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TWO TYPES OF TRACING PROPERTIES IN NON-AUTONOMOUS DYNAMICAL SYSTEM

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Abstract: In this paper, we introduce the notions of asymptotic pseudo-orbit tracing property and pointwise pseudo-orbit tracing property in non-autonomous dynamical system, which are the generalizations of corresponding notions in autonomous dynamical system. We showed that if (X,F) is topologically conjugate to (Y,G), then F has asymptotic pseudo-orbit tracing property if and only if G has the same property. Also we discuss the relationship of pointwise pseudo-orbit tracing property between the product system and its subsystems.

Keywords: pointwise pseudo-orbit; asymptotic pseudo-orbit; topological conjugation; product system.

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1. Introduction

Throughout this paper, by a non-autonomous dynamical system we mean a pair (X,F), where X is a compact metric space with the metric d, and $F=\{f_k\}_{k=1}^{\infty}$ is a sequence of continuous maps on X, i.e. $f_k : X \to X$ is continuous for $k=1,2,\cdots$, and for

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any point $x_1 \in X, x_2 = f_1(x_1)$, $x_3 = f_2(x_2)$, ..., $x_{n+1} = f_n(x_n)$, ..., $n = 1, 2, \ldots$. If $f_k = f$, we call (X, f) an autonomous dynamical system.

Pseudo-orbit and its tracing skills are powerful tools in discussing autonomous dynamic systems. Pseudo-orbit tracing property (POTP) has close relations with chaotic properties of system. The concept of pseudo-orbit in autonomous dynamical system has firstly appeared in the work of Anosov [1-2], and is closed related with the property of stability [3] and chaos [4-5]. Li established the definition of pointwise pseudo-orbit tracing property (PPOTP) in an autonomous dynamical system [6]. The so-called asymptotic pseudo-orbit tracing property (APOTP) in autonomous dynamical system was introduced by S. Y. Pilyugin. And many papers were devoted to its study, see [7-9] for detail.

In this paper, we extend the notions of pointwise pseudo-orbit tracing property and asymptotic pseudo-orbit tracing property to non-autonomous systems, and we show that if the system (X, F) has the asymptotic pseudo-orbit tracing property, then any system (Y, G) conjugated with (X, F) has the same property. Also we investigate the relationship of pointwise pseudo-orbit tracing property between the product system and its subsystem.

Our results extend the scope of the research on discrete dynamical system and generalize the existing results to a very general case. The remainder of this paper is organized as follows. Section 2 introduces some basic definitions and some notations, also several new concepts are given in this part. Main results are established in Section 3.

2. Preliminaries

Firstly we complete some notations and recall some concepts.

Let (X, F) be a non-autonomous dynamical system, for convenience, denote maps $F_k : X \to X$ by $F_k(x) = f_k(f_{k-1}(\ldots f_1(x))) = f_k \circ f_{k-1} \circ \cdots \circ f_1(x), \quad (k = 1, 2, \cdots)$. Then

$x_2 = f_1(x_1) = F_1(x_1), x_3 = f_2 \circ f_1(x_1) = F_2(x_1), \cdots, x_{n+1} = f_n \circ f_{n-1} \circ \cdots \circ f_1(x_1) = F_n(x_1), \cdots$.

The following concept can be found in the classical mathematical theory. For the sake of completeness, they are listed as follows.

Definition 2.1 *The function* $h: X \to Y$ *is called a homeomorphism from a metric space* (X, d_X) *into a metric space* (Y, d_Y) *if, it is one-to-one and onto, and both* h *and* h^{-1} *are continuous.*

Definition 2.2 *Let* (X, F) *and* (Y, G) *be non-autonomous dynamical systems with metrics* d_X and d_Y *respectively, where* $F = \{f_k\}_{k=1}^{\infty}$ *and* $G = \{g_k\}_{k=1}^{\infty}$. $h: X \to Y$ *be a homeomorphism. If for any* $k \ge 1$, $g_k \circ h(x) = h \circ f_k(x), x \in X$, *then* F *and* G *are said to be conjugate or* $h-$*conjugate.*

Let (X, F) and (Y, G) be non-autonomous discrete systems and $X \times Y$ be product space with metric $d^{"}((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$, where d_X and d_Y are metrics, $F = \{f_k\}_{k=1}^{\infty}$ and $G = \{g_k\}_{k=1}^{\infty}$ are sequence of maps on X and Y , respectively. Let the sequence of maps $F \times G = \{f_i \times g_i\}_{i=1}^{\infty}$ be defined by $\forall (x, y) \in X \times Y$, $(f_i \times g_i)(x, y) = (f_i(x), g_i(y)), i = 1, 2, \cdots$. And it is easy to verify that

$$(F \times G)^i(x, y) = (f_i \times g_i) \circ (f_{i-1} \times g_{i-1}) \circ \cdots \circ (f_1 \times g_1)(x, y) = (F_i \times G_i)(x, y).$$

Then we obtain the product system $(X \times Y, F \times G)$ with metric $d^{"}$.

Now, we introduce some new notions in non-autonomous discrete system.

Definition 2.3 *Let* (X, F) *be a non-autonomous discrete systems. If* $\delta > 0$, *and for any* $0 < i < n \le \infty$, $d(f_i(x_i), x_{i+1}) < \delta$, *then the sequence* $\{x_1, x_2 \cdots, x_n\}$ *is called a* δ *-pseudo-orbit of* F *(or* δ *-chain). If for any* $x, y \in X, \delta > 0$, *there exists a finite* δ *pseudo-orbit* $\{x_1, x_2 \cdots, x_n\}$ *of* X , *such that* $x_1 = x, x_n = y$, *then* $\{x_1, x_2 \cdots, x_n\}$ *is called a* δ *-chain from* x *to* y .

Definition 2.4 *Let* (X, F) *be a non-autonomous discrete systems, we say the* *sequence* $\{x_1, x_2 \cdots, x_n\}$ *is* ε *-traced* (ε *>0) by the point* z *in* X *if* $d(F_n(z), x_n) < \varepsilon$ *for every positive integer* n . *We say* F *has pointwise pseudo-orbit tracing property if for any* ε *>0 there exists a real number* $\delta > 0$ *such that for any* δ *-pseudo-orbit* $\{x_1, x_2 \cdots, x_n\}$ *of* F , $\{x_N, x_{N+1} \cdots,\}$ *can be* ε *-traced for some* $N > 0$, *i.e. there exists a point* z *in* X *such that* $d(F_k(z), x_{N+k}) < \varepsilon$ *for* $k > 0$ ..

Definition 2.5 *Let* (X, F) *be a non-autonomous discrete systems. A sequence* $\{x_i\}_{i=1}^{\infty}$ *in* X *is called an asymptotic pseudo-orbit of* F *if* $\lim_{i\to\infty} d(f_i(x_i), x_{i+1}) = 0$. *A sequence* $\{x_i\}_{i=1}^{\infty}$ *is said to be asymptoticcally pseudo-orbit tracing by the point* z *in* X *if* $\lim_{n\to\infty} d(F_n(z), x_n) = 0$. F *is said to have asymptotic pseudo-orbit tracing property if every asymptotic pseudo orbit of* F *can be asymptotically pseudo-orbit tracing by some point in X.*

3. Main results

Theorem 3.1 *Let* X *and* Y *be compact metric spaces with metrics* d_X and d_Y *respectively, and* $F: X \to X$ *and* $G: Y \to Y$ *be sequences of continuous maps. If* F *is topologically conjugate to* G , *then* F *has the asymptotic pseudo-orbit tracing property if and only if* G *has the same property.*

Proof. It is enough to prove the necessity. Suppose F has the asymptotic pseudo-orbit tracing property, and let $\{y_i\}_{i=0}^{\infty}$ be an asymptotic pseudo-orbit of G and h be a conjugate function from X to Y , by the compactness of Y , $h^{-1}: Y \to X$ is uniformly continuous. Then for any $\varepsilon' > 0$, there exists $\delta' > 0$ such that

$$d_Y(y, y') < \delta' \Rightarrow d_X(h^{-1}(y), h^{-1}(y')) < \varepsilon' \text{ for } y, y' \in Y .$$

As $\{y_i\}_{i=1}^{\infty}$ is an asymptotic pseudo-orbit of G , i.e. $\lim_{i\to\infty} d_Y(g_i(y_i), y_{i+1}) = 0$, so there exists a positive integer N_1 such that

$$d_Y(g_i(y_i), y_{i+1}) < \delta' \text{ for each } i \ge N_1 .$$

Hence for each $i \geq N_1$,

$$d_X(h^{-1}(g_i(y_i)), h^{-1}(y_{i+1})) < \varepsilon' .$$

Let $x_i = h^{-1}(y_i)$, the above inequality becomes

$$d_X(h^{-1} \circ g_i \circ h(x_i)), x_{i+1}) < \varepsilon' .$$

By the conjugation of h, we have

$$d_X(f_i(x_i)), x_{i+1}) < \varepsilon' .$$

Hence

$$\lim_{i \to \infty} d_X(f_i(x_i)), x_{i+1}) = 0 .$$

This shows that the sequence $\{x_i\}_{i=1}^{\infty}$ is an asymptotic pseudo-orbit of F, since F has the asymptotic pseudo-orbit tracing property. Therefore there is a point z' in X such that

$$\lim_{i \to \infty} d_X(F_i(z')), x_i) = 0 .$$

To prove that $\{y_i\}_{i=1}^{\infty}$ can be asymptotically traced by some point z in Y, let $z = h(z')$, by the uniformly continuity of h, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x, x') < \delta \Rightarrow d_Y(h(x), h(x')) < \varepsilon \quad \text{for } x, x' \in X .$$

Then as $\lim_{i \to \infty} d_X(F_i(z')), x_i) = 0$, there exists a positive integer N_2 such that

$$d_X(F_i(z')), x_i) < \delta \quad \text{for each } i \geq N_2 .$$

Therefore for each $i \geq N_2$, we have

$$d_Y(h \circ F_i(z')), h(x_i)) < \varepsilon .$$

As $h \circ F_i = G_i \circ h$, the above inequality becomes

$$d_Y(G_i \circ h(z')), h(x_i)) < \varepsilon ,$$

That is $d_Y(G_i(z), y_i) < \varepsilon$, i.e. $\lim_{i \to \infty} d_Y(G_i(z), y_i) = 0$, this shows that G has the asymptotic pseudo-orbit tracing property. It is similar to prove the sufficiency.

This completes the proof.

Theorem 3.2 *Let X and Y be compact metric spaces with metrics* d_X and d_Y

respectively, and $F: X \to X$ *and* $G: Y \to Y$ *be sequences of continuous maps. Then the product map* $F \times G$ *has the pointwise pseudo-orbit tracing property if and only if both* F *and* G *have the pointwise pseudo-orbit tracing properties.*

Proof. Let sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ be δ - pseudo-orbit of F and G respectively, i.e.

$$d_X(f_i(x_i), x_{i+1}) < \delta, d_Y(g_i(y_i), y_{i+1}) < \delta, i = 1, 2, \cdots.$$

Then

$$d''((f_i, g_i)(x_i, y_i), (x_{i+1}, y_{i+1})) = d''((f_i(x_i), g_i(y_i)), (x_{i+1}, y_{i+1}))$$

$$= \max\{d_X(f_i(x_i), x_{i+1}), d_Y(g_i(y_i), y_{i+1})\} < \delta.$$

This means the sequence $\{x_i, y_i\}_{i=1}^{\infty}$ is the δ - pseudo-orbit of $F \times G$. Since $F \times G$ has the pointwise pseudo-orbit tracing property, for each $\varepsilon > 0$, there exists a point (z, z') in $X \times Y$ and a positive integer N such that

$$d''((F \times G)^k(z, z'), (x_N, y_N)) = d''((F^k(z), G^k(z'), (x_N, y_N))$$

$$= \max\{d_X(F^k(z), x_N), d_Y(G^k(z'), y_N)\} < \varepsilon.$$

Thus $d_X(F^k(z), x_N) < \varepsilon$ and $d_Y(G^k(z'), y_N) < \varepsilon$, this shows that F *and* G have the pointwise pseudo-orbit tracing property. It is similar to prove the sufficiency.

The proof of Theorem 3.2 is completed.

As an immediate consequence of Theorems 3.2, we have the following corollary.

Corollary 3.1 *Let* $X_1, \cdots, X_n$ *be compact metric spaces with metrics* d_{X_i} on X_i *respectively, and* $F_i: X_i \to X_i$ *be sequences of continuous maps(* $i = 1, \cdots, n$ *). Then the product map* $F_1 \times \cdots \times F_n$ *has the pointwise pseudo-orbit tracing property if and only if each* F_i *has the pointwise pseudo-orbit tracing property.*

Conflict of Interests

The author declares that there is no conflict of interests.

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