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Engineering Mathematics Letters, 1 (2012), No. 1, 6-17

ISSN 2049-9337

END EDGE DOMINATION IN SUB DIVISION OF GRAPHS

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Abstract: Let $S(G)$ be the subdivision graph of a graph $G=(V,E)$. An edge dominating set D of a sub division graph $S(G)$ is an end edge dominating set if D contains all end edges of $S(G)$. The end edge domination number $\gamma_e^{'} \; S \; G$ of $S(G)$ is the minimum cardinality of an end edge dominating set of $S(G)$. In this paper, some bounds for $\gamma_e^{'}(S(G))$ were obtained and exact values of $\gamma_e^{'}(S(G))$ for some standard graphs were also obtained. Further its relationships with other different domination parameters were obtained. Also we relate split domination and end edge domination numbers in G.

Keywords: Sub division graph; End edge dominating set; End edge domination numbers; Split domination number.

2000 AMS Subject Classification: 05C69; 05C70.

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Received December 24, 2011

1. Introduction

In this paper, we follow the notations of [1]. All the graphs considered here are simple, finite, non-trivial, undirected and connected. As usual p=$|V|$ and q=$|E|$ denote the number of vertices and edges of a graph G , respectively.

In general, we use $\langle X \rangle$ to denote the sub graph induced by the set of vertices X and $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex *v*, respectively.

The notation $\beta_0 \ G$ ($\beta_1 \ G$) is the minimum number of vertices (edges) in a maximal independent set of vertex (edge) of G . Let deg (*v*) is the degree of vertex *v* and as usual $\delta \ G$ ($\Delta \ G$) is the minimum (maximum) degree. The degree of an edge $e = uv$ of G is defined by $\deg e = \deg u + \deg v - 2$ and $\delta' \ G$ ($\Delta' \ G$) is the minimum (maximum) degree among the edges of G .

A vertex of degree one is called a pendent vertex and its neighbor is called a support vertex. A vertex v of V is called a cut vertex if removing it from G increases the number of components of G .

The subdivision graph $S(G)$ of a graph G is the graph obtained by inserting a vertex of degree two to every edge of G .

A spider is a tree with the property that the removal of all end paths of length two of T results in an isolated vertex, called the head of a spider.

A dominating set $D \subseteq V$ is said to be a split dominating set of G, if the induced sub graph $\langle V - D \rangle$ is disconnected. The minimum cardinality of vertices in such a set is called the split domination number of G and is denoted by $\gamma_s(G)$. This concept was introduced by Kulli and Janakiram [3].

A 2-packing in a graph G is a set of vertices of D that are pair wise at distance at least 3 apart i.e., D is 2-packing of G if and only if $d(u,v) \geq 3$ for all distinct $u,v \in D$.

A set $S \subseteq E$ in a graph G is an edge dominating set if every edge in $E - S$ is adjacent to at least one edge in S. The minimum cardinality of edges in such a set is called the edge domination number of G and is denoted by $\gamma'(G)$. Edge domination was introduced by S. Mitchell and S. T. Hedetniemi [4] and is now well studied in graph theory see [2].

An edge dominating set $S \subseteq E$ is said to be an end edge dominating set of G, if S contains all end edges of $E(G)$. The minimum cardinality of edges in such a set is called the end edge domination number of G and is denoted by $\gamma'_e(G)$. This concept was introduced by Muddebihal and Sedamkar [5].

An edge dominating set D of a sub division graph $S(G)$ is an end edge dominating set if D contains all end edges of $S(G)$. The end edge domination number

$\gamma_e'(S(G))$ of $S(G)$ is the minimum cardinality of an end edge dominating set of $S(G)$. In this paper, some bounds for $\gamma_e'(S(G))$ were obtained and exact values of $\gamma_e'(S(G))$ for some standard graphs were also obtained. Further its relationships with other different domination parameters were obtained. Also we relate split domination and end edge domination numbers in G.

2. Results:

We need the following Theorems to prove our later results.

Theorem A.4 [5]: *For any path P_p with $p \geq 2$ vertices,*

$$\gamma_e'(P_p) = p/3 + 1, \text{ if } p \equiv 0 \pmod 3$$

$$= \lceil p/3 \rceil, \textit{ otherwise.}$$

Corollary A [5]: *For any connected graph G, let $A = \{v_1, v_2, ..., v_m\}$, $m \geq 1$, be the set of vertices of degree one. If $A \not\subset V(G)$, then $\gamma_e'(G) = \gamma'(G)$.*

3. Main Results:

We list out end edge domination number for subdivision of some standard graphs.

Theorem 3.1*:*

1) $\gamma'_e\left(S(C_p)\right)=\gamma'_e\left(C_{2p}\right)=\begin{cases}\frac{2p}{3}, & \text{if } p\equiv 0 \ (\text{mod}\, 3)\\ \left\lceil \frac{2p}{3}\right\rceil, & \text{otherwise}\end{cases}$

2) $\gamma'_e\left(S(P_p)\right)=\gamma'_e\left(P_{2p-1}\right)=\begin{cases}\frac{2p}{3}, & \text{if } p\equiv 0 \ (\text{mod}\, 3)\\ \left\lceil \frac{2p}{3}\right\rceil, & \text{otherwise}\end{cases}$

3) $\gamma'_e\left(S(K_p)\right)=p-1$

4) $\gamma'_e\left(S(K_{1,p})\right)=p-1,$ for $p\geq 2$

Remark 3.2: *Subdivision of star* $K_{1,p}, S(K_{1,p}), p\geq 3$ *is always a spider.*

We give the following Lemma to prove our next result.

Lemma 3.3: *For any tree* $T, \beta_1\left(S(T)\right)=q$.

Proof: To prove this result we use induction on q. Let $T=e$, $S(T)=2e$, $\beta_1\left(S(T)\right)=1=q$. Assume the result is tree for any tree with q edges. Let T be a tree with $q+1$ edges and e' be an end edge of T. Then by induction hypothesis, $\beta_1\left(S\left(T-\{e\}\right)\right)=q-1$, further $\beta_1\left(S\left(T\right)\right)=\beta_1\left(S\left(T-\{e\}\right)\right)+1$ and hence $\beta_1\left(S\left(T\right)\right)=q$.

In the following theorem, we obtain an upper bound for $\gamma'_e\left(S(G)\right)$ in terms of number of edges of G.

Theorem 3.4: *For any connected* (p,q) *- graph* G *with* $p>2$, $\gamma'_e\left(S(G)\right)\leq q$.

Proof: For $p=2, \gamma_e'\left(S(G)\right) \nleq q$. Let T be a spanning tree of G . Then by Lemma 1, $\beta_1\left(S(T)\right)=q$ and any collection of q - independent edges of $S(T)$ is an end edge dominating set of $S(G)$. Hence $\gamma_e'\left(S(G)\right) \le q.$

Now we obtain one more upper bound for $\gamma_e'\left(S(T)\right)$ in terms of number of vertices of T.

Theorem 3.5: *For any tree T with $p \ge 3$, $\gamma_e'\left(S(T)\right) \le p-1$. Equality holds if and only if T is isomorphic to sub division of a spider or wounded spider or P_4 or P_5.*

Proof: Let $F=\left\{e_1, e_{2,} \ldots, e_m\right\}$ be the set of all end edges in $S(T)$. Suppose $F'=\left\{e_1, e_{2,} \ldots, e_n\right\}$ denote the set of edges which are adjacent to the edges of F and $E\left(S(T)\right)-F'=I$. Then $H \subseteq I$ is a minimal edge dominating set of I. Clearly, $F \cup H$ is an edge dominating set of $S(T)$ and $\left|F \cup H\right| \le q$. Also by Theorem 2, $\gamma_e'\left(S(T)\right) \le p-1$.

Suppose T is not a spider or wounded spider or P_4 or P_5. since $F \cup H$ is a γ_e'- set of $S(T)$, there exist at least one non end edge $e_k \in N\left(E-F \cup H\right)$ whose at most one end is adjacent to an edge of $F \cup H$. Clearly $|F \cup H| < q$, a contradiction.

Conversely, if T is isomorphic to a spider or wounded spider or P_4 or P_5 . Then by Lemma 1, $|F \cup H| = q$ and hence $\gamma_e'(S(T)) = p-1$.

The following theorem relates $\gamma_e'(T)$ and $\gamma_e'(S(T))$ in terms of vertices of T .

Theorem 3.6: *For any tree* $T, \gamma_e'(T)+\gamma_e'(S(T)) \geq p+1$. *Equality holds if* T *is isomorphic to path* P_p.

Proof: Let S be the γ_e' -set of T. After the sub division of T, let $S' = \{e_1, e_2, \ldots, e_i\}$ denote the end edge dominating set of $S(T)$. Since, there exists at least two end edges common to both T and $S(T)$, also by the Lemma 1, $|S| \cup |S'| \geq q+2$. Hence $\gamma_e'(T)+\gamma_e'(S(T)) \geq p+1$.

Suppose T is isomorphic to path, then by Theorem [A.4], we have

$$\gamma_e'(P_p) = \frac{p}{3}+1, \text{ if } p \equiv 0 \pmod 3$$
$$= \left\lceil \frac{p}{3} \right\rceil, \text{ otherwise}$$

and by 2 of Theorem 1, we have

$$\gamma_e'(S(P_p)) = \frac{2p}{3}, \text{ if } p \equiv 0 \pmod 3$$
$$= \left\lceil \frac{2p}{3} \right\rceil, \text{ otherwise.}$$

By adding these two, the equality holds.

In the following Theorem, we provide characterization of $\gamma_e'(S(G))$ for some standard graphs.

Theorem 3.7:

1) $\gamma_e'\left(S\left(K_p\right)\right) = p-1$.

2) $\gamma_e'\left(S\left(W_p\right)\right) = p-1$.

3) $\gamma_e'\left(S(K_{m,n})\right) = p-1$.

Proof: In view of Theorem 2, it is enough to prove that $\gamma_e'\left(S(G)\right) \geq p-1$, where G is either K_p, W_p or $K_{m,n}$ with $m+n=p$.

Case 1: Suppose G is isomorphic to K_p . Let $V_1 = V\left(K_p\right)$ after the subdivision, let $V_2 = V\left(S\left(K_p\right)\right) - V\left(K_p\right)$. Further, let S be any independent set of $p-2$ edges of $S\left(K_p\right)$ and S' be the set of vertices of $S\left(K_p\right)$ which are incident to the edges of S. Clearly, $|S'| = 2(p-2), |S' \cap V_1| = p-2$ and $|S' \cap V_2| = p-2$. Hence there exist exactly tow vertices u, v in $V_1 - S'$. Now the edges uw, wv, where $w \in S\left(K_p\right)$ that sub divides the edge uv are not dominated by any edge of S. Hence $\gamma'\left(S\left(K_p\right)\right) \geq p-1$. Since by Corollary [A], $\gamma_e' = \gamma'$, it follows that $\gamma_e'\left(S\left(K_p\right)\right) \geq p-1$.

Case 2: Suppose G is isomorphic to W_p. Let $V_1 = V\left(W_p\right)$ and v_k be the centre of W_p. After the sub division of G , let $V_2 = V\left(S\left(W_p\right)\right) - V\left(W_p\right)$. Further, let S be any independent set of $p-2$ edges of $S\left(W_p\right)$ and S' be the set of vertices of $S\left(W_p\right)$ which are incident to the edges of S.

Clearly, $|S'|=2(p-2), |S'\cap V_1|=p-2$ and $|S'\cap V_2|=p-2$. Hence there exists exactly two vertices u,v in V_1-S'. If uv is an edge in W_p, then the edges uw and wv where w is the vertex of $S(W_p)$ that subdivides the edge uv are not dominated by S. Suppose uv is not an edge in W_p. Let w_1, w_2 be the vertices of $S(W_p)$ which sub divide the edge v_1u, v_1v respectively. Since S is independent, at least one of the edges v_1w_1, v_1w_2 does not belong to S. Suppose $v_1w_1 \notin S$, so w_1u is not dominated by S. Thus $\gamma_e'(S(W_p)) \le p-1$.

Case 3: Suppose G is isomorphic to $K_{m,n}$ with $m+n=p$. The proof of this case is similar to that of Case 2.

The following Theorem relates end edge domination and split domination in G.

Theorem 3.8: *For any end edge dominating set S of G, if there exists at least one end edge $e\in S$. Then G has a split dominating set.*

Proof: Let $e=uv\in S$ be an end edge in G. Suppose v is an end vertex of e in G. Then there exist a cut vertex $u\in N(v)$ in G. Let D be a dominating set of G. Further, if $u\in D$, then D is a split dominating set of G. Suppose u is an end vertex, then $v\in D$ is a cut vertex. Hence $D^{-1}=(D-\{v\})\cup\{u\}$ is a split dominating set of G.

Theorem 3.9: *If G is not a tree and S is a γ'_e -set of G. Then for some $e_i \in S$ which are non-end edges, dominates the edges of $E(G)-S$ are also dominated by some $S-e_i$ edges.*

Proof: Let S be the γ'_e -set of G. If possible, assume that there exists at least one non end edge $e \in S$ such that e does not satisfy the given condition. Then $S' = S - \{e\}$ is an end edge dominating set of G, a contradiction.

Hence there exist at least one non end edge $e \in S$, which dominates at least one edge of $E(G)-S$ which is also dominated by some $S-\{e_i\}$ edges.

The following Theorem relates $\gamma'_e\left(S(T)\right)$ and $\gamma'_e(T)$.

Theorem 3.10: *For any tree T, $\gamma'_e\left(S(T)\right) \le 2\cdot\gamma'_e(T)$. Equity holds if T is isomorphic to a spider.*

Proof: Let S be the γ'_e -set of T. Insert a vertex of degree two to each edge of T to obtain $S(T)$. Let $F = \left\{e_1, e_2, \ldots, e_m\right\}$ be the set of edges whose edge degree is one, which are incident to the support vertices and $F' \in N(F)$ in $S(T)$. Suppose H is a γ' -set of $S(T)-\left\{F \cup F'\right\}$, then $F \cup H$ is an end edge dominating set of $S(T)$. Since, each edge is sub divide, $q(S(T)) = 2\cdot q(T)$ and number of end edges in both T and $S(T)$ are same, it follows that, $|F \cup H| \le 2|S|$. Hence, $\gamma'_e(S(T)) \le 2\cdot\gamma'_e(T)$.

Corollary 3.11: *For any tree T, $\gamma'_e(T) \le \gamma'_e\left(S(T)\right) \le 2\cdot\gamma'_e(T)$.*

The following Theorem relates $\gamma'_e\left(S(G)\right)$ and independence number of G.

Theorem 3.12: *For any connected* (p,q) *-graph* G, $\gamma'_e\left(S(G)\right) \le 2\left(p-\beta_1\right)$. *Equity holds if* G *is isomorphic to* K_2.

Proof: Suppose $B=\left\{u_i v_i \,/\, 1\le i\le \beta_1\right\}$ be a maximum independent set of edges of G. Then B is an edge dominating set of G. Let w_i be the vertex of $S(G)$ which is adjacent to both u_i and v_i. Let M be the set of vertices of G which are not incident with any edge of B. If $M=\phi$, then $S\subseteq E(S(G))$ is an end edge dominating set of $S(G)$ such that $|S|\le 2\cdot\beta_1 = 2\left(p-\beta_1\right)$. Hence $\gamma'_e(S(G))\le 2\left(p-\beta_1\right)$. Suppose $M\ne\phi$, let $M=\left\{x_1,x_2,\ldots,x_n\right\}$. Since B is an edge dominating set of G, $\langle M\rangle$ is independent. Furthermore, since G is connected and $\langle M\rangle$ is independent, each vertex x_i in M is adjacent to some $z_j\left(z_j=u_j \text{ or } v_k\right)$ in G. Let y_i be the vertex of $S(G)$ which is adjacent to both x_i and z_i in $S(G)$. Then another set $S_1\subseteq E(S(G))$ forms an end edge dominating set of $S(G)$ such that, $|S_1|\le 2\beta_1+2\left(p-2\beta_1\right)$. Hence $\gamma'_e(S(G))\le 2\left(p-\beta_1\right)$.

Suppose G is isomorphic to K_2. In this case $|S|=p=2$ and $|B|=1$. Clearly $|S|=2\left(p-\beta_1\right)$ and hence $\gamma'_e(S(G))=2\left(p-\beta_1\right)$.

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