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SOME COMMON TRIPLED FIXED POINT THEOREMS IN TWO QUASI-PARTIAL b -METRIC SPACES

NEHA BHATIA

Department of Mathematics, University of Delhi, Delhi-110007, India

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Abstract. In this paper some common tripled fixed-point theorems are proved for mappings defined on a set equipped with two quasi-partial b -metric spaces and some examples are provided to support the results.

Keywords: Common tripled fixed point, Tripled coincidence point, Quasi-partial metric space, w -compatible mappings.

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1. Introduction

Matthews [16] in 1994 introduced the notion of partial metric space which is a generalization of usual metric space obtained by replacing the $d(x,x)=0$ by $d(x,x) \leq d(x,y)$ for all x,y in the definition of metric. He extended the Banach contraction principle from metric spaces to partial metric spaces. Bakhtin [6] introduced the concept of b -metric spaces which was further extended by Czerwick [8]. Later in the year 2013, Shukla [19] generalized both the concept of b -metric and partial metric spaces by introducing the partial b -metric spaces. Many authors ([3,4,5,13,18]) worked on this notion of partial metric spaces and obtained fixed point results for mappings satisfying different contractive conditions.

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2. Preliminaries

In 2012, Karapinar *et al.* [14] introduced the concept of quasi-partial metric spaces. The definition of partial metric space is given as follows:

Definition 2.1. (Matthews, [16]) A partial metric on a nonempty set X is a function $p : X \times X \rightarrow \mathbb{R}^+$ such that for all $x, y, z \in X$:

$$(P_1) \quad x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y),$$

$$(P_2) \quad p(x, x) \leq p(x, y),$$

$$(P_3) \quad p(x, y) = p(y, x),$$

$$(P_4) \quad p(x, y) \leq p(x, z) + p(z, y) - p(z, z).$$

A partial metric space is a pair (X, p) such that X is a non-empty set and p is a partial metric on X . For a partial metric p on X , the function $d_p : X \times X \rightarrow \mathbb{R}^+$ defined by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) \text{ is a metric on } X.$$

Definition 2.2. (Karapinar *et al.* [14]) A *quasi-partial metric* on non-empty set X is a function $q : X \times X \rightarrow \mathbb{R}^+$ which satisfies:

$$(QPM_1) \quad \text{If } q(x, x) = q(x, y) = q(y, y), \text{ then } x = y,$$

$$(QPM_2) \quad q(x, x) \leq q(x, y),$$

$$(QPM_3) \quad q(x, x) \leq q(y, x), \text{ and}$$

$$(QPM_4) \quad q(x, y) + q(z, z) \leq q(x, z) + q(z, y)$$

for all $x, y, z \in X$.

A *quasi-partial metric space* is a pair (X, q) such that X is a non-empty set and q is a quasi-partial metric on X .

Let q be a quasi-partial metric on the set X . Then

$$d_q(x, y) = q(x, y) + q(y, x) - q(x, x) - q(y, y) \text{ is a metric on } X.$$

Lemma 2.1. (Karapinar *et. al* [14]) *Let (X, q) be a quasi-partial metric space. Let (X, p_q) be the corresponding partial metric space, and let (X, d_{p_q}) be the corresponding metric space. Then the following statements are equivalent:*

- (1) (X, q) is complete,
- (2) (X, p_q) is complete,
- (3) (X, d_{p_q}) is complete.

Moreover,

$$\begin{aligned} \lim_{n \rightarrow \infty} d_{p_q}(x, x_n) = 0 &\Leftrightarrow p_q(x, x) = \lim_{n \rightarrow \infty} p_q(x, x_n) = \lim_{n, m \rightarrow \infty} p_q(x_n, x_m) \\ &\Leftrightarrow q(x, x) = \lim_{n \rightarrow \infty} q(x, x_n) = \lim_{n, m \rightarrow \infty} q(x_n, x_m) \\ &= \lim_{n \rightarrow \infty} q(x_n, x) = \lim_{n, m \rightarrow \infty} q(x_m, x_n). \end{aligned}$$

Definition 2.3. (Shukla [19]) A *partial b -metric* on a non-empty set X is a mapping $p_b : X \times X \rightarrow \mathbb{R}^+$ such that for some real number $s \geq 1$ and for all $x, y, z \in X$:

- (P_{b_1}) $x = y$ if and only if $p_b(x, x) = p_b(x, y) = p_b(y, y)$,
- (P_{b_2}) $p_b(x, x) \leq p_b(x, y)$,
- (P_{b_3}) $p_b(x, y) = p_b(y, x)$,
- (P_{b_4}) $p_b(x, y) \leq s[p_b(x, z) + p_b(z, y)] - p_b(z, z)$.

A *partial b -metric space* is a pair (X, p_b) such that X is a non-empty set and p_b is a partial b -metric on X . The number s is called the coefficient of (X, p_b) .

For simplicity, We denote $X \times X \times \dots \times X$ by X^k where $k \in \mathbb{N}$ and X is a non-empty set.

Definition 2.4. (Bhaskar and Lakshmikantham [7]) Let X be a non-empty set. An element $(x, y) \in X^2$ is a *coupled fixed point* of the mapping

$$F : X^2 \rightarrow X \text{ if } F(x, y) = x \text{ and } F(y, x) = y.$$

Definition 2.5. (Lakshmikantham and Ćirić [15]) An element $(x, y) \in X^2$ is called

- (1) a *coupled coincidence point* of the mappings $F : X^2 \rightarrow X$ and $g : X \rightarrow X$ if $F(x, y) = gx$ and $F(y, x) = gy$; in this case (gx, gy) is called *coupled point of coincidence* of mappings F and g ;
- (2) a *common coupled fixed point* of mappings $F : X^2 \rightarrow X$ and $g : X \rightarrow X$ if $F(x, y) = gx = x$ and $F(y, x) = gy = y$.

Definition 2.6. (Samet and Vetro [18]) An element $(x, y, z) \in X^3$ is a *triple fixed point* of the mapping

$$F : X^3 \rightarrow X \text{ if } F(x, y, z) = x, F(y, z, x) = y \text{ and } F(z, x, y) = z.$$

Definition 2.7. (Aydi *et al.* [15]) An element $(x, y, z) \in X^3$ is called

- (1) a *triple coincidence point* of the mappings $F : X^3 \rightarrow X$ and $g : X \rightarrow X$ if $F(x, y, z) = gx$, $F(y, z, x) = gy$ and $F(z, x, y) = gz$; in this case (gx, gy, gz) is called *triple point of coincidence* of mappings F and g ;
- (2) a *common triple fixed point* of mappings $F : X^3 \rightarrow X$ and $g : X \rightarrow X$ if $F(x, y, z) = gx = x$, $F(y, z, x) = gy = y$ and $F(z, x, y) = gz = z$.

Definition 2.8. (Aydi *et al.* [1]) Let X be a non-empty set. The mappings $F : X^3 \rightarrow X$ and $g : X \rightarrow X$ are *w-compatible* if $gF(x, y, z) = F(gx, gy, gz)$ whenever $gx = F(x, y, z)$, $gy = F(y, z, x)$ and $gz = F(z, x, y)$.

Theorem 2.1. [9] Let q_1 and q_2 be two quasi partial metrics on X such that $q_2(x, y) \leq q_1(x, y)$, for all $x, y \in X$, and let $F : X^3 \rightarrow X$, $g : X \rightarrow X$ be two mappings. Suppose that there exists k_1, k_2, k_3, k_4 , and k_5 in $[0, 1)$ with

$$k_1 + k_2 + k_3 + 2k_4 + k_5 < 1$$

such that the condition

$$\begin{aligned} & q_1(F(x, y, z), F(u, v, w)) + q_1(F(y, z, x), F(v, w, u)) + q_1(F(z, x, y), F(w, u, v)) \\ & \leq k_1[q_2(gx, gu) + q_2(gy, gv)] + q_2(gz, gw) \\ & \quad + k_2[q_2(gx, F(x, y, z)) + q_2(gy, F(y, z, x)) + q_2(gz, F(z, x, y))] \\ & \quad + k_3[q_2(gu, F(u, v, w)) + q_2(gv, F(v, w, u)) + q_2(gw, F(w, u, v))] \\ & \quad + k_4[q_2(gx, F(u, v, w)) + q_2(gy, F(v, w, u)) + q_2(gz, F(w, u, v))] \\ & \quad + k_5[q_2(gu, F(x, y, z)) + q_2(gv, F(y, z, x)) + q_2(gw, F(z, x, y))] \end{aligned}$$

holds for all $x, y, z, u, v, w \in X$. Also, suppose we have the following hypotheses:

- (1) $F(X^3) \subseteq g(X)$.
- (2) $g(X)$ is complete subspace of X with respect to the quasi-partial metric q_1 .

Then the mapping F and g have a tripled coincidence point (x, y, z) satisfying $gx = F(x, y, z) = F(y, z, x) = gy = F(z, x, y) = gz$. Moreover, if F and g are w -compatible, then F and g have a unique common tripled fixed point of the form (u, u, u) .

Recently, Gupta and Gautam [11] has introduced quasi-partial b -metric spaces which is the generalization of the concept of quasi-partial-metric spaces.

Definition 2.9. (Gupta and Gautam [11]) A quasi-partial b -metric on a non-empty set X is a mapping $qp_b : X \times X \rightarrow \mathbb{R}^+$ such that for some real number $s \geq 1$ and for all $x, y, z \in X$:

$$(QP_{b_1}) \quad qp_b(x, x) = qp_b(x, y) = qp_b(y, y) \Rightarrow x = y,$$

$$(QP_{b_2}) \quad qp_b(x, x) \leq qp_b(x, y),$$

$$(QP_{b_3}) \quad qp_b(x, x) \leq qp_b(y, x),$$

$$(QP_{b_4}) \quad qp_b(x, y) \leq s[qp_b(x, z) + qp_b(z, y)] - qp_b(z, z).$$

A quasi-partial b -metric space is a pair (X, qp_b) such that X is a non-empty set and qp_b is a quasi-partial b -metric on X .

Let qp_b be a quasi-partial b -metric on the set X . Then

$$d_{qp_b}(x, y) = qp_b(x, y) + qp_b(y, x) - qp_b(x, x) - qp_b(y, y)$$

is a b -metric on X .

Lemma 2.2. (Gupta and Gautam [11]) Every quasi-partial metric space is a quasi-partial b -metric space. But the converse need not be true.

Lemma 2.3. (Gupta and Gautam [11]) Let (X, qp_b) be a quasi-partial b -metric space. Then the following hold:

$$(1) \text{ If } qp_b(x, y) = 0 \text{ then } x = y,$$

$$(2) \text{ If } x \neq y, \text{ then } qp_b(x, y) > 0 \text{ and } qp_b(y, x) > 0.$$

Definition 2.10. (Gupta and Gautam [11]) Let (X, qp_b) be a quasi-partial b -metric space. Then:

(1) A sequence $\{x_n\} \subset X$ converges to $x \in X$ if and only if

$$qp_b(x, x) = \lim_{n \rightarrow \infty} qp_b(x, x_n) = \lim_{n \rightarrow \infty} qp_b(x_n, x)$$

(2) A sequence $\{x_n\} \subset X$ is called a *Cauchy sequence* if and only if

$$\lim_{n,m \rightarrow \infty} qp_b(x_n, x_m) \quad \text{and} \quad \lim_{n,m \rightarrow \infty} qp_b(x_m, x_n) \quad \text{exist (and are finite).}$$

(3) The quasi partial b -metric space (X, qp_b) is said to be *complete* if every Cauchy sequence $\{x_n\} \subset X$ converges with respect to τ_{qp_b} to a point $x \in X$ such that

$$qp_b(x, x) = \lim_{n,m \rightarrow \infty} qp_b(x_m, x_n) = \lim_{n,m \rightarrow \infty} qp_b(x_n, x_m).$$

(4) A mapping $f : X \rightarrow X$ is said to be *continuous* at $x_0 \in X$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0, \delta)) \subset B(f(x_0), \varepsilon)$.

Recently, Aydi and Abbas [2] obtained some tripled coincidence and fixed point theorems in partial metric space. Also, Shatanawi and Pitea [17] derived some common coupled fixed point theorems for a pair of mappings in quasi-partial metric space. Gu and Wang [9,10] obtained some results on coupled and tripled fixed-point theorems in two quasi-partial metric spaces. Very recently, Gupta and Gautam [12] discussed some coupled fixed point results on quasi-partial b -metric spaces. The aim of this paper is to explore some common tripled fixed-point theorems for mappings defined on a set equipped with two quasi-partial b -metric spaces.

3. Main results

In this section we prove our main theorem which gives conditions for existence and uniqueness of a tripled fixed point on quasi-partial b -metric spaces.

Theorem 3.1. *Let qp_{b_1} and qp_{b_2} be two quasi-partial b -metrics on X such that $qp_{b_2}(x, y) \leq qp_{b_1}(x, y)$, for all $x, y \in X$. Let $F : X^3 \rightarrow X$, $g : X \rightarrow X$ be two mappings. Suppose that there exist k_1, k_2, k_3, k_4 , and k_5 in $[0, 1)$ with*

$$k_1 + k_2 + k_3 + 2sk_4 + k_5 < \frac{1}{s} \tag{3.1}$$

such that the condition

$$\begin{aligned}
 & qp_{b_1}(F(x, y, z), F(u, v, w)) + qp_{b_1}(F(y, z, x), F(v, w, u)) + qp_{b_1}(F(z, x, y), F(w, u, v)) \\
 & \leq k_1[qp_{b_2}(gx, gu) + qp_{b_2}(gy, gv)] + qp_{b_2}(gz, gw) \\
 & \quad + k_2[qp_{b_2}(gx, F(x, y, z)) + qp_{b_2}(gy, F(y, z, x)) + qp_{b_2}(gz, F(z, x, y))] \\
 & \quad + k_3[qp_{b_2}(gu, F(u, v, w)) + qp_{b_2}(gv, F(v, w, u)) + qp_{b_2}(gw, F(w, v, u))] \\
 & \quad + k_4[qp_{b_2}(gx, F(u, v, w)) + qp_{b_2}(gy, F(v, w, u)) + qp_{b_2}(gz, F(w, u, v))] \\
 & \quad + k_5[qp_{b_2}(gu, F(x, y, z)) + qp_{b_2}(gv, F(y, z, x)) + qp_{b_2}(gw, F(z, x, y))]
 \end{aligned} \tag{3.2}$$

holds for all $x, y, z, u, v, w \in X$. Also, suppose we have the following hypotheses:

- (1) $F(X^3) \subset g(X)$
- (2) $g(X)$ is a complete subspace of X with respect to the quasi-partial b -metric qp_{b_1} .

Then the mappings F and g have a tripled coincidence point (x, y, z) satisfying $gx = F(x, y, z) = F(y, z, x) = gy = F(z, x, y) = gz$. Moreover, if F and g are w -compatible, then F and g have a unique common tripled fixed point of the form (u, u, u) .

Proof. Let $x_0, y_0, z_0 \in X$. Since $F(X^3) \subset g(X)$, we can choose $x_1, y_1, z_1 \in X$ such that $gx_1 = F(x_0, y_0, z_0)$, $gy_1 = F(y_0, z_0, x_0)$ and $gz_1 = F(z_0, x_0, y_0)$. Similarly, we can choose $x_2, y_2, z_2 \in X$ such that $gx_2 = F(x_1, y_1, z_1)$, $gy_2 = F(y_1, z_1, x_1)$ and $gz_2 = F(z_1, x_1, y_1)$.

Continuing in this manner we can construct three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$$gx_{n+1} = F(x_n, y_n, z_n), gy_{n+1} = F(y_n, z_n, x_n) \text{ and } gz_{n+1} = F(z_n, x_n, y_n) \quad \forall n \geq 0. \tag{3.3}$$

Consider

$$\begin{aligned}
 & qp_{b_1}(gx_n, gx_{n+1}) + qp_{b_1}(gy_n, gy_{n+1}) + qp_{b_1}(gz_n, gz_{n+1}) \\
 & = qp_{b_1}(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n)) \\
 & \quad + qp_{b_1}(F(y_{n-1}, z_{n-1}, x_{n-1}), F(y_n, z_n, x_n)) \\
 & \quad + qp_{b_1}(F(z_{n-1}, x_{n-1}, y_{n-1}), F(z_n, x_n, y_n)).
 \end{aligned}$$

It follows from (3.2), (QP_{b_4}) and (QP_{b_2}) that,

$$\begin{aligned}
& qp_{b_1}(gx_n, gx_{n+1}) + qp_{b_1}(gy_n, gy_{n+1}) + qp_{b_1}(gz_n, gz_{n+1}) \\
& \leq (k_1 + k_2)[qp_{b_2}(gx_{n-1}, gx_n) + qp_{b_2}(gy_{n-1}, gy_n) + qp_{b_2}(gz_{n-1}, gz_n)] \\
& \quad + k_3[qp_{b_2}(gx_n, gx_{n+1}) + qp_{b_2}(gy_n, gy_{n+1}) + qp_{b_2}(gz_n, gz_{n+1})] \\
& \quad + k_4[qp_{b_2}(gx_{n-1}, gx_{n+1}) + qp_{b_2}(gy_{n-1}, gy_{n+1}) + qp_{b_2}(gz_{n-1}, gz_{n+1})] \\
& \quad + k_5[qp_{b_2}(gx_n, gx_n) + qp_{b_2}(gy_n, gy_n) + qp_{b_2}(gz_n, gz_n)]
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
& \leq (k_1 + k_2)[qp_{b_2}(gx_{n-1}, gx_n) + qp_{b_2}(gy_{n-1}, gy_n) + qp_{b_2}(gz_{n-1}, gz_n)] \\
& \quad + k_3[qp_{b_2}(gx_n, gx_{n+1}) + qp_{b_2}(gy_n, gy_{n+1}) + qp_{b_2}(gz_n, gz_{n+1})] \\
& \quad + k_4[s\{qp_{b_2}(gx_{n-1}, gx_n) + qp_{b_2}(gx_n, gx_{n+1})\} - qp_{b_2}(gx_n, gx_n)] \\
& \quad + sk_4[\{qp_{b_2}(gy_{n-1}, gy_n) + qp_{b_2}(gy_n, gy_{n+1})\} - qp_{b_2}(gy_n, gy_n)] \\
& \quad + sk_4[\{qp_{b_2}(gz_{n-1}, gz_n) + qp_{b_2}(gz_n, gz_{n+1})\} - qp_{b_2}(gz_n, gz_n)] \\
& \quad + k_5[qp_{b_2}(gx_n, gx_{n+1}) + qp_{b_2}(gy_n, gy_{n+1}) + qp_{b_2}(gz_n, gz_{n+1})] \\
& \leq (k_1 + k_2 + sk_4)[qp_{b_1}(gx_{n-1}, gx_n) + qp_{b_1}(gy_{n-1}, gy_n) + qp_{b_1}(gz_{n-1}, gz_n)] \\
& \quad + (k_3 + sk_4 + k_5)[qp_{b_1}(gx_n, gx_{n+1}) + qp_{b_1}(gy_n, gy_{n+1}) + qp_{b_1}(gz_n, gz_{n+1})],
\end{aligned} \tag{3.5}$$

which implies that

$$\begin{aligned}
& qp_{b_1}(gx_n, gx_{n+1}) + qp_{b_1}(gy_n, gy_{n+1}) + qp_{b_1}(gz_n, gz_{n+1}) \\
& \leq \frac{k_1 + k_2 + sk_4}{1 - k_3 - sk_4 - k_5} [qp_{b_1}(gx_{n-1}, gx_n) + qp_{b_1}(gy_{n-1}, gy_n) + qp_{b_1}(gz_{n-1}, gz_n)].
\end{aligned}$$

Put $k = \frac{k_1 + k_2 + sk_4}{1 - k_3 - sk_4 - k_5}$. Clearly, $0 \leq k < \frac{1}{s} < 1$. By repetition of inequality (3.4) n times we get

$$\begin{aligned}
& qp_{b_1}(gx_n, gx_{n+1}) + qp_{b_1}(gy_n, gy_{n+1}) + qp_{b_1}(gz_n, gz_{n+1}) \\
& \leq k^n [qp_{b_1}(gx_0, gx_1) + qp_{b_1}(gy_0, gy_1) + qp_{b_1}(gz_0, gz_1)].
\end{aligned}$$

Next, we shall prove that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences in $g(X)$. For each $n, m \in \mathbb{N}$, $m > n$, from (QP_{b_4}) and (3.5), we have

$$\begin{aligned}
 & qp_{b_1}(gx_n, gx_m) + qp_{b_1}(gy_n, gy_m) + qp_{b_1}(gz_n, gz_m) \\
 & \leq \sum_{i=n}^{m-1} s^{m-i} \cdot k^i [qp_{b_1}(gx_0, gx_1) + qp_{b_1}(gy_0, gy_1) + qp_{b_1}(gz_0, gz_1)] \\
 & = \sum_{i=n}^{m-1} \left(\frac{k}{s}\right)^i s^m [qp_{b_1}(gx_0, gx_1) + qp_{b_1}(gy_0, gy_1) + qp_{b_1}(gz_0, gz_1)] \\
 & \leq \sum_{i=n}^{\infty} \left(\frac{k}{s}\right)^i s^m [qp_{b_1}(gx_0, gx_1) + qp_{b_1}(gy_0, gy_1) + qp_{b_1}(gz_0, gz_1)] \\
 & = \frac{\left(\frac{k}{s}\right)^n}{\left(1 - \frac{k}{s}\right)} \cdot s^m [qp_{b_1}(gx_0, gx_1) + qp_{b_1}(gy_0, gy_1) + qp_{b_1}(gz_0, gz_1)].
 \end{aligned} \tag{3.6}$$

Taking limit as $n \rightarrow \infty$ in (3.6) and keeping m fixed, we get

$$\lim_{n \rightarrow \infty} [qp_{b_1}(gx_n, gx_m) + qp_{b_1}(gy_n, gy_m) + qp_{b_1}(gz_n, gz_m)] \leq 0.$$

But

$$\lim_{n \rightarrow \infty} [qp_{b_1}(gx_n, gx_m) + qp_{b_1}(gy_n, gy_m) + qp_{b_1}(gz_n, gz_m)] \geq 0.$$

This gives

$$\lim_{n \rightarrow \infty} [qp_{b_1}(gx_n, gx_m)] = \lim_{n \rightarrow \infty} [qp_{b_1}(gy_n, gy_m)] = \lim_{n \rightarrow \infty} [qp_{b_1}(gz_n, gz_m)] = 0.$$

Now taking limit as $m \rightarrow +\infty$, one has

$$\lim_{n, m \rightarrow \infty} qp_{b_1}(gx_n, gx_m) = \lim_{n, m \rightarrow \infty} qp_{b_1}(gy_n, gy_m) = \lim_{n, m \rightarrow \infty} qp_{b_1}(gz_n, gz_m) = 0. \tag{3.7}$$

Similarly, we can show that

$$\lim_{n, m \rightarrow \infty} qp_{b_1}(gx_m, gx_n) = 0 \quad \text{and} \quad \lim_{n, m \rightarrow \infty} qp_{b_1}(gz_m, gz_n) = 0. \tag{3.8}$$

So, $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences in $(g(X), qp_{b_1})$. Since $(g(X), qp_{b_1})$ is complete, there exist $gx, gy, gz \in g(X)$ such that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ converges to gx, gy and gz

with respect to $\tau_{qp_{b_1}}$, that is,

$$\begin{aligned} qp_{b_1}(gx, gx) &= \lim_{n \rightarrow \infty} qp_{b_1}(gx, gx_n) = \lim_{n \rightarrow \infty} qp_{b_1}(gx_n, gx) \\ &= \lim_{n, m \rightarrow \infty} qp_{b_1}(gx_m, gx_n) = \lim_{n, m \rightarrow \infty} qp_{b_1}(gx_n, gx_m), \end{aligned} \quad (3.9)$$

$$\begin{aligned} qp_{b_1}(gy, gy) &= \lim_{n \rightarrow \infty} qp_{b_1}(gy, gy_n) = \lim_{n \rightarrow \infty} qp_{b_1}(gy_n, gy) \\ &= \lim_{n, m \rightarrow \infty} qp_{b_1}(gy_m, gy_n) = \lim_{n, m \rightarrow \infty} qp_{b_1}(gy_n, gy_m) \text{ and} \end{aligned} \quad (3.10)$$

$$\begin{aligned} qp_{b_1}(gz, gz) &= \lim_{n \rightarrow \infty} qp_{b_1}(gz, gz_n) = \lim_{n \rightarrow \infty} qp_{b_1}(gz_n, gz) \\ &= \lim_{n, m \rightarrow \infty} qp_{b_1}(gz_m, gz_n) = \lim_{n, m \rightarrow \infty} qp_{b_1}(gz_n, gz_m). \end{aligned} \quad (3.11)$$

Combining (3.7)-(3.11), we obtain

$$\begin{aligned} qp_{b_1}(gx, gx) &= \lim_{n \rightarrow \infty} qp_{b_1}(gx, gx_n) = \lim_{n \rightarrow \infty} qp_{b_1}(gx_n, gx) \\ &= \lim_{n, m \rightarrow \infty} qp_{b_1}(gx_m, gx_n) = \lim_{n, m \rightarrow \infty} qp_{b_1}(gx_n, gx_m) = 0, \end{aligned} \quad (3.12)$$

$$\begin{aligned} qp_{b_1}(gy, gy) &= \lim_{n \rightarrow \infty} qp_{b_1}(gy, gy_n) = \lim_{n \rightarrow \infty} qp_{b_1}(gy_n, gy) \\ &= \lim_{n, m \rightarrow \infty} qp_{b_1}(gy_m, gy_n) = \lim_{n, m \rightarrow \infty} qp_{b_1}(gy_n, gy_m) = 0 \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} qp_{b_1}(gz, gz) &= \lim_{n \rightarrow \infty} qp_{b_1}(gz, gz_n) = \lim_{n \rightarrow \infty} qp_{b_1}(gz_n, gz) \\ &= \lim_{n, m \rightarrow \infty} qp_{b_1}(gz_m, gz_n) = \lim_{n, m \rightarrow \infty} qp_{b_1}(gz_n, gz_m) = 0. \end{aligned} \quad (3.14)$$

By QP_{b_4} , we have

$$\begin{aligned} qp_{b_1}(gx_{n+1}, F(x, y, z)) &\leq s\{qp_{b_1}(gx_{n+1}, gx) + qp_{b_1}(gx, F(x, y, z))\} - qp_{b_1}(gx, gx) \\ &\leq s\{qp_{b_1}(gx_{n+1}, gx) + qp_{b_1}(gx, F(x, y, z))\} \\ &\leq s[qp_{b_1}(gx_{n+1}, gx) + s\{qp_{b_1}(gx, gx_{n+1}) \\ &\quad + qp_{b_1}(gx_{n+1}, F(x, y, z))\} - qp_{b_1}(gx_{n+1}, gx_{n+1})] \\ &\leq s[qp_{b_1}(gx_{n+1}, gx)] + s^2[qp_{b_1}(gx, gx_{n+1})] \\ &\quad + s^2[qp_{b_1}(gx_{n+1}, F(x, y, z))]. \end{aligned}$$

Taking limit as $n \rightarrow \infty$ in the above inequalities and using (3.14), we have

$$\begin{aligned} \frac{1}{s}qp_{b_1}(gx, F(x, y, z)) &\leq \lim_{n \rightarrow \infty} qp_{b_1}(gx_{n+1}, F(x, y, z)) \\ &\leq sqp_{b_1}(gx, F(x, y, z)). \end{aligned} \quad (3.15)$$

Similarly using (3.15), one has

$$\begin{aligned} \frac{1}{s}qp_{b_1}(gy, F(y, z, x)) &\leq \lim_{n \rightarrow \infty} qp_{b_1}(gy_{n+1}, F(y, z, x)) \\ &\leq sqp_{b_1}(gy, F(y, z, x)). \end{aligned} \quad (3.16)$$

and

$$\begin{aligned} \frac{1}{s}qp_{b_1}(gz, F(z, x, y)) &\leq \lim_{n \rightarrow \infty} qp_{b_1}(gz_{n+1}, F(z, x, y)) \\ &\leq sqp_{b_1}(gz, F(z, x, y)). \end{aligned} \quad (3.17)$$

Now, we prove that $F(x, y, z) = gx$, $F(y, z, x) = gy$ and $F(z, x, y) = gz$. In fact, it follows from (3.1) and (3.2) that

$$\begin{aligned} &qp_{b_1}(gx_{n+1}, F(x, y, z)) + qp_{b_1}(gy_{n+1}, F(y, z, x)) + qp_{b_1}(gz_{n+1}, F(z, x, y)) \\ &= qp_{b_1}(F(x_n, y_n, z_n), F(x, y, z)) + qp_{b_1}(F(y_n, z_n, x_n), F(y, z, x)) \\ &\quad + qp_{b_1}(F(z_n, x_n, y_n), F(z, x, y)) \\ &\leq k_1[qp_{b_2}(gx_n, gx) + qp_{b_2}(gy_n, gy) + qp_{b_2}(gz_n, gz)] \\ &\quad + k_2[qp_{b_2}(gx_n, F(x_n, y_n, z_n)) + qp_{b_2}(gy_n, F(y_n, z_n, x_n)) + qp_{b_2}(gz_n, F(z_n, x_n, y_n))] \\ &\quad + k_3[qp_{b_2}(gx, F(x, y, z)) + qp_{b_2}(gy, F(y, z, x)) + qp_{b_2}(gz, F(z, x, y))] \\ &\quad + k_4[qp_{b_2}(gx_n, F(x, y, z)) + qp_{b_2}(gy_n, F(y, z, x)) + qp_{b_2}(gz_n, F(z, x, y))] \\ &\quad + k_5[qp_{b_2}(gx, F(x_n, y_n, z_n)) + qp_{b_2}(gy, F(y_n, z_n, x_n)) + qp_{b_2}(gz, F(z_n, x_n, y_n))] \\ &\leq k_1[qp_{b_1}(gx_n, gx) + qp_{b_1}(gy_n, gy) + qp_{b_1}(gz_n, gz)] \\ &\quad + k_2[qp_{b_1}(gx_n, gx_{n+1}) + qp_{b_1}(gy_n, gy_{n+1}) + qp_{b_1}(gz_n, gz_{n+1})] \\ &\quad + k_3[qp_{b_1}(gx, F(x, y, z)) + qp_{b_1}(gy, F(y, z, x)) + qp_{b_1}(gz, F(z, x, y))] \\ &\quad + k_4[qp_{b_1}(gx_n, F(x, y, z)) + qp_{b_1}(gy_n, F(y, z, x)) + qp_{b_1}(gz_n, F(z, x, y))] \\ &\quad + k_5[qp_{b_1}(gx, gx_{n+1}) + qp_{b_1}(gy, gy_{n+1}) + qp_{b_1}(gz, gz_{n+1})]. \end{aligned}$$

Taking limit as $n \rightarrow \infty$ in the above inequality, using (3.12)-(3.17), we get

$$\begin{aligned}
& \lim_{n \rightarrow \infty} [qp_{b_1}(gx_{n+1}, F(x, y, z)) + qp_{b_1}(gy_{n+1}, F(y, z, x)) + qp_{b_1}(gz_{n+1}, F(z, x, y))] \\
& \leq \lim_{n \rightarrow \infty} \{ [k_1(qp_{b_1}(gx_n, gx) + qp_{b_1}(gy_n, gy) + qp_{b_1}(gz_n, gz))] \\
& \quad + k_2[qp_{b_1}(gx_n, gx_{n+1}) + qp_{b_1}(gy_n, gy_{n+1}) + qp_{b_1}(gz_n, gz_{n+1})] \\
& \quad + k_3[qp_{b_1}(gx, F(x, y, z)) + qp_{b_1}(gy, F(y, z, x)) + qp_{b_1}(gz, F(z, x, y))] \\
& \quad + k_4[qp_{b_1}(gx_n, F(x, y, z)) + qp_{b_1}(gy_n, F(y, z, x)) + qp_{b_1}(gz_n, F(z, x, y))] \\
& \quad + k_5[qp_{b_1}(gx, gx_{n+1}) + qp_{b_1}(gy, gy_{n+1}) + qp_{b_1}(gz, gz_{n+1})] \}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \lim_{n \rightarrow \infty} [qp_{b_1}(gx_{n+1}, F(x, y, z)) + qp_{b_1}(gy_{n+1}, F(y, z, x)) + qp_{b_1}(gz_{n+1}, F(z, x, y))] \\
& \leq k_1[qp_{b_1}(gx, gx) + qp_{b_1}(gy, gy) + qp_{b_1}(gz, gz)] + k_2[qp_{b_1}(gx, gx) + qp_{b_1}(gy, gy) + qp_{b_1}(gz, gz)] \\
& \quad + k_3[qp_{b_1}(gx, F(x, y, z)) + qp_{b_1}(gy, F(y, z, x)) + qp_{b_1}(gz, F(z, x, y))] \\
& \quad + \lim_{n \rightarrow \infty} k_4[qp_{b_1}(gx_n, F(x, y, z)) + qp_{b_1}(gy_n, F(y, z, x)) + qp_{b_1}(gz_n, F(z, x, y))] \\
& \quad + k_5[qp_{b_1}(gx, gx) + qp_{b_1}(gy, gy) + qp_{b_1}(gz, gz)] \\
& = k_3[qp_{b_1}(gx, F(x, y, z)) + qp_{b_1}(gy, F(y, z, x)) + qp_{b_1}(gz, F(z, x, y))] \\
& \quad + \lim_{n \rightarrow \infty} k_4[qp_{b_1}(gx_n, F(x, y, z)) + qp_{b_1}(gy_n, F(y, z, x)) + qp_{b_1}(gz_n, F(z, x, y))].
\end{aligned} \tag{3.19}$$

By using (3.12)-(3.17), we get

$$\begin{aligned}
& \lim_{n \rightarrow \infty} [qp_{b_1}(gx_{n+1}, F(x, y, z)) + qp_{b_1}(gy_{n+1}, F(y, z, x)) + qp_{b_1}(gz_{n+1}, F(z, x, y))] \\
& \leq k_3[qp_{b_1}(gx, F(x, y, z)) + qp_{b_1}(gy, F(y, z, x)) + qp_{b_1}(gz, F(z, x, y))] \\
& \quad + k_4 \cdot s [qp_{b_1}(gx, F(x, y, z)) + qp_{b_1}(gy, F(y, z, x)) + qp_{b_1}(gz, F(z, x, y))] \\
& = (k_3 + sk_4)[qp_{b_1}(gx, F(x, y, z)) + qp_{b_1}(gy, F(y, z, x)) + qp_{b_1}(gz, F(z, x, y))].
\end{aligned}$$

And also

$$\begin{aligned}
& \frac{1}{s} [qp_{b_1}(gx, F(x, y, z)) + qp_{b_1}(gy, F(y, z, x)) + qp_{b_1}(gz, F(z, x, y))] \\
& \leq (k_3 + sk_4) [qp_{b_1}(gx, F(x, y, z)) + qp_{b_1}(gy, F(y, z, x)) + qp_{b_1}(gz, F(z, x, y))] \\
\Rightarrow & \left[\frac{1}{s} - k_3 - sk_4 \right] [qp_{b_1}(gx, F(x, y, z)) + qp_{b_1}(gy, F(y, z, x)) + qp_{b_1}(gz, F(z, x, y))] \leq 0.
\end{aligned} \tag{3.18}$$

Since $k_3 + sk_4 < \frac{1}{s}$. Thus it follows from (3.18) that

$$qp_{b_1}(gx, F(x, y, z)) = qp_{b_1}(gy, F(y, z, x)) = qp_{b_1}(gz, F(z, x, y)) = 0.$$

By Lemma 2.3, we get $F(x, y, z) = gx$, $F(y, z, x) = gy$ and $F(z, x, y) = gz$. Hence, (gx, gy, gz) is a tripled point of coincidence of mappings F and g .

Next, we will show that the tripled point of coincidence is unique. Suppose that $(x', y', z') \in X^3$ with $F(x', y', z') = gx'$, $F(y', z', x') = gy'$ and $F(z', x', y') = gz'$.

Using (3.2), (3.14)-(3.16), and (QP_{b_3}) , we obtain

$$\begin{aligned}
& qp_{b_1}(gx, gx') + qp_{b_1}(gy, gy') + qp_{b_1}(gz, gz') \\
& = qp_{b_1}(F(x, y, z), F(x', y', z')) + qp_{b_1}(F(y, z, x), F(y', z', x')) \\
& \quad + qp_{b_1}(F(z, x, y), F(z', x', y')) \\
& \leq k_1 [qp_{b_2}(gx, gx') + qp_{b_2}(gy, gy') + qp_{b_2}(gz, gz')] \\
& \quad + k_2 [qp_{b_2}(gx, F(x, y, z)) + qp_{b_2}(gy, F(y, z, x)) + qp_{b_2}(gz, F(z, x, y))] \\
& \quad + k_3 [qp_{b_2}(gx', F(x', y', z')) + qp_{b_2}(gy', F(y', z', x')) + qp_{b_2}(gz', F(z', x', y'))] \\
& \quad + k_4 [qp_{b_2}(gx, F(x', y', z')) + qp_{b_2}(gy, F(y', z', x')) + qp_{b_2}(gz, F(z', x', y'))] \\
& \quad + k_5 [qp_{b_2}(gx', F(x, y, z)) + qp_{b_2}(gy', F(y, z, x)) + qp_{b_2}(gz', F(z, x, y))]
\end{aligned}$$

$$\begin{aligned}
&= k_1[qp_{b_2}(gx, gx') + qp_{b_2}(gy, gy') + qp_{b_2}(gz, gz')] \\
&\quad + k_2[qp_{b_2}(gx, gx) + qp_{b_2}(gy, gy) + qp_{b_2}(gz, gz)] \\
&\quad + k_3[qp_{b_2}(gx', gx') + qp_{b_2}(gy', gy') + qp_{b_2}(gz', gz')] \\
&\quad + k_4[qp_{b_2}(gx, gx') + qp_{b_2}(gy, gy') + qp_{b_2}(gz, gz')] \\
&\quad + k_5[qp_{b_2}(gx', gx) + qp_{b_2}(gy', gy) + qp_{b_2}(gz', gz)] \\
&\leq (k_1 + k_4)[qp_{b_1}(gx, gx') + qp_{b_1}(gy, gy') + qp_{b_1}(gz, gz')] \\
&\quad + k_2[qp_{b_1}(gx, gx) + qp_{b_1}(gy, gy) + qp_{b_1}(gz, gz)] \\
&\quad + k_3[qp_{b_1}(gx', gx') + qp_{b_1}(gy', gy') + qp_{b_1}(gz', gz')] \\
&\quad + k_5[qp_{b_1}(gx', gx) + qp_{b_1}(gy', gy) + qp_{b_1}(gz', gz)] \\
&\leq (k_1 + k_3 + k_4)[qp_{b_1}(gx, gx') + qp_{b_1}(gy, gy') + qp_{b_1}(gz, gz')] \\
&\quad + k_5[qp_{b_1}(gx', gx) + qp_{b_1}(gy', gy) + qp_{b_1}(gz', gz)].
\end{aligned}$$

This implies that

$$\begin{aligned}
&qp_{b_1}(gx, gx') + qp_{b_1}(gy, gy') + qp_{b_1}(gz, gz') \\
&\leq \frac{k_5}{1 - k_1 - k_3 - k_4}[qp_{b_1}(gx', gx) + qp_{b_1}(gy', gy) + qp_{b_1}(gz', gz)].
\end{aligned} \tag{3.19}$$

Similarly, we have

$$\begin{aligned}
&qp_{b_1}(gx', gx) + qp_{b_1}(gy', gy) + qp_{b_1}(gz', gz) \\
&\leq \frac{k_5}{1 - k_1 - k_3 - k_4}[qp_{b_1}(gx, gx') + qp_{b_1}(gy, gy') + qp_{b_1}(gz, gz')].
\end{aligned} \tag{3.20}$$

Substituting (3.20) into (3.19), we obtain

$$\begin{aligned}
&qp_{b_1}(gx, gx') + qp_{b_1}(gy, gy') + qp_{b_1}(gz, gz') \\
&\leq \left(\frac{k_5}{1 - k_1 - k_3 - k_4} \right)^2 [qp_{b_1}(gx, gx') + qp_{b_1}(gy, gy') + qp_{b_1}(gz, gz')].
\end{aligned} \tag{3.21}$$

Since $\frac{k_5}{1 - k_1 - k_3 - k_4} < 1$, from (2.21), we must have

$$qp_{b_1}(gx, gx') = qp_{b_1}(gy, gy') = qp_{b_1}(gz, gz') = 0.$$

By Lemma 2.3, we get $gx = gx'$, $gy = gy'$ and $gz = gz'$. This gives the uniqueness of the tripled point of coincidence of F and g , that is, (gx, gy, gz) .

Next, we will show that $gx = gy = gz$. In fact, from (3.2), (3.14)-(3.16), we have

$$\begin{aligned}
 & qp_{b_1}(gx, gy) + qp_{b_1}(gy, gz) + qp_{b_1}(gz, gx) \\
 &= qp_{b_1}(F(x, y, z), F(y, z, x)) + qp_{b_1}(F(y, z, x), F(z, x, y)) + qp_{b_1}(F(z, x, y), F(x, y, z)) \\
 &\leq k_1[qp_{b_2}(gx, gy) + qp_{b_2}(gy, gz)] + qp_{b_2}(gz, gx) \\
 &\quad + k_2[qp_{b_2}(gx, F(x, y, z)) + qp_{b_2}(gy, F(y, z, x)) + qp_{b_2}(gz, F(z, x, y))] \\
 &\quad + k_3[qp_{b_2}(gy, F(y, z, x)) + qp_{b_2}(gz, F(z, x, y)) + qp_{b_2}(gx, F(x, y, z))] \\
 &\quad + k_4[qp_{b_2}(gx, F(y, z, x)) + qp_{b_2}(gy, F(z, x, y)) + qp_{b_2}(gz, F(x, y, z))] \\
 &\quad + k_5[qp_{b_2}(gy, F(x, y, z)) + qp_{b_2}(gz, F(y, z, x)) + qp_{b_2}(gx, F(z, x, y))] \\
 &= k_1[qp_{b_2}(gx, gy) + qp_{b_2}(gy, gz)] + qp_{b_2}(gz, gx) \\
 &\quad + k_2[qp_{b_2}(gx, gx) + qp_{b_2}(gy, gy) + qp_{b_2}(gz, gz)] \\
 &\quad + k_3[qp_{b_2}(gy, gy) + qp_{b_2}(gz, gz) + qp_{b_2}(gx, gx)] \\
 &\quad + k_4[qp_{b_2}(gx, gy) + qp_{b_2}(gy, gz) + qp_{b_2}(gz, gx)] \\
 &\quad + k_5[qp_{b_2}(gy, gx) + qp_{b_2}(gz, gy) + qp_{b_2}(gx, gz)] \\
 &\leq k_1[qp_{b_1}(gx, gy) + qp_{b_1}(gy, gz) + qp_{b_1}(gz, gx)] \tag{3.22} \\
 &\quad + k_2[qp_{b_1}(gx, gx) + qp_{b_1}(gy, gy) + qp_{b_1}(gz, gz)] \\
 &\quad + k_3[qp_{b_1}(gy, gy) + qp_{b_1}(gz, gz) + qp_{b_1}(gx, gx)] \\
 &\quad + k_4[qp_{b_1}(gx, gy) + qp_{b_1}(gy, gz) + qp_{b_1}(gz, gx)] \\
 &\quad + k_5[qp_{b_1}(gy, gx) + qp_{b_1}(gz, gy) + qp_{b_1}(gx, gz)] \\
 &= (k_1 + k_4 + k_5)[qp_{b_1}(gx, gy) + qp_{b_1}(gy, gz) + qp_{b_1}(gz, gx)].
 \end{aligned}$$

Since $k_1 + k_4 + k_5 < 1$ from (3.22) we have

$$qp_{b_1}(gx, gy) = qp_{b_1}(gy, gz) = qp_{b_1}(gz, gx) = 0.$$

By Lemma 2.3, we get $gx = gy = gz$.

Finally, assume that g and F are w -compatible. Let $u = gx$, then we have $u = gx = F(x, y, z) = gy = F(y, z, x) = gz = F(z, x, y)$, so that

$$gu = ggx = g(F(x, y, z)) = F(gx, gy, gz) = F(u, u, u). \quad (3.23)$$

Consequently, (u, u, u) is a tripled coincidence point of F and g , and therefore (gu, gu, gu) is a tripled point of coincidence of F and g , and by its uniqueness, we get $gu = gx$. Thus, we obtain $F(u, u, u) = gu = u$. Therefore, (u, u, u) is the unique common tripled fixed point of F and g . This completes the proof.

Corollary 3.1. *Let qp_b be a quasi-partial b -metrics on X , $F : X^3 \rightarrow X$, $g : X \rightarrow X$ be two mappings. Suppose that there exist k_1, k_2, k_3, k_4 , and k_5 in $[0, 1)$ with*

$$k_1 + k_2 + k_3 + 2sk_4 + k_5 < \frac{1}{s} \quad (3.1.1)$$

such that the condition

$$\begin{aligned} & qp_b(F(x, y, z), F(u, v, w)) + qp_b(F(y, z, x), F(v, w, u)) + qp_b(F(z, x, y), F(w, u, v)) \\ & \leq k_1[qp_b(gx, gu) + qp_b(gy, gv)] + qp_b(gz, gw) \\ & + k_2[qp_b(gx, F(x, y, z)) + qp_b(gy, F(y, z, x)) + qp_b(gz, F(z, x, y))] \\ & + k_3[qp_b(gu, F(u, v, w)) + qp_b(gv, F(v, w, u)) + qp_b(gw, F(w, v, u))] \\ & + k_4[qp_b(gx, F(u, v, w)) + qp_b(gy, F(v, w, u)) + qp_b(gz, F(w, u, v))] \\ & + k_5[qp_b(gu, F(x, y, z)) + qp_b(gv, F(y, z, x)) + qp_b(gw, F(z, x, y))] \end{aligned} \quad (3.1.2)$$

holds for all $x, y, z, u, v, w \in X$. Also, suppose we have the following hypotheses:

- (1) $F(X^3) \subset g(X)$
- (2) $g(X)$ is a complete subspace of X with respect to the quasi-partial b -metric qp_b .

Then the mappings F and g have a tripled coincidence point (x, y, z) satisfying $gx = F(x, y, z) = F(y, z, x) = gy = F(z, x, y) = gz$.

Moreover, if F and g are w -compatible, then F and g have a unique common tripled fixed point of the form (u, u, u) .

Corollary 3.2. *Let qp_{b_1} and qp_{b_2} be two quasi-partial b -metrics on X and $qp_{b_2}(x, y) \leq qp_{b_1}(x, y)$, for all $x, y \in X$. Let $F : X^3 \rightarrow X$, $g : X \rightarrow X$ be two mappings. Suppose that there exist $a_i \in [0, 1)$*

($i = 1, 2, 3, \dots, 15$) with

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + 2s(a_{10} + a_{11} + a_{12}) + a_{13} + a_{14} + a_{15} < \frac{1}{s} \quad (3.2.1)$$

such that the condition

$$\begin{aligned} & qp_{b_1}(F(x, y, z), F(u, v, w)) + qp_{b_1}(F(y, z, x), F(v, w, u)) + qp_{b_1}(F(z, x, y), F(w, u, v)) \\ & \leq a_1 qp_{b_2}(gx, gu) + a_2 qp_{b_2}(gy, gv) + a_3 qp_{b_2}(gz, gw) \\ & \quad + a_4 qp_{b_2}(gx, F(x, y, z)) + a_5 qp_{b_2}(gy, F(y, z, x)) + a_6 qp_{b_2}(gz, F(z, x, y)) \\ & \quad + a_7 qp_{b_2}(gu, F(u, v, w)) + a_8 qp_{b_2}(gv, F(v, w, u)) + a_9 qp_{b_2}(gw, F(w, u, v)) \\ & \quad + a_{10} qp_{b_2}(gx, F(u, v, w)) + a_{11} qp_{b_2}(gy, F(v, w, u)) + a_{12} qp_{b_2}(gz, F(w, u, v)) \\ & \quad + a_{13} qp_{b_2}(gu, F(x, y)) + a_{14} qp_{b_2}(gv, F(y, z, x)) + a_{15} qp_{b_2}(gw, F(z, y, x)) \end{aligned} \quad (3.2.2)$$

holds for all $x, y, z, u, v, w \in X$. Also suppose we have the following hypotheses:

- (1) $F(X^3) \subseteq g(X)$
- (2) $g(X)$ is a complete subspace of X with respect to the quasi-partial b -metric qp_{b_1} .

Then the mappings F and g have a tripled coincidence point (x, y, z) satisfying $gx = F(x, y, z) = F(y, z, x) = gy = F(z, x, y) = gz$.

Moreover, if F and g are w -compatible, then F and g have a unique common tripled fixed point of the form (u, u, u) .

Proof. Given $x, y, z, u, v, w \in X$, it follows from (3.2.2) that

$$\begin{aligned} & qp_{b_1}(F(x, y, z), F(u, v, w)) \\ & \leq a_1 qp_{b_2}(gx, gu) + a_2 qp_{b_2}(gy, gv) + a_3 qp_{b_2}(gz, gw) \\ & \quad + a_4 qp_{b_2}(gx, F(x, y, z)) + a_5 qp_{b_2}(gy, F(y, z, x)) + a_6 qp_{b_2}(gz, F(z, x, y)) \\ & \quad + a_7 qp_{b_2}(gu, F(u, v, w)) + a_8 qp_{b_2}(gv, F(v, w, u)) + a_9 qp_{b_2}(gw, F(w, u, v)) \\ & \quad + a_{10} qp_{b_2}(gx, F(u, v, w)) + a_{11} qp_{b_2}(gy, F(v, w, u)) + a_{12} qp_{b_2}(gz, F(w, u, v)) \\ & \quad + a_{13} qp_{b_2}(gu, F(x, y)) + a_{14} qp_{b_2}(gv, F(y, z, x)) + a_{15} qp_{b_2}(gw, F(z, y, x)) \end{aligned} \quad (3.2.3)$$

holds for all $x, y, z, u, v, w \in X$. Also suppose we have the following hypotheses: and

$$\begin{aligned}
& qp_{b_1}(F(y, z, x), F(v, w, u)) \\
& \leq a_1 qp_{b_2}(gy, gv) + a_2 qp_{b_2}(gz, gw) + a_3 qp_{b_2}(gx, gu) \\
& \quad + a_4 qp_{b_2}(gy, F(y, z, x)) + a_5 qp_{b_2}(gz, F(z, x, y)) + a_6 qp_{b_2}(gx, F(x, y, z)) \\
& \quad + a_7 qp_{b_2}(gv, F(v, w, u)) + a_8 qp_{b_2}(gw, F(w, u, v)) + a_9 qp_{b_2}(gu, F(u, v, w)) \\
& \quad + a_{10} qp_{b_2}(gy, F(v, w, u)) + a_{11} qp_{b_2}(gz, F(w, u, v)) + a_{12} qp_{b_2}(gx, F(u, v, w)) \\
& \quad + a_{13} qp_{b_2}(gv, F(y, z, x)) + a_{14} qp_{b_2}(gw, F(z, y, x)) + a_{15} qp_{b_2}(gu, F(x, y))
\end{aligned} \tag{3.2.4}$$

holds for all $x, y, z, u, v, w \in X$. Also suppose we have the following hypotheses:

$$\begin{aligned}
& qp_{b_1}(F(z, x, y), F(w, u, v)) \\
& \leq a_1 qp_{b_2}(gz, gw) + a_2 qp_{b_2}(gx, gu) + a_3 qp_{b_2}(gy, gv) \\
& \quad + a_6 qp_{b_2}(gy, F(y, z, x)) + a_4 qp_{b_2}(gz, F(z, x, y)) + a_5 qp_{b_2}(gx, F(x, y, z)) \\
& \quad + a_7 qp_{b_2}(gw, F(w, u, v)) + a_8 qp_{b_2}(gu, F(u, v, w)) + a_9 qp_{b_2}(gv, F(v, w, u)) \\
& \quad + a_{10} qp_{b_2}(gz, F(w, u, v)) + a_{11} qp_{b_2}(gx, F(u, v, w)) + a_{12} qp_{b_2}(gy, F(v, w, u)) \\
& \quad + a_{13} qp_{b_2}(gw, F(z, y, x)) + a_{14} qp_{b_2}(gu, F(x, y)) + a_{15} qp_{b_2}(gv, F(y, z, x))
\end{aligned} \tag{3.2.5}$$

holds for all $x, y, z, u, v, w \in X$. Adding inequalities (3.2.3) and (3.2.4) to inequality (3.2.5), we get

$$\begin{aligned}
& qp_{b_1}(F(x, y, z), F(u, v, w)) + qp_{b_1}(F(y, z, x), F(v, w, u)) + qp_{b_1}(F(z, x, y), F(w, u, v)) \\
& \leq (a_1 + a_2 + a_3)[qp_{b_2}(gx, gu) + qp_{b_2}(gy, gv) + qp_{b_2}(gz, gw)] \\
& \quad + (a_4 + a_5 + a_6)[qp_{b_2}(gx, F(x, y, z)) + qp_{b_2}(gy, F(y, z, x)) + qp_{b_2}(gz, F(z, x, y))] \\
& \quad + (a_7 + a_8 + a_9)[qp_{b_2}(gu, F(u, v, w)) + qp_{b_2}(gv, F(v, w, u)) + qp_{b_2}(gw, F(w, u, v))] \\
& \quad + (a_{10} + a_{11} + a_{12})[qp_{b_2}(gx, F(u, v, w)) + qp_{b_2}(gy, F(v, w, u)) + qp_{b_2}(gz, F(w, u, v))] \\
& \quad + (a_{13} + a_{14} + a_{15})[qp_{b_2}(gu, F(x, y, z)) + qp_{b_2}(gv, F(y, z, x)) + qp_{b_2}(gw, F(z, x, y))].
\end{aligned}$$

Therefore, letting $a_1 + a_2 + a_3 = k_1$, $a_4 + a_5 + a_6 = k_2$, $a_7 + a_8 + a_9 = k_3$, $a_{10} + a_{11} + a_{12} = k_4$, $a_{13} + a_{14} + a_{15} = k_5$, the result follows from Theorem 3.1.

Corollary 3.3. *Let qp_{b_1} and qp_{b_2} be two quasi-partial b -metrics such that $qp_{b_2}(x, y) \leq qp_{b_1}(x, y)$, for all $x, y \in X$. Let $F : X^3 \rightarrow X$, $g : X \rightarrow X$ be two mappings. Suppose that there exists $k \in [0, 1)$ such that the condition*

$$\begin{aligned} & qp_{b_1}(F(x, y, z), F(u, v, w)) + qp_{b_1}(F(y, z, x), F(v, w, u)) + qp_{b_1}(F(z, x, y), F(w, u, v)) \\ & \leq k[qp_{b_2}(gx, gu) + qp_{b_2}(gy, gv) + qp_{b_2}(gz, gw)] \end{aligned}$$

holds for all $x, y, z, u, v, w \in X$. Also, suppose we have the following hypotheses:

- (1) $F(X^3) \subseteq g(X)$
- (2) $g(X)$ is a complete subspace of X with respect to the quasi-partial b -metric qp_{b_1} .

Then the mappings F and g have a tripled coincidence point (x, y, z) satisfying $gx = F(x, y, z) = F(y, z, x) = gy = F(z, x, y) = gz$.

Moreover, if F and g are w -compatible, then F and g have a unique common tripled fixed point of the form (u, u, u) .

Proof. By putting $k_1 = k$ and $k_2 = k_3 = k_4 = k_5 = 0$ in Theorem 3.1 we get the result.

Corollary 3.4. *Let qp_{b_1} and qp_{b_2} be two quasi-partial b -metrics on X such that $qp_{b_2}(x, y) \leq qp_{b_1}(x, y)$, for all $x, y \in X$. Let $F : X^3 \rightarrow X$, $g : X \rightarrow X$ be two mappings. Suppose that there exists $k \in \left[0, \frac{1}{2s}\right)$ such that the condition*

$$\begin{aligned} & qp_{b_1}(F(x, y, z), F(u, v, w)) + qp_{b_1}(F(y, z, x), F(v, w, u)) + qp_{b_1}(F(z, x, y), F(w, u, v)) \\ & \leq k[qp_{b_2}(gx, F(u, v, w)) + qp_{b_2}(gy, F(v, w, u)) + qp_{b_2}(gz, F(w, u, v))] \end{aligned} \quad (3.2)$$

holds for all $x, y, z, u, v, w \in X$. Also, suppose we have the following hypotheses:

- (1) $F(X^3) \subseteq g(X)$
- (2) $g(X)$ is a complete subspace of X with respect to the quasi-partial b -metric qp_{b_1} .

Then the mappings F and g have a tripled coincidence point (x, y) satisfying $gx = F(x, y, z) = F(y, z, x) = gy = F(z, x, y) = gz$.

Moreover, if F and g are w -compatible, then F and g have a unique common tripled fixed point of the form (u, u, u) .

Proof. By putting $k_4 = k$ and $k_1 = k_2 = k_3 = k_5 = 0$ in Theorem 3.1 we get the desired result.

Example 3.1. Let $X = [0, 1]$ and two quasi-partial b -metrics qp_{b_1} and qp_{b_2} on X be given as

$$qp_{b_1}(x, y) = |x - y| + x \quad \text{and} \quad qp_{b_2}(x, y) = \frac{1}{2}(|x - y| + x)$$

for all $x, y \in X$. Also, define $F : X^3 \rightarrow X$ and $g : X \rightarrow X$ as $F(x, y) = \frac{x + y + z}{36}$ and $g(x) = \frac{x}{2}$ for all $x, y \in X$. Then

- (1) (X, qp_{b_1}) is a complete quasi-partial b -metric space.
- (2) $F(X^3) \subseteq g(X)$
- (3) F and g is w -compatible.
- (4) For any $x, y, z, u, v, w \in X$, we have

$$\begin{aligned} & qp_{b_1}(F(x, y, z), F(u, v, w)) + qp_{b_1}(F(y, z, x), F(v, w, u)) + qp_{b_1}(F(z, x, y), F(w, u, v)) \\ & \leq \frac{1}{3}(qp_{b_2}(gx, gu) + qp_{b_2}(gy, gv) + qp_{b_2}(gz, gw)). \end{aligned}$$

Proof. The proof of (i), (ii) and (iii) are clear. Next, we prove (iv). For $x, y, z, u, v, w \in X$, we have

$$\begin{aligned} & qp_{b_1}(F(x, y, z), F(u, v, w)) + qp_{b_1}(F(y, z, x), F(v, w, u)) + qp_{b_1}(F(z, x, y), F(w, u, v)) \\ & = qp_{b_1}\left(\frac{x + y + z}{36}, \frac{u + v}{36}\right) + qp_{b_1}\left(\frac{y + z + x}{36}, \frac{v + w + u}{36}\right) + qp_{b_1}\left(\frac{z + x + y}{36}, \frac{w + u + v}{36}\right) \\ & = \left|\frac{x + y + z}{36} - \frac{u + v + w}{36}\right| + \left|\frac{y + z + x}{36} - \frac{v + w + u}{36}\right| \\ & + \left|\frac{z + x + y}{36} - \frac{w + u + v}{36}\right| + \frac{3(x + y + z)}{36} \\ & = \frac{1}{12} [|(x + y + z) - (u + v + w)| + (x + y + z)] \\ & = \frac{1}{12} [|x - u| + |y - v| + |z - w| + (x + y + z)] \\ & \leq \frac{1}{12} [|x - u| + |y - v| + |z - w| + (x + y + z)] \\ & = \frac{1}{3} \left[\frac{1}{4}|x - u| + \frac{1}{4}|y - v| + \frac{1}{4}|z - w| + \frac{x}{4} + \frac{y}{4} + \frac{z}{4} \right] \\ & = \frac{1}{3}(qp_{b_2}\left(\frac{x}{2}, \frac{u}{2}\right) + qp_{b_2}\left(\frac{y}{2}, \frac{v}{2}\right) + qp_{b_2}\left(\frac{z}{2}, \frac{w}{2}\right)) \\ & = \frac{1}{3}(qp_{b_2}(gx, gu) + qp_{b_2}(gy, gv) + qp_{b_2}(gz, gw)). \end{aligned}$$

Thus, F and g satisfy all the hypotheses of Corollary 2.4. So, F and g have a unique common tripled fixed point. Here $(0,0,0)$ is the unique common tripled fixed point of F and g .

Example 3.2. Let $X = [0, 1]$ and two quasi-partial b -metrics qp_{b_1} and qp_{b_2} on X be given as

$$qp_{b_1}(x, y) = qp_{b_2}(x, y) = |x - y| + x$$

for all $x, y \in X$. Also, define $F : X^3 \rightarrow X$ and $g : X \rightarrow X$ as $F(x, y) = \frac{x+y+z}{3^nm}$ and $g(x) = \frac{x}{m}$ for all $x, y \in X$ and $n, m \in \mathbb{N}$. Then

(1) (X, qp_{b_1}) is a complete quasi-partial b -metric space.

(2) $F(X^3) \subseteq g(X)$

(3) F and g is w -compatible.

(4) For any $x, y, z, u, v, w \in X$, we have

$$\begin{aligned} & qp_{b_1}(F(x, y, z), F(u, v, w)) + qp_{b_1}(F(y, z, x), F(v, w, u)) + qp_{b_1}(F(z, x, y), F(w, u, v)) \\ & \leq \frac{1}{3^{n-1}}(qp_{b_2}(gx, gu) + qp_{b_2}(gy, gv) + qp_{b_2}(gz, gz)). \end{aligned}$$

Conflict of Interests

The author declares that there is no conflict of interests.

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