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BIFURCATION AND CHAOS OF A PARTICLE MOTION SYSTEM WITH HOLONOMIC CONSTRAINT

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Abstract. This paper investigates a holonomic constrained system of a particle moving on a horizontal smooth plane. The equilibrium points, bifurcations and chaotic attractors of the system are analyzed. It shows that the rich dynamic behaviors of the particle motion system, including the degenerate Hopf bifurcations at multiple equilibrium points, the chaotic behaviors of the particle motion. The numerical simulations are carried out to verify theoretical analyses and to exhibit the rich chaotic behaviors.

Keywords: bifurcation; particle motion; chaotic attractor.

2010 AMS Subject Classification: 70K50, 34C23.

1. Introduction

Some research shows that the the particle motion becomes complex because of the existence of external force, such as shear force [1] and creep force [2]. Junhong and his cooperators studied a particle motion under external force and investigated the influences of two nonlinear

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nonholonomic constraints on the particle motion [3]. In this paper, we will discuss the particle motion system of Ref. [3] with holonomic constraint

$x^2+y^2=M^2$ $(M>0)$.

By applying Lagrange's method, the equations of holonomic system can be obtained as follows

$$\begin{cases} \ddot{x}=ax-bx^2-cx_1y+2\lambda x, \\ \ddot{y}=ay-by^2-cxy_1+2\lambda y, \end{cases} \tag{1}$$

Here λ is Lagrange's multiplier. Differentiate the holonomic constraint equation twice with respect to time and obtain $\dot{x}^2+x\ddot{x}+\dot{y}^2+y\ddot{y}=0$, then $\lambda=-\frac{x_1^2+y_1^2+aM^2-bx^3-by^3-cx_1xy-cxyy_1}{2M^2}$. Thus, the equations of motion of the constrained system become

$$\begin{cases} \dot{x}=x_1, \\ \dot{x}_1=-bx_1{}^2-cx_2x_3+ax_1-P_1(x,x_1,y,y_1), \\ \dot{y}=y_1, \\ \dot{y}_1=-bx_3{}^2-cx_1x_4+ax_3-P_2(x,x_1,y,y_1). \end{cases} \tag{2}$$

where

$P_1(x,x_1,y,y_1)=\frac{x_1\left(\left(-bx_1{}^2-cx_2x_3+ax_1\right)x_1+\left(-bx_3{}^2-cx_1x_4+ax_3\right)x_3+x_2{}^2+x_4{}^2\right)}{M^2}$,
$P_2(x,x_1,y,y_1)=\frac{x_3\left(\left(-bx_1{}^2-cx_2x_3+ax_1\right)x_1+\left(-bx_3{}^2-cx_1x_4+ax_3\right)x_3+x_2{}^2+x_4{}^2\right)}{M^2}$.

The stability of equilibrium points, Hopf bifurcation and chaos of the constrained system are investigated as follows.

2. Dynamic analysis

By computations, we can obtain the equilibrium points as follows

$E_0=(0,0,0,0)$, $E_1=(0,0,M,0)$, $E_2=(0,0,-M,0)$, $E_3=(0,0,\frac{a}{b},0)$,
$E_4=(M,0,0,0)$, $E_5=(-M,0,0,0)$, $E_6=(\frac{a}{b},0,0,0)$, $E_7=(\frac{a}{b},0,\frac{a}{b},0)$,
$E_8=(-\frac{M}{\sqrt{2}},0,-\frac{M}{\sqrt{2}},0)$, $E_9=(\frac{M}{\sqrt{2}},0,\frac{M}{\sqrt{2}},0)$.

The characteristic equations at equilibrium points are obtained as follows

$f_{E_0}(\lambda)=\lambda^4-a\lambda^2-\lambda^3+a\lambda$,

$f_{E_1}(\lambda)=\lambda^4+(cM-1)\lambda^3+(-bM-cM)\lambda^2+bM\lambda$,

$f_{E_2}(\lambda)=\lambda^4+(-cM-1)\lambda^3+(bM+cM)\lambda^2-bM\lambda$,

$$f_{E_3}(\lambda) = \lambda^4 + \tfrac{(ca-b)\lambda^3}{b} - \tfrac{a(b+c)\lambda^2}{b} + a\lambda,$$
$$f_{E_4}(\lambda) = \lambda^4 + (cM-1)\lambda^3 + (-2bM - cM + 2a)\lambda^2 + \left(-2M^2bc + 2Mac + 2bM - 2a\right)\lambda$$
$$+2cM(bM-a),$$
$$f_{E_5}(\lambda) = \lambda^4 + (-cM-1)\lambda^3 + (2bM + cM + 2a)\lambda^2 + \left(-2M^2bc - 2Mac - 2bM - 2a\right)\lambda$$
$$+2cM(bM+a),$$
$$f_{E_6}(\lambda) = \lambda^4 + \tfrac{(ca-b)\lambda^3}{b} + \tfrac{a\left(b^2M^2 - M^2bc - a^2\right)\lambda^2}{b^2M^2} + \tfrac{a\left(b^2M^2 - a^2\right)(ca-b)\lambda}{b^3M^2} - \tfrac{ca^2\left(b^2M^2 - a^2\right)}{b^3M^2},$$
$$f_{E_7}(\lambda) = \lambda^4 - \tfrac{(ca+b)\lambda^3}{b} - \tfrac{a(3b^2M^2 - M^2bc - 7a^2)\lambda^2}{b^2M^2} + \tfrac{a(3b^2M^2 - 7a^2)(ca+b)\lambda}{b^3M^2} - \tfrac{ca^2(3b^2M^2 - 7a^2)}{b^3M^2}.$$
$$f_{E_8}(\lambda) = \lambda^4 + (cM\sqrt{2}/2 - 1)\lambda^3 + (-cM\sqrt{2}/2 - bM\sqrt{2}/4 + a)\lambda^2 + (-M^2bc/2$$
$$+M\sqrt{2}ac/2 + bM\sqrt{2}/4 - a)\lambda + M^2bc/2 - 1/2M\sqrt{2}ac,$$
$$f_{E_9}(\lambda) = \lambda^4 + (-cM\sqrt{2}/2 - 1)\lambda^3 + (cM\sqrt{2}/2 + bM\sqrt{2}/4 + a)\lambda^2 + (-M^2bc/2$$
$$-M\sqrt{2}ac/2 - bM\sqrt{2}/4 - a)\lambda + M^2bc/2 + M\sqrt{2}ac/2.$$

From the expression of $f_{E_6}(\lambda)$, we can obtain the eigenvalues are $\pm\frac{\sqrt{ab^2M^2-a^3}}{bM}i$, 1, $\frac{ac}{b}$, where $\frac{a}{Mb} < 1$. In this case, the system occurs Hopf bifurcation at the equilibrium E_6. For E_6, let

$$\begin{pmatrix} x - \frac{a}{b} \\ x_1 \\ y \\ y_1 \end{pmatrix} = \begin{bmatrix} 0 & \frac{bM}{\sqrt{ab^2M^2-a^3}} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{ca+b}{ba} \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}.$$

Then, the system (2) becomes

$$(3)\qquad \begin{cases} \dot{u}_1 = -\frac{\sqrt{ab^2M^2-a^3}}{bM}u_2 + G_1(u_1,u_2,u_3,u_4), \\ \dot{u}_2 = \frac{\sqrt{ab^2M^2-a^3}}{bM}u_1 + G_2(u_1,u_2,u_3,u_4), \\ \dot{u}_3 = -cMu_3 + G_3(u_1,u_2,u_3,u_4), \\ \dot{u}_4 = u_4 + U_4(u_1,u_2,u_3,u_4), \end{cases}$$

where

$$G_1(u_1,u_2,u_3,u_4) = -\tfrac{b^3M^2{u_2}^2}{M^2ab^2-a^3} - \tfrac{cu_1(ac+b)u_4}{ba} + \tfrac{abMu_2}{\sqrt{M^2ab^2-a^3}} - \tfrac{bu_2}{M\sqrt{M^2ab^2-a^3}}\Big(\tfrac{bMu_2}{\sqrt{M^2ab^2-a^3}}$$
$$\times\Big(-\tfrac{b^3M^2u_2^2}{M^2ab^2-a^3} - \tfrac{cu_1(ac+b)u_4}{ba} + \tfrac{abMu_2}{\sqrt{M^2ab^2-a^3}}\Big) + \tfrac{(ac+b)u_4}{ba}\Big(-\tfrac{(ac+b)^2{u_4}^2}{ba^2} - \tfrac{cbMu_2(u_3+u_4)}{\sqrt{M^2ab^2-a^3}}$$
$$+\tfrac{(ac+b)u_4}{b} + u_1^2 + (u_3+u_4)^2\Big) + \tfrac{\sqrt{M^2ab^2-a^3}u_2}{bM},$$

$$G_2(u_1,u_2,u_3,u_4)=0,$$

$$G_3(u_1,u_2,u_3,u_4)=cMu_3-\frac{(ac+b)^2{u_4}^2}{ba^2}-\frac{cbMu_2(u_3+u_4)}{\sqrt{M^2ab^2-a^3}}+\frac{(ac+b)u_4}{b}-\frac{(ac+b)u_4}{abM^2}(\frac{bMu_2}{\sqrt{M^2ab^2-a^3}} \times(-\frac{b^3M^2{u_2}^2}{M^2ab^2-a^3}-\frac{cu_1(ac+b)u_4}{ba}+\frac{abMu_2}{\sqrt{M^2ab^2-a^3}})+\frac{(ac+b)u_4}{ba}(-\frac{(ac+b)^2{u_4}^2}{ba^2}-\frac{cbMu_2(u_3+u_4)}{\sqrt{M^2ab^2-a^3}} +\frac{(ac+b)u_4}{b})+u_1^2+(u_3+u_4)^2),$$

$$G_4(u_1,u_2,u_3,u_4)=0.$$

Furthermore,

$$g_{11}=\frac{1}{4}(-\frac{2b^3M^2}{M^2ab^2-a^3}),\ g_{02}=\frac{1}{4}(\frac{2b^3M^2}{M^2ab^2-a^3}),$$

$$g_{20}=\frac{1}{4}(\frac{2b^3M^2}{M^2ab^2-a^3}),\quad g_{21}=\frac{1}{8}(\frac{6b^3Ma}{\left(M^2ab^2-a^3\right)^{3/2}}+\frac{2b}{M\sqrt{M^2ab^2-a^3}}i),$$

thus

$$\mathrm{Re}c_1(0)=\mathrm{Re}(\frac{Mbi}{\sqrt{M^2ab^2-a^3}}(g_{20}g_{11}-2|g_{11}|^2-\frac{1}{3}|g_{02}|^2)+\frac{1}{2}g_{21})=\frac{1}{8}\frac{b^2}{M^2ab^2-a^3}>0.$$

Based on the above analysis and the theorem in [4], we can obtain the the conclusion as follows.

Theorem 1. The system (2) undergoes degenerate Hopf bifurcations at E_6 and the bifurcating periodic solution is unstable.

According to Routh-Hurwitz criteria, E_0, E_1, E_5, E_7 and E_9 are unstable points. For the other equilibria, we can obtain the same conclusions using the method in [4] when the Hopf bifurcations occur.

3. Chaos and simulations

The above results show that the particle motion system has complex dynamic behaviors. In this section, we give some simulations to study the particle motion and select the parameters $a=5$, $c=0.001$, $b=1$, and initial values $(x_0,x_{10},y_0,y_{10})=(0.1,0.001,0.1,0.1)$. In this case, the system (2) has ten equilibrium points. Table.1 indicates the eigenvalues of corresponding Jacobian matrix and the equilibria type and shows the unstable manifold and stable manifold at the equilibrium points of the particle motion system when $M=0.2$.

In chaos theory, the equilibrium points of the system are of great importance to understand its nonlinear dynamics [5]. It has been long supposed that the existence of chaotic behaviour in the microscopic motions is responsible for their equilibrium and nonequilibrium properties

[6] and the interconversion of the stable manifolds and the unstable manifolds which can cause complicated dynamics in the system (2)[7-8]. The Lyapunov exponents are 0, -0.2 -0.3 and 0.6 using the method in [7], thus the system (2) is chaotic. Fig 1. shows the particle motion trajectory and the chaotic attractor of the system. The poincare maps and chaotic attractors in $x-y$ plane and x_1-y_1 plane are given in Fig. 1-6. If the constrained parameter $M=1$, the particlemotion system also occur chaotic phenomena. Fig. 7-8 show the chaotic ayyractor in $x-y$ plane and $x-x_1$ plane.

TABLE 1. The eigenvalues of corresponding Jacobian matrix and the equilibria type.

equilibrium points	eigenvalues of Jacobian matrix	equilibria type
(0,0,0,0)	0, 1,2.2361, -2.2361	unstable equilibrium point
(0,0,0.2,0)	0, 1, 0.44731, -0.44731	unstable equilibrium point
(0,0,-0.2,0)	0, 1, ±0.44721i	Hopf bifurcation
(0,0,5,0)	0, 1, -2.23857, 2.23357	unstable equilibrium point
(0.2,0,0,0)	1, 0.0024, 0.0612±3.9763i	unstable equilibrium point
(-0.2,0,0,0)	1, 0.0026, -0.0637±3.22442i	unstable equilibrium point
(5,0,0,0)	1, -0.005, 55.8569, -55.8569	unstable equilibrium point
(5,0, 5,0)	1, 59.069, -0.01, -52.819	unstable equilibrium point
$(-\frac{0.2}{\sqrt{2}}$,0,-$\frac{0.2}{\sqrt{2}}$,0)	1, 0.0001,±2.25182i	Hopf bifurcation
($\frac{0.2}{\sqrt{2}}$,0,$\frac{0.2}{\sqrt{2}}$,0)	1, -0.000139, 0.0000001±2.2201i	unstable equilibrium point

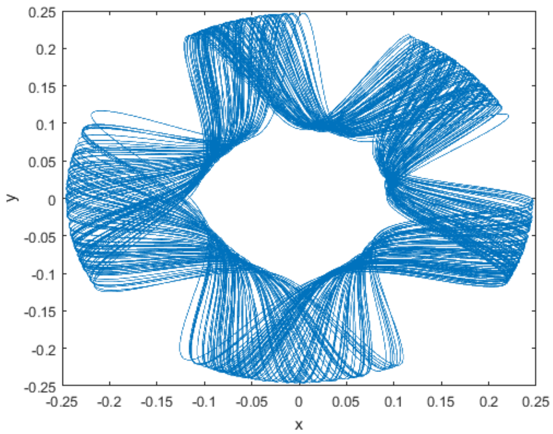

Fig. 1. The chaotic attractor in $x-y$ plane.

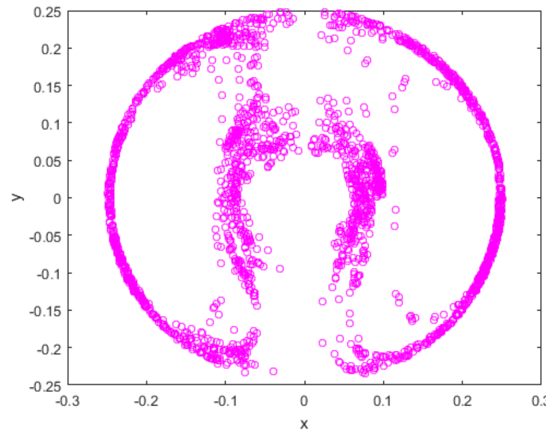

Fig. 2. The poincare map in $x-y$ plane.

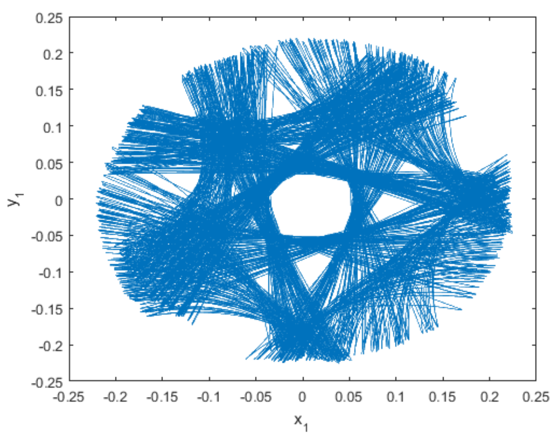

Fig. 3. The chaotic attractor in x_1-y_1 plane.

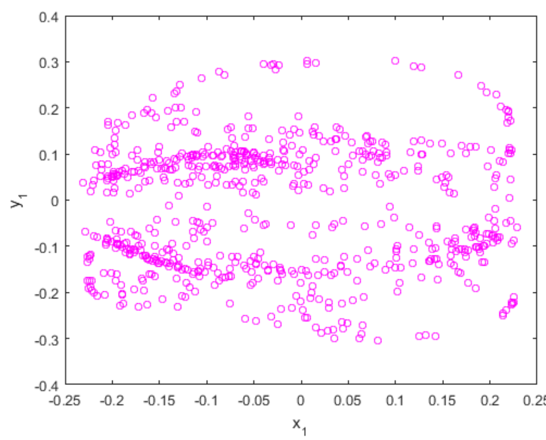

Fig. 4. The poincare map in x_1-y_1 plane.

4. Conclusion

The results show that the rich dynamic behaviors of the particle motion system, including Hopf bifurcations, interconversion of the stable manifolds and the unstable manifolds at multiple equilibrium points and chaotic attractors. Thus, the particle motion trajectory has complex dynamic behaviors under the holonomic constraint.

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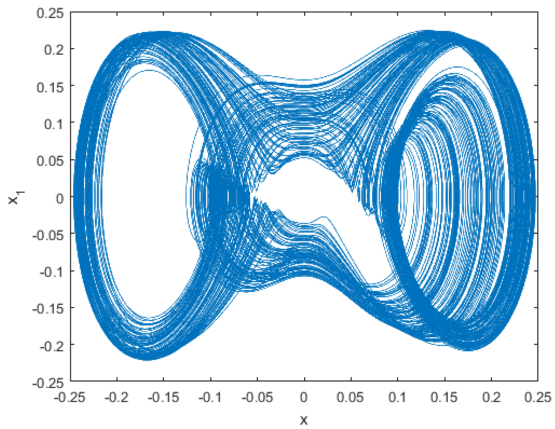

Fig. 5. The chaotic attractor in $x - x_1$ plane.

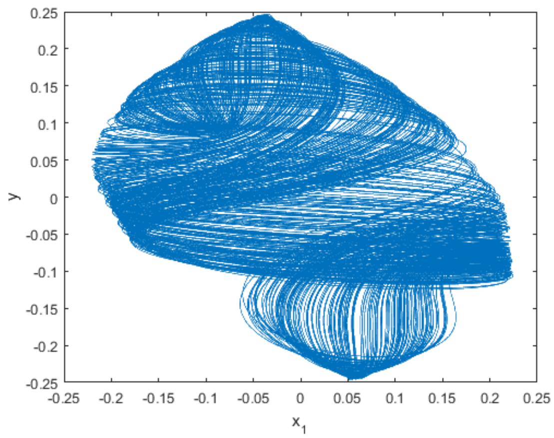

Fig. 6. The chaotic attractor in $x_1 - y$ plane.

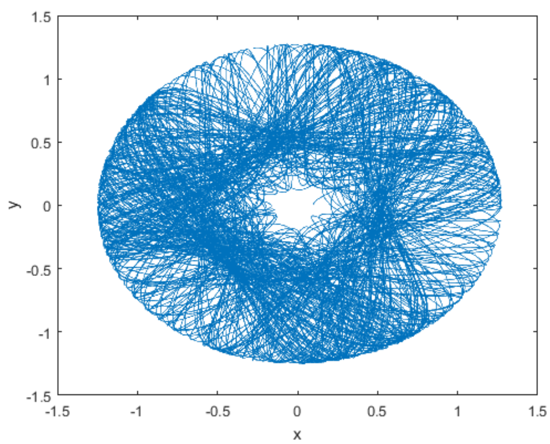

Fig. 7. The chaotic attractor in $x - y$ plane.

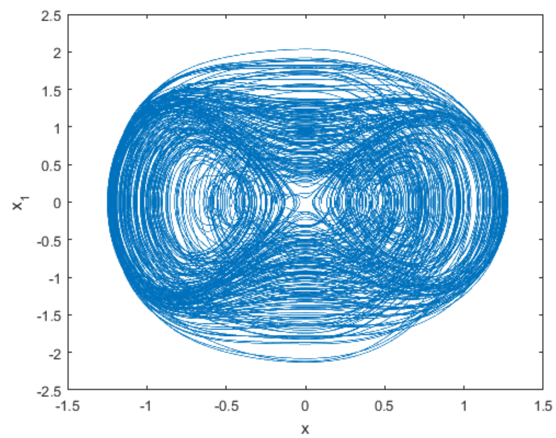

Fig. 8. The chaotic attractor in $x - x_1$ plane.

Conflict of Interests

The authors declare that there is no conflict of interests.

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