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A CLASS OF ANALYTIC FUNCTIONS BASED ON AN EXTENSION OF GENERALIZED SALAGEAN OPERATOR

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Abstract. In this paper, using generalized Salagean operator, we introduce a new subclass of analytic functions. Coefficient inequalities and distortion theorems and extreme points are studied. Furthermore, we discuss certain application of fractional derivatives for $f \in TSV_\lambda^n(\alpha,\beta,\gamma)$.

Keywords: Hadamard product; Salagean operator; univalent functions; fractional derivative.

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1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad (a_j \geq 0), \tag{1.1}$$

which are analytic in the open disc $\Delta = \{z : |z| < 1\}$ and normalized by $f(0) = 0$, $f'(0) = 1$. Let S be the subclass of A consisting of univalent functions $f(z)$ of the form (1.1) Further denote by

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T the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \quad (a_j \geq 0). \tag{1.2}$$

We recall the Al-Oboudi operator (see [1]) denoted by D_λ^n, $(n \in N; \lambda \geq 0)$ given by $D_\lambda^n : A \to A$,

$$D_\lambda^0 f(z) = f(z), \tag{1.3}$$

$$D_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z), \;\; \lambda \geq 0, \tag{1.4}$$

$$D_\lambda^n f(z) = D_\lambda (D_\lambda^{n-1} f(z)). \tag{1.5}$$

From (1.4) and (1.5), we see that

$$D_\lambda^n f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^n a_j z^j. \tag{1.6}$$

It is of interest to note that for $\lambda = 1$, we have Salagean operator [11].

Remark 1.1. *If $f \in T$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, then*

$$D_\lambda^n f(z) = z - \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^n a_j z^j.$$

Further we recall the well-known subclasses of S, the class consisting of functions starlike of order $\alpha (0 \leq \alpha < 1)$ denoted by $S^*(\alpha)$ and convex of order $\alpha (0 \leq \alpha < 1)$, denoted by $K(\alpha)$. For convenience, we write $S^*(0) = S^*$ and $K(0) = K$. Silverman [12] investigated functions in the classes $T^*(\alpha) = T \cap S^*(\alpha)$ and $C(\alpha) = T \cap K(\alpha)$. In 1991, Goodman [3, 4], introduced the classes UCV and UST of uniformly convex and uniformly starlike functions respectively. Further Ma and Minda [5] and Ronning [9] independently gave the one-variable characterization for the classes UCV and UST as given below.

A function $f(z)$ of the form (1.1) is in UCV if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|, \;\; z \in \Delta.$$

Ronning (see [9, 10]), introduced the class S_p consisting of starlike functions $zf'(z)$, $f \in UCV$ and the class $S_p(\alpha)$ of functions of the form (1.1) for which

$$\Re\left(\frac{zf'(z)}{f(z)}\right) \geq \left|\frac{zf'(z)}{f(z)} - 1\right|, \;\; \alpha \in [-1, 1), z \in \Delta.$$

Ronning [10] also defined the class $UCV(\alpha)$, of uniformly convex functions of order α for which $zf' \in S_p(\alpha)$. Subramanian *et al.* [16] introduced the classes $TS_p(\alpha)$ and $TV(\alpha)$, $\alpha \in [-1,1)$ as follows: A function $f(z)$ of the form (1.2) is in $TS_p(\alpha)$, $\alpha \in [-1,1)$ if

$$\Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) \geq \left|\frac{zf'(z)}{f(z)} - 1\right|$$

and is in $TV(\alpha)$ if $zf' \in TS_p(\alpha)$.

In this paper, using the operator $D_\lambda^n f(z)$, we define the following new class motivated by Murugusundaramoorthy and Magesh [6].

Definition 1.2. The function $f(z)$ of the form (1.1) is in the class $SV_\lambda^n(\alpha,\beta,\gamma)$ if it satisfies the inequality

$$Re\left\{\frac{z(D_\lambda^n f(z))'}{(1-\gamma)D_\lambda^n f(z) + \gamma z(D_\lambda^n(f(z))'} - \alpha\right\}$$
$$> \beta\left|\frac{z(D_\lambda^n f(z))'}{(1-\gamma)D_\lambda^n f(z) + \gamma z(D_\lambda^n(f(z))'} - 1\right|$$

for $0 \leq \gamma < 1$, $-1 \leq \alpha < 1$, $\beta \geq 0$, $\lambda \geq 0$.

Further we define $TSV_\lambda^n(\alpha,\beta,\gamma) = SV_\lambda^n(\alpha,\beta,\gamma) \cap T$.

Remark 1.3.

(1) $TSV_0^0(\alpha,\beta,0) = TS(\alpha,\beta)$ *(Bharathi et al.* [2]*).*

(2) $TSV_0^0(\alpha,0,1) = T^*(\alpha)$ *(Silverman* [12]*).*

(3) $TSV_0^0(\alpha,1,0) = TS_p(\alpha)$ *(Subramanian et al.* [16]*).*

(4) $TSV_0^0(\alpha,\beta,\gamma) = TS_p(\alpha,\beta,\gamma)$ *(Murugusundaramoorthy and Magesh* [6]*).*

For $f \in TSV_\lambda^n(\alpha,\beta,\gamma)$ we obtain coefficient characterization ,Distortion theorems , extreme points and modified hadmard product results. Furthermore, we discussed certain application of fractional derivatives for $f \in TSV_\lambda^n(\alpha,\beta,\gamma)$.

2. Properties of the class $TSV_\lambda^n(\alpha,\beta,\gamma)$

Theorem 2.1. *The function $f(z)$ of the form (1.1) is in $TSV_\lambda^n(\alpha,\beta,\gamma)$ if*

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^n[j(1+\beta)-(\alpha+\beta)(\gamma j-\gamma+1)]a_j < 1-\alpha. \tag{2.1}$$

Proof. It suffices to show that

$$\beta\left|\frac{z(D_\lambda^n f(z))'}{(1-\gamma)D_\lambda^n f(z)+\gamma z(D_\lambda^n(f(z))'}-1\right| \\ -Re\left\{\frac{z(D_\lambda^n f(z))'}{(1-\gamma)D_\lambda^n f(z)+\gamma z(D_\lambda^n(f(z))'}-\alpha\right\}\leq 1-\alpha.$$

Now we consider

$$\begin{aligned}&\beta\left|\frac{z(D_\lambda^n f(z))'}{(1-\gamma)D_\lambda^n f(z)+\gamma z(D_\lambda^n(f(z))'}-1\right| \\ &\quad -Re\left\{\frac{z(D_\lambda^n f(z))'}{(1-\gamma)D_\lambda^n f(z)+\gamma z(D_\lambda^n(f(z))'}-\alpha\right\} \\ &\leq (1+\beta)\left|\frac{z(D_\lambda^n f(z))'}{(1-\gamma)D_\lambda^n f(z)+\gamma z(D_\lambda^n(f(z))'}-1\right| \\ &\leq \frac{(1+\beta)\sum_{j=2}^{\infty}[1+(j-1)\lambda]^n(\gamma-\gamma j+j-1)|a_j||z|^{j-1}}{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^n(\gamma j-\gamma+1)|a_j||z|^{j-1}},\end{aligned}$$

which is bounded above by $(1-\alpha)$ if

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^n[j(1+\beta)-(\alpha+\beta)(\gamma j-\gamma+1)]a_j<1-\alpha.$$

Hence the proof is completed.

Theorem 2.2. *A function $f(z)$ of the form (1.2) is in $TSV_\lambda^n(\alpha,\beta,\gamma)$ if and only if*

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^n[j(1+\beta)-(\alpha+\beta)(\gamma j-\gamma+1)]a_j<1-\alpha.$$

Proof. We only need to prove the necessary condition. If $f(z) \in TSV_\lambda^n(\alpha,\beta,\gamma)$ and z is real, then

$$\frac{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^n j a_j z^{j-1}}{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^n(\gamma j-\gamma+1)]a_j z^{j-1}}-\alpha$$
$$>\beta\left|\frac{\sum_{j=2}^{\infty}[1+(j-1)\lambda]^n(\gamma j-\gamma-j+1)]a_j z^{j-1}}{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^n(\gamma j-\gamma+1)]a_j z^{j-1}}\right|.$$

Letting $z\to 1$ along real axis, we obtain

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^n[j(1+\beta)-(\alpha+\beta)(\gamma j-\gamma+1)]a_j<1-\alpha$$

and the proof is completed.

Remark 2.3. *If $f(z)\in TSV_\lambda^n(\alpha,\beta,\gamma)$, then*

$$a_j \le \frac{1-\alpha}{[1+(j-1)\lambda]^n[j(1+\beta)-(\alpha+\beta)(\gamma j-\gamma+1)]} \text{ for } j=2,3,\dots \tag{2.2}$$

and equality holds for

$$f(z)=z-\frac{1-\alpha}{[1+(j-1)\lambda]^n[j(1+\beta)-(\alpha+\beta)(\gamma j-\gamma+1)]}z^j \text{ for } j=2,3,\dots \tag{2.3}$$

Theorem 2.4. *Let the function $f(z)$ defined by (1.2) be in the class $TSV_\lambda^n(\alpha,\beta,\gamma)$. Then for $|z|<r=1$,*

$$r-\frac{1-\alpha}{2(1+\beta)-(\alpha+\beta)(\gamma+1)}r^2\le|D_\lambda^n f(z)|\le r+\frac{1-\alpha}{2(1+\beta)-(\alpha+\beta)(\gamma+1)}r^2. \tag{2.4}$$

The result is attained for the function $f(z)$ given by (2.3) for $z=\pm r$.

Proof. By Theorem 2.1, we have

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^n[j(1+\beta)-(\alpha+\beta)(\gamma j-\gamma+1)]a_j<1-\alpha.$$

Then we get

$$\begin{aligned}
&(1+\lambda)^n[2(1+\beta)-(\alpha+\beta)(\gamma+1)]\sum_{j=2}^{\infty}a_j\\
&\leq \sum_{j=2}^{\infty}[1+(j-1)\lambda]^n[j(1+\beta)-(\alpha+\beta)(\gamma j-\gamma+1)]a_j\\
&< 1-\alpha.
\end{aligned}$$

Therefore

$$\sum_{j=2}^{\infty}a_j \leq \frac{1-\alpha}{(1+\lambda)^n[2(1+\beta)-(\alpha+\beta)(\gamma+1)]}.$$

Hence

$$\begin{aligned}
|D_\lambda^n f(z)| &\leq |z|+|z|^2(1+\lambda)^n\sum_{j=2}^{\infty}a_j\\
&\leq r+r^2(1+\lambda)^n\sum_{j=2}^{\infty}a_j\\
&\leq r+r^2\frac{1-\alpha}{[2(1+\beta)-(\alpha+\beta)(\gamma+1)]}
\end{aligned}$$

and

$$|D_\lambda^n f(z)| \geq r-r^2\frac{1-\alpha}{[2(1+\beta)-(\alpha+\beta)(\gamma+1)]}.$$

Thus

$$\begin{aligned}
r-\frac{1-\alpha}{2(1+\beta)-(\alpha+\beta)(\gamma+1)}r^2 &\leq |D_\lambda^n f(z)|\\
&\leq r+\frac{1-\alpha}{2(1+\beta)-(\alpha+\beta)(\gamma+1)}r^2.
\end{aligned}$$

Further, we have

$$\begin{aligned}
|(D_\lambda^n f(z))'| &\leq 1+2r(1+\lambda)^n\sum_{j=2}^{\infty}a_j\\
&\leq 1+\frac{2(1-\alpha)}{[2(1+\beta)-(\alpha+\beta)(\gamma+1)]}r
\end{aligned}$$

and also

$$|(D_\lambda^n f(z))'| \geq 1-\frac{2(1-\alpha)}{[2(1+\beta)-(\alpha+\beta)(\gamma+1)]}r.$$

The result is sharp for the function $f(z)$, defined by

$$f(z)=z-\frac{1-\alpha}{[2(1+\beta)-(\alpha+\beta)(\gamma+1)]}r^2,\ \ z=\pm r.$$

This completes the proof.

The determination of the extreme points of a family of univalent functions enables us to solve many extremal problems.

Theorem 2.5. *Let* $f_1(z)=z$ *and*

$$f_j(z)=\frac{1-\alpha}{[1+(j-1)\lambda]^n[j(1+\beta)-(\alpha+\beta)(\gamma j-\gamma+1)]}z^j$$

for all $j=2,3,4,\ldots$. *Then* $f(z)\in TSV_\lambda^n(\alpha,\beta,\gamma)$ *if and only if* $f(z)$ *can be expressed in the form* $f(z)=\sum\limits_{j=1}^{\infty}\mu_j f_j(z)$ *where* $\mu_j\geq 0$ *and* $\sum\limits_{j=1}^{\infty}\mu_j=1$.

Proof. Suppose $f(z)$ can be written in the form

$$\begin{aligned} f(z) &= \sum_{j=1}^{\infty}\mu_j f_j(z) \\ &= \sum_{j=1}^{\infty}\mu_j z-\sum_{j=2}^{\infty}\mu_j\left[\frac{1-\alpha}{[j(1+\beta)-(\alpha+\beta)(\gamma j-\gamma+1)]}z^j\right]. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{j=2}^{\infty}\frac{\mu_j(1-\alpha)}{[j(1+\beta)-(\alpha+\beta)(\gamma j-\gamma+1)]}\times\frac{[j(1+\beta)-(\alpha+\beta)(\gamma j-\gamma+1)]}{(1-\alpha)} \\ &= \sum_{j=2}^{\infty}\mu_j \\ &= 1-\mu_1 \\ &\leq 1. \end{aligned}$$

In view of Theorem 2.1, this shows that $f(z)\in TSV_\lambda^n(\alpha,\beta,\gamma)$. Conversely, suppose that $f(z)\in TSV_\lambda^n(\alpha,\beta,\gamma)$. Then

$$a_j\leq\frac{1-\alpha}{[1+(j-1)\lambda]^n[j(1+\beta)-(\alpha+\beta)(\gamma j-\gamma+1)]},\ \ j\geq 2.$$

Putting

$$\mu_j=\frac{[1+(j-1)\lambda]^n[j(1+\beta)-(\alpha+\beta)(\gamma j-\gamma+1)]}{1-\alpha}a_j,\ \ j=2,3,4,\ldots$$

and $\mu_1 = 1 - \sum_{j=2}^{\infty} \mu_j$, we see that $f(z) = \sum_{j=1}^{n} \mu_j f_j(z)$.

Definition 2.6. For two functions $f(z), g(z) \in T$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$ $(a_j \geq 0)$ and $g(z) = z - \sum_{j=2}^{\infty} b_j z^j$ $(b_j \geq 0)$ the modified Hadamard product $f * g$ is defined by

$$(f * g)(z) = z - \sum_{j=2}^{\infty} a_j b_j z^j.$$

Theorem 2.7. *Let* $f(z) = z - \sum_{j=2}^{\infty} a_j z^j \in TSV_\lambda^n(\alpha, \beta, \gamma)$, $(a_j \geq 0)$ *and* $g(z) \in T$ *with* $g(z) = z - \sum_{j=2}^{\infty} b_j z^j \in TSV_\lambda^n(\alpha, \beta, \gamma)$, $(b_j \geq 0)$, $\alpha \in [-1, 1)$, $\lambda \geq 0$, $\beta \geq 0$, $\gamma \geq 0$. *Then* $f(z) * g(z) \in TSV_\lambda^n(\alpha, \beta, \gamma)$.

Proof. Since $f(z), g(z) \in TSV_\lambda^n(\alpha, \beta, \gamma)$, we have

$$\sum_{j=2}^{\infty} [1 + (j-1)\lambda]^n [j(1+\beta) - (\alpha+\beta)(\gamma j - \gamma + 1)] a_j < 1 - \alpha$$

and

$$\sum_{j=2}^{\infty} [1 + (j-1)\lambda]^n [j(1+\beta) - (\alpha+\beta)(\gamma j - \gamma + 1)] b_j < 1 - \alpha.$$

We know that $f(z) * g(z) = z - \sum_{j=2}^{\infty} a_j b_j z^j$. From $g(z) \in T$, we have $\sum_{j=2}^{\infty} j b_j \leq 1$ and we notice that $b_j < 1$ for $j = 2, 3, \ldots$. Thus

$$\begin{aligned} &\sum_{j=2}^{\infty} [1 + (j-1)\lambda]^n [j(1+\beta) - (\alpha+\beta)(\gamma j - \gamma + 1)] a_j b_j \\ &< \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^n [j(1+\beta) - (\alpha+\beta)(\gamma j - \gamma + 1)] a_j \\ &< 1 - \alpha. \end{aligned}$$

This implies that $f(z) * g(z) \in TSV_\lambda^n(\alpha, \beta, \gamma)$.

3. Application of the Fractional Calculus

We begin with the statements of the following definitions of fractional calculus (that is, fractional derivatives and fractional integrals) which were defined by Owa ([7, 8]) and were used by many researchers ([13, 14, 15]).

Definition 3.1. The fractional integral of order δ is defined for a function $f(z)$, by

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(t)}{(z-t)^{1-\delta}} dt \quad (\delta > 0), \tag{3.1}$$

where $f(z)$ is analytic in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\delta-1}$ is removed by requiring $log(z-t)$ to be real when $(z-t) > 0$.

Definition 3.2. The fractional derivative of order δ is defined for a function $f(z)$, by

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\delta}} dt \quad (0 \leq \delta < 1), \tag{3.2}$$

where $f(z)$ is as in Definition 3.1.

Definition 3.3. (Under the condition of Definition 3.2) The fractional derivative of order $k+\delta$ $(k = 0, 1, 2, \ldots)$ is defined by

$$D_z^{k+\delta} f(z) = \frac{d^k}{dz^k} D_z^{\delta} f(z), \quad (0 \leq \delta < 1) \tag{3.3}$$

From Definitions 3.1 and 3.2 by applying simple calculation, we get

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(2+\delta)} z^{\delta+1} - \sum_{j=2}^{\infty} \frac{\Gamma(j+1)}{\Gamma(j+1+\delta)} a_j z^{j+\delta}. \tag{3.4}$$

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(2-\delta)} z^{1-\delta} - \sum_{j=2}^{\infty} \frac{\Gamma(j+1)}{\Gamma(j+1-\delta)} a_j z^{j+\delta}. \tag{3.5}$$

Now making use of (3.4) and (3.5), we state the following theorems.

Theorem 3.4. *Let $f(z) \in TSV_\lambda^n(\alpha, \beta, \gamma)$. Then*

$$|D_z^{-\delta} f(z)| \leq \frac{|z|^{\delta+1}}{\Gamma(2+\delta)} \left[1 + \frac{2(1-\alpha)}{(2+\delta)(1+\lambda)^n[2(1+\beta)-(\alpha+\beta)(\gamma+1)]} |z|\right] \tag{3.6}$$

and

$$|D_z^{-\delta} f(z)| \geq \frac{|z|^{\delta+1}}{\Gamma(2+\delta)} \left[1 - \frac{2(1-\alpha)}{(2+\delta)(1+\lambda)^n[2(1+\beta)-(\alpha+\beta)(\gamma+1)]} |z|\right], \tag{3.7}$$

for $\delta > 0$ and $z \in \Delta$. The result is sharp.

Proof. Using (3.4), we have

$$\begin{aligned} D_z^{-\delta} f(z) &= \frac{1}{\Gamma(2+\delta)} z^{\delta+1} - \sum_{j=2}^{\infty} \frac{\Gamma(j+1)}{\Gamma j+1+\delta} a_j z^{j+\delta} \\ &= \frac{1}{\Gamma(2+\delta)} z^{\delta} \left[z - \sum_{j=2}^{\infty} \ell(j,\delta) a_j z^j \right], \end{aligned} \tag{3.8}$$

where $\ell(j,\delta) = \dfrac{\Gamma(j+1)\Gamma(2+\delta)}{\Gamma(j+1+\delta)}$. We have

$$0 < \ell(j,\delta) \leq \ell(2,\delta) = \frac{2}{2+\delta}. \tag{3.9}$$

In view of Theorem 2.1, we have

$$\begin{aligned} &(1+\lambda)^n [2(1+\beta) - (\alpha+\beta)(\gamma+1)] \sum_{j=2}^{\infty} a_j \\ &\leq \sum_{j=2}^{\infty} [1+(j-1)\lambda]^n [j(1+\beta) - (\alpha+\beta)(\gamma j - \gamma + 1)] a_j \\ &< 1-\alpha, \end{aligned}$$

which evidently yields

$$\sum_{j=2}^{\infty} a_j \leq \frac{1-\alpha}{(1+\lambda)^n [2(1+\beta) - (\alpha+\beta)(\gamma+1)]}. \tag{3.10}$$

Using (3.8) and (3.10), we have

$$\begin{aligned} |D_z^{-\delta} f(z)| &\leq \frac{|z|^{\delta}}{\Gamma(2+\delta)} \left[|z| + \ell(2,\delta)|z|^2 \sum_{j=2}^{\infty} a_j \right] \\ &\leq \frac{|z|^{\delta}}{\Gamma(2+\delta)} \left[|z| + |z|^2 \frac{2(1-\alpha)}{(2+\delta)(1+\lambda)^n [2(1+\beta) - (\alpha+\beta)(\gamma+1)]} \right] \end{aligned}$$

or

$$|D_z^{-\delta} f(z)| \leq \frac{|z|^{\delta+1}}{\Gamma(2+\delta)} \left[1 + \frac{2(1-\alpha)}{(2+\delta)(1+\lambda)^n [2(1+\beta) - (\alpha+\beta)(\gamma+1)]} |z| \right].$$

We also have

$$|D_z^{-\delta} f(z)| \geq \frac{|z|^{\delta+1}}{\Gamma(2+\delta)} \left[1 - \frac{2(1-\alpha)}{(2+\delta)(1+\lambda)^n [2(1+\beta) - (\alpha+\beta)(\gamma+1)]} |z| \right].$$

This completes the proof.

Theorem 3.5. *Let* $f(z) \in TSV_{\lambda}^{n}(\alpha, \beta, \gamma)$. *Then*

$$|D_z^{\delta} f(z)| \leq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)} \left[1 + \frac{2(1-\alpha)}{(2-\delta)(1+\lambda)^n [2(1+\beta) - (\alpha+\beta)(\gamma+1)]} |z| \right] \tag{3.11}$$

and

$$|D_z^{\delta} f(z)| \geq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)} \left[1 - \frac{2(1-\alpha)}{(2-\delta)(1+\lambda)^n [2(1+\beta) - (\alpha+\beta)(\gamma+1)]} |z| \right], \tag{3.12}$$

for $0 \leq \delta < 1$ *and* $z \in \Delta$. *The result is sharp.*

Proof. Using (3.5), we have

$$\begin{aligned} D_z^{\delta} f(z) &= \frac{z^{1-\delta}}{\Gamma(2-\delta)} - \sum_{j=2}^{\infty} \frac{\Gamma j+1}{\Gamma j+1-\delta} a_j z^{j+\delta} \\ &= \frac{z^{-\delta}}{\Gamma(2-\delta)} \left[z - \sum_{j=2}^{\infty} m(j,\delta) a_j z^j \right], \end{aligned} \tag{3.13}$$

where $m(j,\delta) = \dfrac{\Gamma(j+1)\Gamma(2-\delta)}{\Gamma(j+1-\delta)}$. We have

$$0 < m(j,\delta) \leq m(2,\delta) = \frac{2}{2-\delta}. \tag{3.14}$$

In view of Theorem 2.1, we have

$$\begin{aligned} &(1+\lambda)^n [2(1+\beta) - (\alpha+\beta)(\gamma+1)] \sum_{j=2}^{\infty} a_j \\ &\leq \sum_{j=2}^{\infty} [1+(j-1)\lambda]^n [j(1+\beta) - (\alpha+\beta)(\gamma j - \gamma + 1)] a_j \\ &< 1-\alpha, \end{aligned}$$

which gives

$$\sum_{j=2}^{\infty} a_j \leq \frac{1-\alpha}{(1+\lambda)^n [2(1+\beta) - (\alpha+\beta)(\gamma+1)]}. \tag{3.15}$$

Using (3.13) and (3.15), we have

$$\begin{aligned} |D_z^{\delta} f(z)| &\leq \frac{|z|^{-\delta}}{\Gamma(2-\delta)} \left[|z| + m(2,\delta)|z|^2 \sum_{j=2}^{\infty} a_j \right] \\ &\leq \frac{|z|^{-\delta}}{\Gamma(2-\delta)} \left[|z| + \frac{2(1-\alpha)}{(1+\lambda)^n (2-\delta)[2(1+\beta) - (\alpha+\beta)(\gamma+1)]} |z|^2 \right] \end{aligned}$$

or

$$|D_z^\delta f(z)| \leq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)}\left[1+\frac{2(1-\alpha)}{(2-\delta)(1+\lambda)^n[2(1+\beta)-(\alpha+\beta)(\gamma+1)]}|z|\right].$$

Also, we get

$$|D_z^\delta f(z)| \geq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)}\left[1-\frac{2(1-\alpha)}{(2-\delta)(1+\lambda)^n[2(1+\beta)-(\alpha+\beta)(\gamma+1)]}|z|\right].$$

Hence the proof is completed.

Corollary 3.6. *For every $f(z) \in TSV_\lambda^n(\alpha,\beta,\gamma)$,*

(1) When $\delta = 1$, from Definition 3.1 and Theorem 3.4, we obtain

$$\frac{|z|^2}{2}\left[1-\frac{2(1-\alpha)}{3(1+\lambda)^n[2(1+\beta)-(\alpha+\beta)(\gamma+1)]}|z|\right] \leq \left|\int_0^z f(t)dt\right|$$
$$\leq \frac{|z|^2}{2}\left[1+\frac{2(1-\alpha)}{3(1+\lambda)^n[2(1+\beta)-(\alpha+\beta)(\gamma+1)]}|z|\right]$$

(2) When $\delta = 0$, from Definition 3.2 and Theorem 3.5, we obtain

$$|z|\left[1-\frac{1-\alpha}{(1+\lambda)^n[2(1+\beta)-(\alpha+\beta)(\gamma+1)]}|z|\right] \leq |f(z)|$$
$$\leq |z|\left[1+\frac{1-\alpha}{(1+\lambda)^n[2(1+\beta)-(\alpha+\beta)(\gamma+1)]}|z|\right]$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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