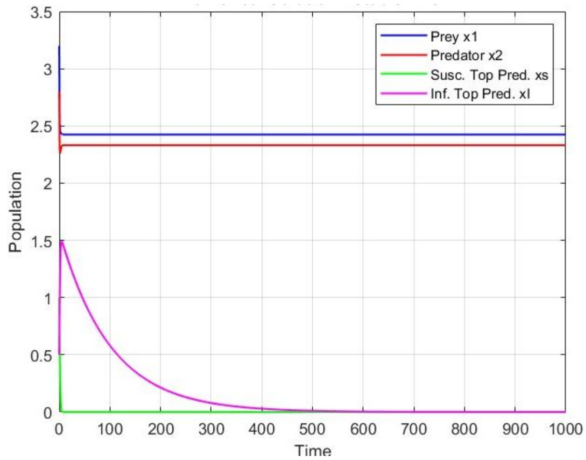


Figure -10 (a)- The solution time series for system (1) which approach to $M_6 = (2.671, 1.224, 2.712, 0)$ when β=0.001, (b) the solution time series for system (1) which approach to $M_9 = (2.481, 2.128, 0.835, 3.01)$ when β=0.012, (c) the solution time series for system (1) which approach to $M_3 = (2.425, 2.331, 0, 0)$ when β = 0.8.

8. CONCLUSION AND DISCUSSION

This study included the presentation and analysis of the dynamic behavior of a modified Sokol-Howell prey-predator model, which integrates ecological factors such as fear, hunting cooperation, anti-predators, environmental toxins, and diseased in top predator undergoing treatment. Our objectives are to examine:

- The effects of fear, anti-predator and toxin secreted on predator in prey population.
- The effects of hunting cooperation and environmental toxicity in predator population.
- The effects of hunting cooperation and toxicity from eating predator in susceptible top predator population.

System (1) has eleven equilibria, all of which are locally asymptotically stable under suitable conditions. Five equilibria are examined globally (M_0, M_1, M_2, M_3, M_8), while ($M_4, M_5, M_6, M_7, M_9, M_{10}, M_{11}$) possess unique initial neighborhoods while sharing identical local stability criteria. Ultimately, we used numerical verification to validate our analytical findings for the data delineated in (27) and defined.

1. The parameters a_1, f_1, ρ_1, d_1, a_2, k_2, f_2, $b_2, c_2, \gamma_1, \eta, \rho_2, \delta_2, d_2, g_2, \delta_3, \beta, d_3$ and d_4 have a significant impact on maintaining system (1).
2. The parameters k_1, b_1, c_1, δ_1, g_1, e_1 and e_2 the solutions are still moving toward the point of positive equilibrium.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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