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A QUARTER-STEP COMPUTATIONAL HYBRID BLOCK METHOD FOR FIRST-ORDER MODELED DIFFERENTIAL EQUATIONS USING LAGUERRE POLYNOMIAL

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Abstract. In this paper, we present the derivation and implementation of a new quarter-step computational hybrid block method for first-order modeled differential equations. The block method was developed using Laguerre polynomial of degree five as our basis function via interpolation and collocation techniques. We went further to apply the quater-step method developed on some modeled first order differential equations. The paper also analysed the basic properties of the method derived. From the results obtained, it is obvious that the method is computationally reliable.

Keywords: Computational Method, Hybrid, Laguerre Polynomial, Quarter-step, Model.

2010 AMS Subject Classification: 65L05, 65L06, 65D30.

1. Introduction

In recent times, classic application of differential equations is found in many areas of science and technology. They can be used for modeling of physical, technical or biological processes such as in the study of growth, decay, epidemic, electricity, among others. The main questions

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of modern technology are how to increase the accuracy of calculations considering short computational time and how to decrease necessary mathematical operations.This paper presents a quarter-step computational hybrid block method for the integration of modeled first order problems of the form,

$$y' = f(x,y), \;\; y(a) = \eta, \;\; f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \tag{1}$$

The following standard theorem lays down sufficient conditions for a unique solution of (1) to exist; we shall always assume that the hypotheses of this theorem are satisfied.

Theorem 1.1 (Lambert [1]): *Let $f(x,y)$, where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, be defined and continuous for all (x,y) in the region Ddefined by $a \leq x \leq b, \; -\infty < y < \infty$, where a and b are finite and let there exist a constant L such that,*

$$\|f(x,y) - f(x,y^*)\| \leq L \|y - y^*\| \tag{2}$$

holds for every $(x,y),(x,y^*) \in D$. Then for $\eta \in \mathbb{R}$,there exists a unique solution $y(x)$ of the problem (1), where $y(x)$ is continuous and differentiable for all $(x,y^*) \in D$. The requirement (2) is known as Lipchitz condition and the constant L as a Lipchitz constant.

It is important to note that, researchers have proposed different computational methods for the solution of problems of the form (1) ranging from predictor-corretor methods to hybrid methods. Despite the success recorded by the predictor-corrector methods, its major setbacks are that the predictors are in reducing order of accuracy, high cost of developing separate predictor for the corrector, high cost of human and computer time involved in the execution, Sunday *et al.* [2].Block methods were later proposed to carter for some of the setbacks of the predictor-corrector methods. It is important to state that Milne in 1953 first developed block method to serve as a predictor to a predictor-corrector algorithm before it was later adopted as a full method. Block method has the advantage of generating simultaneous numerical approximations at different grid points within the interval of integration, Sunday [3]. Another advantage of the block method is the fact that it is less expensive in terms of the number of function evaluations compared to the linear multistep and the Runge-Kutta methods. Its major setback however is that the order of interpolation points must not exceed the order of the differential equations, thus when equations of lower order are developed, the accuracy of the developed method is reduced. This led to the development of hybrid methods which permit the incorporation of function evaluation at off-step points which affords the opportunity of circumventing the "Dahlquist Zero-Stabilty Barrier" and it is actually possible to obtain convergent $k-$step methods with order $2k+1$ up to $k=7$.The method is also useful in reducing the step number of a method and still remain zero-stable, see Sunday *et al.* [4], Adesanya *et al.* [5], Sunday *et al.* [6], Sunday *et al.* [7] and Sunday *et al.* [8].

Definition 1.1 Jain *et al.* [9] : Laguerre polynomial $y_n(x)$ is defined as,

$$\sum_{n=0}^{5} y_n(x) = \sum_{n=0}^{5} \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}) \tag{3}$$

In particular, $y_0(x)=1, y_1(x)=x-1,\ y_2(x)=x^2-4x+2,...$The Laguerre polynomial $y_n(x)$ are orthogonal with respect to the weight function $w(x)=e^{-x}$ on $[0,\infty)$.

Many scholars used different basis functions for the solution of problems of the form (1). For instance, Sunday *et al.* [10], Sunday *et al.* [11] and Sunday *et al.* [12] used basis functions which are the combination of power series and exponential functions to develop block integrators for the solution of (1). Sunday *et al.* [7] and Sunday *et al.* [6] also used Chebyshev and Legendre polynomials as basis functions respectively to develop hybrid methods for the solution of (1).

In this paper, we shall employ Laguerre polynomial as a basis function in developing the new quater-step computational hybrid block method for the solution of (1).

2. Preliminaries: Derivation of the Quarter-step Computational Hybrid Method

We shall derive a new quarter-step hybrid method of the form,

$$A^{(0)}Y_m = Ey_n + hdf(y_n) + hbF(Y_m) \tag{4}$$

using a Laguerre polynomial of degree 5 as our basis function. This is given by,

$$y_5(x) = 720 - 1800x + 1200x^2 - 300x^3 + 30x^4 - x^5 \tag{5}$$

We interpolate (5) at point $x_{n+s},\ s = 0$ and collocate its first derivative at points $x_{n+r},\ r = 0\left(\frac{1}{16}\right)\frac{1}{4}$, where s and r are the numbers of interpolation and collocation points respectively. This leads to the system of equatons,

$$XA = U \tag{6}$$

where

$$A = [a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5]^T$$

$$U = \left[y_n \quad f_n \quad f_{n+\frac{1}{16}} \quad f_{n+\frac{1}{8}} \quad f_{n+\frac{3}{16}} \quad f_{n+\frac{1}{4}}\right]^T$$

and

$$X = \begin{bmatrix} 720 & -1800x_n & 1200x_n^2 & -300x_n^3 & 30x_n^4 & -x_n^5 \\ 0 & -1800 & 2400x_n & -900x_n^2 & 120x_n^3 & -5x_n^4 \\ 0 & -1800 & 2400x_{n+\frac{1}{16}} & -900x^2_{n+\frac{1}{16}} & 120x^3_{n+\frac{1}{16}} & -5x^4_{n+\frac{1}{16}} \\ 0 & -1800 & 2400x_{n+\frac{1}{8}} & -900x^2_{n+\frac{1}{8}} & 120x^3_{n+\frac{1}{8}} & -5x^4_{n+\frac{1}{8}} \\ 0 & -1800 & 2400x_{n+\frac{3}{16}} & -900x^2_{n+\frac{3}{16}} & 120x^3_{n+\frac{3}{16}} & -5x^4_{n+\frac{3}{16}} \\ 0 & -1800 & 2400x_{n+\frac{1}{4}} & -900x^2_{n+\frac{1}{4}} & 120x^3_{n+\frac{1}{4}} & -5x^4_{n+\frac{1}{4}} \end{bmatrix}$$

Solving (6), for $a_j's$, $j = 0(1)5$ and substituting back into the basis function gives a continuous linear multistep method of the form,

$$y(x) = \alpha_0(x)y_n + h\sum_{j=0}^{\frac{1}{4}} \beta_j(x) f_{n+j},\ j = 0\left(\frac{1}{16}\right)\frac{1}{4} \tag{7}$$

where

(8)
$$\begin{bmatrix} \alpha_0(t)=1 \\ \beta_0(t)=\frac{1}{45}(24576t^5-19200t^4+5600t^3-750t^2+45t) \\ \beta_{\frac{1}{16}}(t)=\frac{32}{45}(-3072t^5+2160t^4-520t^3+45t^2) \\ \beta_{\frac{1}{8}}(t)=-\frac{8}{15}(-6144t^5+3840t^4-760t^3+45t^2) \\ \beta_{\frac{3}{16}}(t)=\frac{32}{45}(-3072t^5+1680t^4-280t^3+15t^2) \\ \beta_{\frac{1}{4}}(t)=-\frac{2}{45}(-12288t^5+5760t^4-880t^3+45t^2) \end{bmatrix}$$

$t=\frac{x-x_n}{h}$, $\alpha(t)$ and $\beta(t)$ are continuous functions. Evaluating (7) at $t=\frac{1}{16}\left(\frac{1}{16}\right)\frac{1}{4}$ gives a discrete computational method of the form (4), where

$$Y_m=\left[y_{n+\frac{1}{16}} \quad y_{n+\frac{1}{8}} \quad y_{n+\frac{3}{16}} \quad y_{n+\frac{1}{4}}\right]^T, \quad y_n=\left[y_{n-\frac{3}{16}} \quad y_{n-\frac{1}{8}} \quad y_{n-\frac{1}{16}} \quad y_n\right]^T$$

$$F(Y_m)=\left[f_{n+\frac{1}{16}} \quad f_{n+\frac{1}{8}} \quad f_{n+\frac{3}{16}} \quad f_{n+\frac{1}{4}}\right]^T, \quad f(y_n)=\left[f_{n-\frac{3}{16}} \quad f_{n-\frac{1}{8}} \quad f_{n-\frac{1}{16}} \quad f_n\right]^T$$

$$A^{(0)}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$d=\begin{bmatrix} 0 & 0 & 0 & \frac{251}{11520} \\ 0 & 0 & 0 & \frac{29}{1440} \\ 0 & 0 & 0 & \frac{27}{1280} \\ 0 & 0 & 0 & \frac{7}{360} \end{bmatrix}, \; b=\begin{bmatrix} \frac{323}{5760} & \frac{-11}{480} & \frac{53}{5760} & \frac{-19}{11520} \\ \frac{31}{360} & \frac{1}{60} & \frac{1}{360} & \frac{-1}{1440} \\ \frac{51}{640} & \frac{9}{160} & \frac{21}{640} & \frac{-3}{1280} \\ \frac{4}{45} & \frac{1}{30} & \frac{4}{45} & \frac{7}{360} \end{bmatrix}$$

It is important to note here that the computational method developed above is implicit in nature, meaning that it requires some starting values before it can be implemented. Starting values for $y_{n+j},\; j=\frac{1}{16}\left(\frac{1}{16}\right)\frac{1}{4}$ are predicted using the Taylor series up to the order of each individual scheme.

3. Aalysis of Basic Properties of the Quarter-Step Computational Method

To justify the applicability and accuracy of the proposed computational method, we need to examine its basic properties which include order of accuracy, consistency, root condition, convergence, symmetry and region of absolute stability.

3.1. Order of Accuracy and Error Constant. The block method (4) is said to be of uniform accurate order p, if p is the larget positive integer for which $\overline{c}_0 = \overline{c}_1 = \overline{c}_2 = \ldots = \overline{c}_p = 0$ but $\overline{c}_{p+1} \neq 0$, Lambert [1]. Thus, $\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \bar{c}_3 = \bar{c}_4 = \bar{c}_5 = 0$, $\bar{c}_6 = [-1.1176 \times 10^{-9}\ \ 6.6227 \times 10^{-10}\ \ 1.1176 \times 1$

.

Therefore, the quarter-step computational method is of accurate fifth order.

3.2. Root Condition and Zero Stability. Definition 3.1 (Lambert [1]): The block method (4) is said to satisfy root condition, if the roots $z_s, s = 1, 2, \ldots, k$ of the first characteristic polynomial $\rho(z)$ defined by $\rho(z) = \det(zA^{(0)} - E)$ satisfies $|z_s| \leq 1$ and every root satisfying $|z_s| = 1$ have multiplicity not exceeding the order of the differential equation. The method (4) is said to be zero-stable if it satisfies the root condition. Moreover, as $h \to 0, \rho(z) = z^{r-\mu}(z-1)^{\mu}$, where μ is the order of the differential equation, r is the order of the matrices $A^{(0)}$ and E (see Awoyemi *et al.* [13] for details). We shall now verify whether or not our quarter-step computational method satisfies root condition.

$$\rho(z) = \left| z \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0 \tag{9}$$

$\rho(z) = z^3(z-1) = 0 \Longrightarrow z_1 = z_2 = z_3 = 0,\ z_4 = 1$. Hence, the quarter-step computational method (4) is said to satisfy root condition.

Theorem 3.1 (Lambert [1]) : *The necessary and sufficient condition for the method given by* (4) *to be zero-stable is that it satisfies the root condition.*

3.3. Consistency. According to Fatunla [14], consistency controls the magnitude of the local truncation error committed at each stage of the computation.The computational quarter-step method (4) is consistent since it has order $p = 5 \geq 1$.

3.4. Convergence. The quarter-step computational method (4) is convergent by consequence of Dahlquist theorem below.

Theorem 3.2: *(Dahlquist $[15]$): The necessary and sufficient conditions that a continuous LMM be convergent are that it be consistent and zero-stable.*

3.5. Region of Absolute Stability. **Definition 3.3** (Lambert $[1]$): The linear multistep method (7) is said to have region of absolute stability R_A, where R_A is a region of the complex $\bar{h}-$plane, if it is absolutely stable for all $\bar{h} \in R_A$. The intersection of R_A with the real axis is called the interval of absolute stability.

In ploting the stability region, we shall adopt the boundary locus method. The stability polynomial of the newly derived quarter-step computational method is given by,

$$\bar{h}(w) = -h^4\left(\frac{1}{327680}w^3 - \frac{1}{327680}w^4\right) - h^3\left(\frac{5}{24576}w^4 + \frac{5}{24576}w^3\right) \tag{10}$$

$$-h^2\left(\frac{7}{1024}w^3 - \frac{7}{1024}w^4\right) - h\left(\frac{1}{8}w^4 + \frac{1}{8}w^3\right) + w^4 - w^3 \tag{11}$$

The stability region is shown in Figure 1.

Lambert $[1]$ showed that the stability region for L-stable schemes must encroach into the positive half of the complex plane. Thus, the stability region in the Figure 1 is L-stable.

4. Main results: Numerical Experiments

We shall consider the following real-life problem by modelling them into equations of the form (1). We shall use the following notation in the the tables below.

ERR=|Exact Solution - Computed Solution|

Problem 4.1 (Mixture Model):

In an oil refinery, a storage tank contains 2000 gal of gasoline that initially has 100 *lb* of an additive dissolved in it. In the preparation for winter, gasoline containing 2 *lb* of additive per gallon is pumped into the tank at a rate of 40 gal/min. The well-mixed solution is pumped out

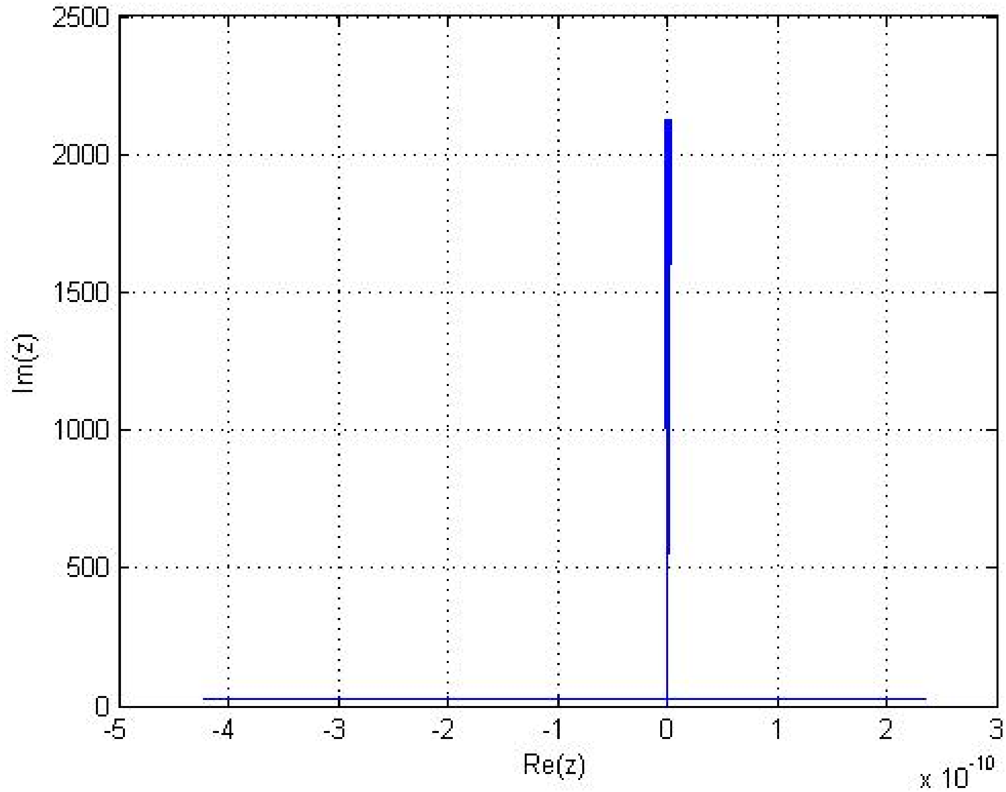

FIGURE 1. Stability region of the quarter-step method

at a rate of 45 gal/min. Using a numerical method, how much of the additive is in the tank 0.1 min, 0.5 min and 1 min after the pumping process begins?

Let y be the amount (in pounds) of additive in the tank at time t. We know that $y = 100$ when $t = 0$. Thus, the initial value problem modeling the mixture process is,

$$\frac{dy}{dt} = 80 - \frac{45}{(2000-5t)}y, \quad y(0) = 100 \tag{12}$$

with the theoretical solution,

$$y(t) = 2(2000-5t) - \frac{3900}{(2000)^9}(2000-5t)^9 \tag{13}$$

Source: [10]

The numerical and graphical results for problem 4.1 is presented in Table 4.1 and Figure 2 respectively.

Problem 4.2 (Decay Model):

A certain radioactive material is known to decay at a rate proportional to the amount present. If initially there is 50 milligrams of the material present and after two hours it is observed that the material has lost 10 percent of its original mass. Calculate the mass of the material remaining at time t: $0 \leq t \leq 1$.

Let N denote the amount of the material present at time t. The initial value problem modeling the problem above is,

$$\frac{dN}{dt} = -0.053N, \quad N(0) = 50 \tag{14}$$

with the exact solution,

$$N(t) = 50e^{-0.053t} \tag{15}$$

Source: [16]

The numerical and graphical results for problem 4.2 is presented in Table 4.2 and Figure 3 respectively.

Problem 4.3 (Growth Model):

A bacteria culture is known to grow at a rate proportional to the amount present. After one hour, 1000 strands of the bacteria are observed in the culture; and after four hours, 3000 strands. Find the number of strands of the bacteria present in the culture at time $t: 0 \leq t \leq 1$.

Let $N(t)$ denote the number of bacteria strands in the culture at time t, the initial value problem modeling this problem is given by,

$$\frac{dN}{dt} = 0.366N, \quad N(0) = 694 \tag{16}$$

The exact solution is given by

$$N(t) = 694e^{0.366t} \tag{17}$$

Source: [16]

The numerical and graphical results for problem 4.3 is presented in Table 4.3 and Figure 4 respectively.

Problem 4.4 (SIR Model):

The SIR model is an epidemiological model that computes the theoretical number of people infected with a contagious illness in a closed population over time t. The name of this class of models derives from the fact that they involve coupled equations relating the number of susceptible people $S(t)$, number of people infected $I(t)$ and the number of people who have recovered $R(t)$. This is a good and simple model for many infectious diseases including measles, mumps and rubella [17]. It is given by the following three coupled equations,

$$\left\{ \begin{array}{c} \frac{dS}{dt} = \mu(1-S) - \beta IS \\ \frac{dI}{dt} = -\mu I - \gamma I + \beta IS \\ \frac{dR}{dt} = -\mu R + \gamma I \end{array} \right\} \tag{18}$$

where μ, γ and β are positive parameters. Define y to be,

$$y = S + I + R \tag{19}$$

and adding the equations in (17), we obtain the following evolution equation for y,

$$y' = \mu(1-y) \tag{20}$$

Taking $\mu = 0.5$ and attaching an initial condition $y(0) = 0.5$ (for a particular closed population), we obtain,

$$\frac{dy}{dt} = 0.5(1-y), \;\; y(0) = 0.5 \tag{21}$$

whose exact solution is,

$$y(t) = 1 - 0.5e^{-0.5t} \tag{22}$$

Source: [10]

The numerical and graphical results for problem 4.4 is presented in Table 4.4 and Figure 5 respectively.

Table 4.1 : Showing the result for problem 4.1

t	Exact Solution	Computed Solution	ERR	t/sec
0.1000	107.7662301168311400	107.7662301168309500	$1.847411e-013$	0.0069
0.2000	115.5149409193027200	115.5149409193028600	$1.421085e-013$	0.0087
0.3000	123.2461630508842100	123.2461630508845500	$3.410605e-013$	0.0105
0.4000	130.9599271090915000	130.9599271090911000	$3.979039e-013$	0.0125
0.5000	138.6562636455414600	138.6562636455413700	$8.526513e-014$	0.0144
0.6000	146.3352031660151600	146.3352031660153600	$1.989520e-013$	0.0162
0.7000	153.9967761305115300	153.9967761305114800	$5.684342e-014$	0.0180
0.8000	161.6410129533037400	161.6410129533038600	$1.136868e-013$	0.0198
0.9000	169.2679440029992300	169.2679440029996000	$3.694822e-013$	0.0216
1.0000	176.8775996025960900	176.8775996025958600	$2.273737e-013$	0.0233

Table 4.2 : Showing the result for problem 4.2

t	Exact Solution	Computed Solution	ERR	t/sec
0.1000	49.7357010110004440	49.7357010110004370	$7.105427e-015$	0.0238
0.2000	49.4727991011126060	49.4727991011125990	$7.105427e-015$	0.0255
0.3000	49.2112868854045620	49.2112868854045540	$7.105427e-015$	0.0271
0.4000	48.9511570179809750	48.9511570179809610	$1.421085e-014$	0.0288
0.5000	48.6924021917767500	48.6924021917767430	$7.105427e-015$	0.0304
0.6000	48.4350151383518220	48.4350151383518150	$7.105427e-015$	0.0320
0.7000	48.1789886276869340	48.1789886276869200	$1.421085e-014$	0.0338
0.8000	47.9243154679805560	47.9243154679805340	$2.131628e-014$	0.0355
0.9000	47.6709885054468930	47.6709885054468640	$2.842171e-014$	0.0372
1.0000	47.4190006241149080	47.4190006241148790	$2.842171e-014$	0.0388

Table 4.3 : Showing the result for problem 4.3

t	Exact Solution	Computed Solution	ERR	t/sec
0.1000	719.8709504841319800	719.8709504841319800	$0.000000e+000$	0.0165
0.2000	746.7063189494632500	46.7063189494632500	$0.000000e+000$	0.0181
0.3000	774.5420569951836600	774.5420569951836600	$0.000000e+000$	0.0197
0.4000	803.4154564251550700	803.4154564251550700	$0.000000e+000$	0.0214
0.5000	833.3651992080965600	833.3651992080965600	$0.000000e+000$	0.0232
0.6000	864.4314093001880800	864.4314093001878500	$2.273737e-013$	0.0248
0.7000	896.6557063995159100	896.6557063995156800	$2.273737e-013$	0.0264
0.8000	930.0812617043808400	930.0812617043804900	$3.410605e-013$	0.0281
0.9000	964.7528557501631200	964.7528557501628900	$2.273737e-013$	0.0297
1.0000	1000.7169384022342000	1000.7169384022338000	$3.410605e-013$	0.0314

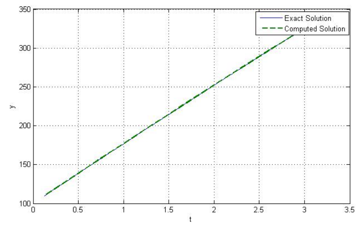

FIGURE 2. Graphical results for problem 4.1

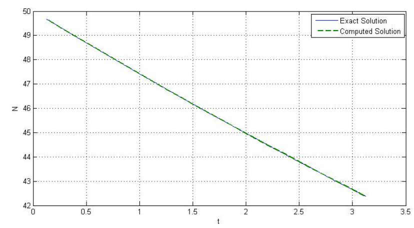

FIGURE 3. Graphical results for problem 4.2

Table 4.4 : Showing the result for problem 4.4

t	Exact Solution	Computed Solution	ERR	$t/\sec$
0.1000	0.5243852877496430	0.5243852877496430	$0.000000e+000$	0.0518
0.2000	0.5475812909820201	0.5475812909820201	$0.000000e+000$	0.0537
0.3000	0.5696460117874711	0.5696460117874710	$1.110223e-016$	0.0554
0.4000	0.5906346234610092	0.5906346234610089	$2.220446e-016$	0.0570
0.5000	0.6105996084642976	0.6105996084642974	$2.220446e-016$	0.0587
0.6000	0.6295908896591411	0.6295908896591409	$2.220446e-016$	0.0604
0.7000	0.6476559551406433	0.6476559551406431	$2.220446e-016$	0.0623
0.8000	0.6648399769821805	0.6648399769821802	$2.220446e-016$	0.0640
0.9000	0.6811859241891134	0.6811859241891132	$2.220446e-016$	0.0656
1.0000	0.6967346701436834	0.6967346701436831	$3.330669e-016$	0.0674

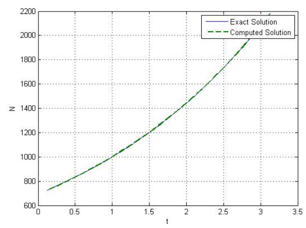

FIGURE 4. Graphical results for problem 4.3

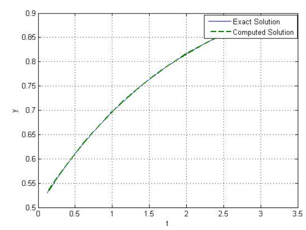

FIGURE 5. Graphical results for problem 4.4

4.1. Discussion of Results. We considered four real-life modeled first-order problems of the form (1). From the results obtained in the tables above, it is obvious that the quarter-step method derived is compuationally reliable. The graphical results obtained also buttress the fact that the computed results converge toward the exact solution.

5. Conclusion

We developed a quarter-step computational hybrid method for the solution modeled first-order problems of the form (1) using Laguerree polynomial of degree five as our basis function. The method developed was found to be L-stable and that explains why it performed well on this class of problems. The method was also found to be zero-stable,consistent, convergent and computationally reliable.

Conflict of Interests

The authors declare that there is no conflict of interests.

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