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SOME OPERATIONS ON CONVEX AND CONCAVE FUNCTIONS

W. T. SULAIMAN*

Department of Computer engineering, College of Engineering, University of Mosul, Iraq

Abstract: Several new results concerning operations on convex and concave functions are obtained.

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1. Introduction

A real-valued function f is said to be convex on a closed interval I if $f(tx+(1-t)y)\leq tf(x)+(1-t)f(y),$ for all $x,y\in I,\ 0\leq t\leq 1.$ If the inequality is reversed, the f is called concave. It is known that f is convex if $f''(x)\geq 0.$

The inequality

$$f\left(\frac{a+b}{2}\right)\leq \frac{1}{b-a}\int_a^b f(x)\,dx\leq \frac{f(a)+f(b)}{2} \tag{1}$$

which holds for all convex mapping $f:[a,b]\rightarrow \Re$, is known in the literature as Hadamard inequality. In [2], Fejér generalized Hadamard's inequality by giving the following :

Theorem 1.1. If $g:[a,b]\rightarrow \Re$ is non-negative integrable and symmetric to $x=\frac{a+b}{2}$, and if f is convex on [a,b], then

*Corresponding author
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$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \le \int_a^b f(x)\,g(x)\,dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \tag{2}$$

2. Lemmas

The following lemmas are needed for our aim

Lemma 2.1. *Let*

$$(a-b)(c-d) \ge 0, \tag{3}$$

then

$$\frac{(a+b)}{2}\frac{(c+d)}{2} \le \frac{ac+bd}{2}\ . \tag{4}$$

Proof. By (3),

$$ad+bc \le ac+bd$$

$$\Rightarrow ac+bc+ad+bd \le 2(ac+bd)$$

$$\Rightarrow \frac{(a+b)}{2}\frac{(c+d)}{2} \le \frac{ac+bd}{2}\ .$$

Lemma 2.2. *If* $c,\ d > 0$, *and*

$$a+b \le a\frac{d}{c}+b\frac{c}{d}, \tag{5}$$

then

$$\frac{a+b}{c+d} \le \frac{ad+bc}{2cd}. \tag{6}$$

Proof. By (6),

$$acd+bcd \le ad^2+bc^2$$

$$\Rightarrow 2acd+2bcd \le acd+bcd+ad^2+bc^2$$

$$\Rightarrow 2cd(a+b) \le (c+d)(ad+bc)$$

$$\Rightarrow \frac{a+b}{c+d} \le \frac{ad+bc}{2c}.$$

3. Results

Theorem 3.1. *Let* $f, g : I \subset \Re \to \Re$ *be positive convex functions such that for all* $a, b \in I$,

$$\big(f(a) - f(b)\big)\big(g(a) - g(b)\big) \geq 0, \tag{7}$$

then fg *is convex. If*

$$\big(f(a) - f(b)\big)\big(gb) - g(a)\big) \geq 0, \tag{8}$$

then fg *is concave.*

Proof. Applying Lemma 2.1, we have

$$fg\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}\,\frac{g(a)+g(b)}{2}$$

$$\leq \frac{f(a)g(a)+f(b)g(b)}{2} = \frac{(fg)(a)+(fg)(b)}{2}.$$

The proof of the other part is similar.

Theorem 3.2. *Let* $f,\ g : I \subset \Re \to \Re$, *be positive functions,* f *is convex and* g *is concave,* $g(a), g(b) \neq 0$ *and satisfying*

$$f(a) + f(b) \leq f(a)\frac{g(b)}{g(a)} + f(b)\frac{g(a)}{g(b)}, \quad \forall\, a, b \in I. \tag{9}$$

Then f/g *is convex.*

Proof. Since

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}, \tag{10}$$

$$\frac{g(a)+g(b)}{2} \leq g\left(\frac{a+b}{2}\right), \tag{11}$$

then on multiplying (9) and (10), we obtain

$$f\left(\frac{a+b}{2}\right)\left(\frac{g(a)+g(b)}{2}\right) \leq \frac{f(a)+f(b)}{2}\,g\left(\frac{a+b}{2}\right),$$

which implies

$$\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \le \frac{\frac{f(a)+f(b)}{2}}{\frac{g(a)+g(b)}{2}}.$$

Therefore, by Lemma 2.2,

$$(f/g)\left(\frac{a+b}{2}\right) = \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \le \frac{f(a)+f(b)}{g(a)+g(b)} \le \frac{f(a)g(b)+f(b)g(a)}{2g(a)g(b)}$$

$$= \frac{1}{2}\left(\frac{f(a)}{g(a)} + \frac{f(b)}{g(b)}\right) = \frac{(f/g)(a)+(f/g)(b)}{2}.$$

A positive function is said to be log-convex if *logf* is convex. Concerning this type of functions, we have the following result

Theorem 3.3. *If* $f : I \subset \Re \to \Re$ *is a positive convex function and if* $c \ge 1$, *then* $c^{f(x)}$ *is convex.*

Proof.

$$c^{f\left(\frac{a+b}{2}\right)} \le c^{\frac{f(a)+f(b)}{2}} = c^{\frac{f(a)}{2}} c^{\frac{f(b)}{2}} \le \frac{1}{2}\left(\left(c^{\frac{f(a)}{2}}\right)^2 + \left(c^{\frac{f(b)}{2}}\right)^2\right) = \frac{c^{f(a)} + c^{f(b)}}{2} .$$

Corollary 3.4. *Let* $f, g : I^+ \subset \Re \to \Re$, *f is log-convex and g is convex. If*

$$\big((\log f)(a) - (\log f)(b)\big)\big(g(a) - g(b)\big) \ge 0, \quad \forall a, b \in I^+ \tag{12}$$

then the function f^g *is convex.*

Proof. By Theorem 3.1, $(\log f)g$ is convex. The result follows by an application of Theorem 3.3, with $c = e$.

Theorem 3.5. *Let* $f : I \subset \Re \to \Re$ *be positive concave. Then* $1/f$ *is convex.*

Proof. For $a, b \in I$, we have

$$2f(a)f(b) \le f^2(a) + f^2(b)$$

$$\Rightarrow \; 4f(a)f(b) \le \big(f(a) + f(b)\big)^2$$

$$\Rightarrow \frac{2f(a)f(b)}{f(a)+f(b)} \leq \frac{f(a)+f(b)}{2} \leq f\left(\frac{a+b}{2}\right)$$

$$\Rightarrow (1/f)\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2f(a)f(b)} = \frac{(1/f)(a)+(1/f)(b)}{2}.$$

Theorem 3.6. *Let* $f : I \subset \Re \to \Re$ *be positive convex such that* f^{-1} *exist. Then* f^{-1} *is concave. If* f *is concave, then* f^{-1} *is convex.*

Proof. We have for $a, b \in I$,

$$\frac{a+b}{2} = \frac{f\left(f^{-1}(a)\right)+f\left(f^{-1}(b)\right)}{2}$$

$$\geq f\left(\frac{f^{-1}(a)+f^{-1}(b)}{2}\right),$$

which implies

$$f^{-1}\left(\frac{a+b}{2}\right) \geq (f^{-1}f)\left(\frac{f^{-1}(a)+f^{-1}(b)}{2}\right) = \frac{f^{-1}(a)+f^{-1}(b)}{2}.$$

The proof of the other part is similar.

Theorem 3.7. *Let* $f, g : I \subset \Re \to \Re$ *be positive convex functions such that If for all* $a, b \in I$,

$$\frac{f^2(a)+f^2(b)+g^2(a)+g^2(b)}{4} \leq f(a)g(a)+f(b)g(b) + \left(f(a)-g(b)\right)\left(g(a)-f(b)\right) \qquad (13)$$

is satisfied, then fg *is convex. If both* f *and* g *are concave and* (13) *reverses, then* fg *is concave .*

Proof. We have, by (13), via simple application,

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{f^2\left(\frac{a+b}{2}\right)+g^2\left(\frac{a+b}{2}\right)}{2}$$

$$\leq \frac{\left(f(a)+f(b)\right)^2+\left(g(a)+g(b)\right)^2}{8}$$

$$= \frac{f^2(a) + f^2(b) + g^2(a) + g^2(b)}{4} + \frac{f(a)f(b) + g(a)g(b)}{4}$$

$$\leq \frac{f(a)g(a) + f(b)g(b)}{2}$$

Corollary 3.8. *Let* $f, g : I \subset \Re \to \Re$ *be positive functions such that f is convex and g is concave, then f/g is convex, provided the following is satisfied for all* $a, b \in I$,

$$\left(f(a) - f(b)\right)\left(\frac{1}{g}(a) - \frac{1}{g}(b)\right) \geq 0, \tag{14}$$

Proof. As g is concave, then by Theorem 3.5, $1/g$ is convex. The result follows by Theorem 3.1 via (14) .

Theorem 2.9. (a). *If* f *is convex and* g *is concave, then f-g is convex.*
(b). *If* f *is concave and* g *is convex, then f-g is concave.*

Proof. (a).

$$(f-g)\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right) - g\left(\frac{a+b}{2}\right)$$

$$\leq \frac{f(a) + f(b)}{2} - \frac{g(a) + g(b)}{2}$$

$$= \frac{(f-g)(a) + (f-g)(b)}{2}.$$

Theorem 3.10. *Let* $f, g : I \subset \Re \to \Re$ *be positive convex functions . Let* $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$. *If for all* $a, b \in I$,

$$\frac{\left(f(a) + f(b)\right)^p}{p\,2^p} + \frac{\left(g(a) + g(b)\right)^q}{q 2^q} \leq \frac{f(a)g(a) + f(b)g(b)}{2}, \tag{15}$$

then fg is convex. If f *and* g *are concave such that* $0 < p < 1$, *and (15) reversed, then fg is concave.*

Proof.

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\leq\frac{1}{p}f^{p}\left(\frac{a+b}{2}\right)+\frac{1}{q}g^{q}\left(\frac{a+b}{2}\right)$$

$$\leq\frac{1}{p}\left(\frac{f(a)+f(b)}{2}\right)^{p}+\frac{1}{q}\left(\frac{g(a)+g(b)}{2}\right)^{q}$$

$$\leq\frac{f(a)g(a)+f(b)g(b)}{2}.$$

Theorem 3.11. *Let* $f, g: I\subset\Re\to\Re$ *be positive functions such that* f *convex and* g *concave and for all* $a,b\in I$,

$$\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\geq g\left(\frac{a+b}{2}\right)-\frac{g(a)+g(b)}{2}, \tag{16}$$

then $f+g$ *is convex.*

Proof. We have

$$\frac{f(a)+f(b)}{2}+\frac{g(a)+g(b)}{2}-f\left(\frac{a+b}{2}\right)-g\left(\frac{a+b}{2}\right)$$

$$=\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)+\frac{g(a)+g(b)}{2}-g\left(\frac{a+b}{2}\right)$$

$$\geq 0.$$

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