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ITERATIVE SEQUENCES WITH ERRORS FOR GENERALIZED CONTRACTIVE MAPPINGS IN METRIC SPACES

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Abstract: Many mathematicians studied the convergence theorem of iterative sequence with errors in Banach space and in Hilbert space. In 2004, Liu Qihou defined the iterative sequence with errors in metric spaces. In this paper, the article mainly discusses the convergence of iterative sequence with errors for extended contractive mappings in metric space. The result improves and extends the known results.

Keywords: iterative sequence with errors; contractive mappings; metric space.

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1. Introduction

Definition: let X be a set and T is a self-mapping, if for any $a \in X$, such that Ta=a, then a be called a fixed point.

Suppose $x_1 \in X$, $x_{n+1} = Tx_n$, ($\forall n \in N$) is Picard iterative sequence and it is the most primal iterative sequence.

In 1953, W.R Mann [1] and in 1974, S. Ishikawa [2] put the following two iterative sequences:

Let X be a linear space, M is a subset on X, T is a self-mapping on M

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(1). Let $x_1 \in M$, iterative sequence

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \forall n \in N, \tag{1}$$

(where 0<= α_n <=1), is called Mann iterative sequence.

(2). Let $x_1 \in M$, iterative sequence

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n,$$

$$y_n = (1-\beta_n)x_n + \beta_n T x_n, \forall n \in N, \tag{2}$$

(Where $0 \le \alpha_n \le 1, 0 \le \beta_n \le 1$) is called Ishikawa iterative sequence.

It is well known that the consideration of errors terms in any approximate method is an important part of the method. In 1995, Lishan Liu [3] introduce what he called Ishikawa and Mann iterative sequence with error terms to the formulas (1) and (2)

(3). For a nonempty subset K of a Banach space X and a mapping T: $K \to K$,the sequence $\{x_n\}$ on K is defined by

$$x_0 \in K$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n$$

$$y_n = (1-\beta_n)x_n + \beta_n T x_n + v_n, \qquad n \ge 0$$

(where $\{u_n\}$ and $\{v_n\}$ are two summable sequence in X, i.e $\sum_{n=0}^{x} \|u_n\| < \infty$, $\sum_{n=0}^{x} \|v_n\| < \infty$, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequence in [0,1] satisfying certain restriction) is called Ishikawa iterative sequence with errors.

The conditions he placed on the error terms imply that they go to zero as n goes to infinity. In 1991, Yugang Xu [4] introduced the following more satisfactory definitions:

(4) Let K a nonempty convex subset of a normed linear space E, and a mapping T: $K \to K$. For any given $x_1 \in K$ the sequence $\{x_n\}$ on K is defined iteratively by

$$x_{n+1} = a_n x_n + b_n T y_n + c_n u_n$$

$$y_n = a_n{}' x_n + b_n{}' T x_n + c_n{}' u_n \qquad n \geq 0$$

Where $\{u_n\}$ $\{v_n\}$ are bounded sequences in K and $\{a_n\}$ $\{b_n\}$ $\{c_n\}$ $\{a_n{}'\}$ $\{b_n{}'\}$ $\{c_n{}'\}$ Are sequences in [0,1] such that $a_n + b_n + c_n = a_n{}' + b_n{}' + c_n{}' = 1, \forall n \geq 1$ is called the Ishikawa iterative sequence with errors.

(5) If, with the same notations and definitions as in part 4, $b_n{}' = c_n{}' = 0$ for all integers n>=1, then the resulting sequence is called the Mann iteration sequence with errors.

Many mathematicians have discusses the convergence of iterative sequence for mapping classes above in Banach space and in Hilbert sequence space. In 2004, Liu Qihou [5] introduced Ishikawa iterative sequence of the mapping in metric space.

Let K be a nonempty complete metric space E and T be a self-mapping of E, F(T) denotes the set of fixed point of T, and let F(T) is nonempty. For any given $x_1 \in K$, the sequence $\{x_n\}$ on K is defined iteratively by

$$d(x_{n+1}, p) = a_n d(x_n, p) + b_n d(T^n y_n, p) + c_n \varepsilon_n$$

$$d(y_n, p) = \overline{a_n} d(x_n, p) + \overline{b_n} (T^n x_n, p) + \overline{c_n}\,\overline{\varepsilon_n}$$

$\forall n \in N, \forall p \in F(T)$, $a_n + b_n + c_n = \overline{a_n} + \overline{b_n} + \overline{c_n} = 1, \forall n \geq 1$ where ε_n, $\overline{\varepsilon_n}$ is bounded and do not depend on p ($p \in F(T)$), $0 \leq a_n, b_n, c_n, \overline{a_n}, \overline{b_n}, \overline{c_n} \leq 1$,

$$\sum_{n=1}^{\infty} c_n < +\infty \qquad \sum_{n=1}^{\infty} \overline{c_n} < +\infty$$

As we know, the fixed point theory is an effective method to study solutions of some kind of equations, and iterative sequence is an important tool to find the mappings' fixed point. So it becomes the hot spot in the field of mathematics in recent years. In 1977, the founder of the fixed point theory-the university professor of American Indiana B.E Rhoades [6] induced 25 kinds of the basic contraction mappings in his thesis, and also extended these definitions. This paper mainly discusses the convergence of iterative sequence with errors about some contractive mappings in metric space.

2. Preliminaries

Definition the 43 type contractive mapping:

exist nonnegative number a_1, a_2, a_3, a_4, a_5, and $\sum_{i=1}^{5} a_i < 1$, and nonnegative integer p ,

such that $\forall x, y \in X$,

$$d(T^p x, T^p y) \le a_1 d(x, y) + a_2 d(x, T^p x) + a_3 d(y, T^p y) + a_4 d(x, T^p y) + a_5 d(y, T^p x)$$

Definition the 18 type contractive mapping:

exist nonnegative number a_1, a_2, a_3, a_4, a_5, and $\sum_{i=1}^{5} a_i < 1$, such that $x, y \in X$

$$d(Tx, Ty) \le a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx)$$

Definition the 49 type contractive mapping:

exist $h \in (0,1)$.and nonnegative integer p, such that for any $x, y \in X$:,

$$d(T^p x, T^p y) \le h \max\{d(x, y), d(x, T^p x), d(y, T^p y), d(x, T^p y), d(y, T^p x)\}$$

Definition the 24 type contractive mapping:

exist $h \in (0,1)$.such that for any $x, y \in X$:

$$d(Tx, Ty) \le h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

Lemma (see the lemma on P302, Liu Qihou [9])

Let sequence $\{x_n\}_{n=1}^{\infty}$ satisfy $x_{n+1} \le \alpha x_n + \beta_n$, where $x_n \ge 0$, $x_n \ge 0$ and $\lim_{n\to\infty} \beta_n = 0$, $0 \le \alpha < 1$, then we have $\lim_{n\to\infty} x_n = 0$

3. Main results

Theorem 1 :Let X be a nonempty complete metric space，T is contractive mapping on X .satisfied the condition of the 43 type contractive mapping, then there must exist an unique fixed point on X and for some $x_1 \in X$,if the sequence $\{x_n\}_{n=1}^{\infty}$ satisfy $d(x_{n+1}, T^p x_n) \le \varepsilon_n \ \forall n \in N$ and $\varepsilon_n \ge 0, 且 \lim_{n\to\infty} \varepsilon_n = 0$ then we have $\{x_n\}_{n=1}^{\infty}$ converge to the fixed point of T .

Theorem 2: Let X be a nonempty complete metric space，T is contractive mapping on X .satisfied the condition of the 18 type contractive mapping, then there must exist an unique fixed point on X and for some $x_1 \in X$,if the sequence $\{x_n\}_{n=1}^{\infty}$ satisfy $d(x_{n+1}, Tx_n) \le \varepsilon_n$, $\forall n \in N$ and $\varepsilon_n \ge 0, 且 \lim_{n\to\infty} \varepsilon_n = 0$, then we have $\{x_n\}_{n=1}^{\infty}$ converge to the fixed point of T .

Theorem 3: Let X be a nonempty complete metric space，T is contractive mapping on X .satisfied exist nonnegative number a_1, a_2, a_3, and nonnegative integer p, and $\sum_{i=1}^{3} a_i < 1$, such that $d(Tx, Ty) \le ad(x, y) + bd(x, T^p x) + cd(y, T^p y)$, then there must exist an unique fixed point on X and for some $x_1 \in X$, if the sequence $\{x_n\}_{n=1}^{\infty}$ satisfy $d(x_{n+1}, T^p x_n) \le \varepsilon_n$, $\forall n \in N$ and $\varepsilon_n \ge 0, 且 \lim_{n\to\infty} \varepsilon_n = 0$, then we have $\{x_n\}_{n=1}^{\infty}$ converge to the fixed point of T .

Theorem 4 :Let X be a nonempty complete metric space，T is contractive mapping on X .satisfied exist nonnegative number a_1, a_2, a_3, and $\sum_{i=1}^{3} a_i < 1$, such that

$d(Tx, Ty) \le ad(x, y) + bd(x, Tx) + cd(y, Ty)$. Then there must exist an unique fixed point on X and for some $x_1 \in X$,if the sequence $\{x_n\}_{n=1}^{\infty}$ satisfy $d(x_{n+1}, Tx_n) \le \varepsilon_n$, $\forall n \in N$ and $\varepsilon_n \ge 0, 且 \lim_{n\to\infty} \varepsilon_n = 0$, then we have $\{x_n\}_{n=1}^{\infty}$ converge to the fixed point of T .

Theorem 5: Let X be a nonempty complete metric space，T is contractive mapping on X .satisfied the condition of the 49 type contractive mapping. Then there must exist an unique fixed point on X and for some $x_1 \in X$, if the sequence $\{x_n\}_{n=1}^{\infty}$ satisfy

$d(x_n, x_{n+1}) \le \beta_n \; d(x_{n+1}, T^p x_n) \le \varepsilon_n$, $\forall n \in N$ and $\beta_n \ge 0$ $\varepsilon_n \ge 0,$ 且 $\lim\limits_{n\to\infty} \varepsilon_n = 0$

$\lim\limits_{n\to\infty} \beta_n = 0$, then we have $\{x_n\}_{n=1}^{\infty}$ converge to the fixed point of T.

Theorem 6: Let X be a nonempty complete metric space，T is contractive mapping on X .satisfied the condition of the 24 type contractive mapping，then there must exist an unique fixed point on X and for some $x_1 \in X$, if the sequence $\{x_n\}_{n=1}^{\infty}$ satisfy $d(x_n, x_{n+1}) \le \beta_n$ $d(x_{n+1}, Tx_n) \le \varepsilon$, $\forall n \in N$ and $\beta_n \ge 0$, $\varepsilon_n \ge 0,$ 且 $\lim\limits_{n\to\infty} \varepsilon_n = 0$, $\lim\limits_{n\to\infty} \beta_n = 0$, then we have $\{x_n\}_{n=1}^{\infty}$ converge to the fixed point of T

Theorem 7:Let X be a nonempty complete metric space，T is contractive mapping on X .satisfied $d(Tx, Ty) \le h\{d(x, Tx) + d(y, Ty)\}$ $h \in (0, \frac{1}{2})$, for some $x, y \in X$, then there must exist an unique fixed point on X and for some $x, y \in X$, if the sequence $\{x_n\}_{n=1}^{\infty}$ satisfy $d(x_{n+1}, Tx_n) \le \varepsilon_n$, $\forall n \in N$ and $\varepsilon_n \ge 0,$ 且 $\lim\limits_{n\to\infty} \varepsilon_n = 0$,then we have $\{x_n\}_{n=1}^{\infty}$ converge to the fixed point of T .

Theorem 8: Let X be a nonempty complete metric space, T is contractive mapping on X .satisfied $d(Tx, Ty) \le hd(x, y)$, $h \in (0,1)$, for some $x, y \in X$, then there must exist a unique fixed point on X and for some $x_1 \in X$, if the sequence $x_1 \in X$, satisfy $d(x_{n+1}, Tx_n) \le \varepsilon_n$, $\forall n \in N$ and $\varepsilon_n \ge 0,$ 且 $\lim\limits_{n\to\infty} \varepsilon_n = 0$, then we have $\{x_n\}_{n=1}^{\infty}$ converge to the fixed point of T .

Proof of theorem 1:

We have already known that if T satisfies the condition of the 43 type contractive mapping, then T must have an unique fixed point. Let this fixed point is x_* , then we also have $T^p x_* = x_*$

Because $d(x_{n+1}, x_*) \le d(x_{n+1}, T^p x_n) + d(T^p x_n, x_*)$

$$= d(x_{n+1}, T^p x_n) + d(T^p x_n, T^p x_*) \qquad (1)$$

According to the condition

$$d(T^p x, T^p y) \le a_1 d(x,y) + a_2 d(x, T^p x) + a_3 d(y, T^p y) + a_4 d(x, T^p y) + a_5 d(y, T^p x)$$

So we have

$$d(T^p x_n, T^p x_*) \le a_1 d(x_n, x_*) + a_2 d(x_n, T^p x_n) + a_3 d(x_*, Tx_*)$$
$$+ a_4 d(x_n, Tx_*) + a_5 d(x_*, T^p x_n)$$

$$= a_1 d(x_n, x_*) + a_2 d(x_n, T^p x_n) + a_4 d(x_n, Tx_*) + a_5 d(x_*, T^p x_n),$$

$$\le a_1 d(x_n, x_*) + a_2 d(x_n, x_*) + a_2 d(x_*, T^p x_n) + a_4 d(x_n, Tx_*) + a_5 d(x_*, T^p x_n)$$

$$= (a_1 + a_2 + a_4) d(x_n, Tx_*) + (a_2 + a_5) d(x_*, T^p x_n) \qquad (2)$$

$$d(T^p x_*, T^p x_n) \le a_1 d(x_*, x_n) + a_2 d(x_*, T^p x_*) + a_3 d(x_n, T^p x_n)$$
$$+ a_4 d(T^p x_*, x_n) + a_5 d(T^p x_n, x_*)$$

$$= \left(a_1 + a_3 + a_5\right) d\left(x_n, x_*\right) + \left(a_3 + a_4\right) d\left(x_*, T^p x_n\right) \qquad (3)$$

Add (3) to (2), we have that:

$$\left(2 - a_2 - a_3 - a_4 - a_5\right) d\left(x_*, T^p x_n\right) \le \left(2a_1 + a_2 + a_3 + a_4 + a_5\right) d\left(x_n, x_*\right)$$

$$d\left(x_*, T^p x_n\right) \le \frac{2a_1 + a_2 + a_3 + a_4 + a_5}{2 - a_2 - a_3 - a_4 - a_5} d\left(x_n, x_*\right)$$

$$= h\, d\left(x_n, x_*\right) \qquad (4)$$

And because $\sum_{i=1}^{5} a_i < 1$ so $2(a_1 + a_2 + a_3 + a_4 + a_5) \le 2$

So $0 < \frac{2a_1 + a_2 + a_3 + a_4 + a_5}{2 - a_2 - a_3 - a_4 - a_5} = h < 1$

So $d(x_{n+1}, x_*) = d(x_{n+1}, T^p x_n) + d(T^p x_n, T^p x_*)$

$$\le d(x_{n+1}, Tx_n) + h d(x_n, x_*)$$

Because $d\left(x_{n+1}, T^p x_n\right) \le \varepsilon_n$ for any $n \in N$

according to the lemma：

satisfy $d\left(x_n, x_*\right) \ge 0$ $\varepsilon_n \ge 0$ and $\lim_{n\to\infty} \beta_n = 0$, $0 < h < 1$

So we have $\lim_{n\to\infty} d\left(x_{n_i}, x_*\right) = 0$. It is that $\lim_{n\to\infty} x_n = x_*$

In the theorem 1, let p=1 then we can obtain the theorem 2.

Let $a_4 = a_5$ =0, and let a_1=a, a_2 =b, a_3=c, then we obtain the theorem 3

Let $a_4 = a_5$ =0, and let a_1=a, a_2 =b, a_3=c, and let p=1,then we obtain the theorem 4

Proof of theorem 5:

We also have already known that if T satisfies the condition, then T must have an unique fixed point. Let this fixed point is x_* , then we also have $T^p x_* = x_*$

Because $d(x_{n+1}, x_*) \le d(x_{n+1}, T^p x_n) + d(T^p x_n, x_*)$

$$= d(x_{n+1}, T^p x_n) + d(T^p x_n, T^p x_*) \qquad (1)$$

According to the condition

$$d(T^p x, T^p y) \le h \max\{d(x, y), d(x, T^p x), d(y, T^p y), d(x, T^p y), d(y, T^p x)\}$$

then we have:

$$d(T^p x_n, T^p x_*) \le h \max\{d(x_n, x_*), d(x_n, T^p x_n), d(x_*, T^p x_*), d(x_n, T^p x_*), d(x_*, T^p x_n)\}$$

$$= h \max\{d(x_n, x_*), d(x_n, T^p x_n), d(x_*, T^p x_n)\}$$

$$\le h \max\{d(x_n, x_*), d(x_n, T^p x_n)\}$$

$$\le hd(x_n, x_*) + hd(x_n, T^p x_n) \qquad (2)$$

So put (2) in (1), then

$$d(x_{n+1}, x_*) \le d(x_{n+1}, T^p x_n) + d(T^p x_n, T^p x_*)$$

$$\le d(x_{n+1}, T^p x_n) + hd(x_n, x_*) + hd(x_n, T^p x_n)$$

: $$\le d(x_{n+1}, T^p x_n) + hd(x_n, x_*) + hd(x_n, x_{n+1}) + hd(x_{n+1}, T^p x_n)$$

$$\le (1+h) d(x_{n+1}, T^p x_n) + h d(x_n, x_*) + h d(x_n, x_{n+1})$$

Because $d(x_n, x_{n+1}) \le \beta_n$ and $d(x_{n+1}, T^p x_n) \le \varepsilon_n$

So $d(x_{n+1}, x_*) \le (1+h)\varepsilon_n + hd(x_n, x_*) + h\beta_n$

And because $\beta_n \ge 0$, $\varepsilon_n \ge 0,$ 且 $\lim_{n\to\infty} \varepsilon_n = 0$, and $\lim_{n\to\infty} \beta_n = 0$

From the lemma, we have $\lim_{n\to\infty} d(x_{n+1}, x_*) = 0,$ i.e $\lim_{n\to\infty} x_n = x_*$

In the theorem 5, let p=1 then we can obtain the theorem 6

In the theorem 2, let $a_3 = a_4 = a_5$ =0, and let $a_1 = a_2$ =h, $0 \le h < 1$, then we have obtain the theorem 7

In the theorem 2, let $a_2 = a_3 = a_4 = a_5$ =0, and let a_1=h, $0 \le h < 1$, then we have obtain the theorem 8

When we let ε_n =0 in the above theorems, then we can obtain some corollaries[10];

Corollary 1:Let X be a nonempty complete metric space，T is contractive mapping on X .satisfied the condition of the 43 type contractive mapping, then (1) there must exist a unique fixed point $x_* \in X$, (2) for any $x \in X$, $\lim_{n\to\infty} T^n x = x_*$.

Corollary 2: Let X be a nonempty complete metric space，T is contractive mapping on X .satisfied the condition of the 18 type contractive mapping, then (1) there must exist an unique fixed point $x_* \in X$, (2) for any $x \in X$, $\lim_{n\to\infty} T^n x = x_*$.

Corollary 3: Let X be a nonempty complete metric space，T is contractive mapping on X .satisfied exist nonnegative number a_1, a_2, a_3, and nonnegative integer p, and $\sum_{i=1}^{3} a_i < 1$, such that $d(Tx,Ty) \le ad(x,y) + bd(x,T^p x) + cd(y,T^p y)$, then there must exist an unique fixed point $x_* \in X$, (2) for any $x \in X$, $\lim_{n\to\infty} T^n x = x_*$

Corollary 4: Let X be a nonempty complete metric space，T is contractive mapping on X .satisfied exist nonnegative number a_1, a_2, a_3, and $\sum_{i=1}^{3} a_i < 1$, such that

$d(Tx,Ty) \le ad(x,y) + bd(x,Tx) + cd(y,Ty)$. Then (1) there must exist an unique fixed point $x_* \in X$, (2) for any $x \in X$, $\lim_{n\to\infty} T^n x = x_*$

Corollary 5: Let X be a nonempty complete metric space，T is contractive mapping on X .satisfied the condition of the 49 type contractive mapping. Then (1) there must exist an unique fixed point $x_* \in X$, (2) for any $x \in X$, $\lim_{n\to\infty} T^n x = x_*$

Corollary 6: Let X be a nonempty complete metric space，T is contractive mapping on X .satisfied the condition of the 24 type contractive mapping，then (1) there must exist an unique fixed point $x_* \in X$, (2) for any $x \in X$, $\lim_{n\to\infty} T^n x = x_*$

Corollary 7:Let X be a nonempty complete metric space，T is contractive mapping on X .satisfied $d(Tx,Ty) \le h\{d(x,Tx)+d(y,Ty)\}$ $h \in (0,\frac{1}{2})$, for some $x, y \in X$, then (1)there must exist an unique fixed point $x_* \in X$, (2) for any $x \in X$, $\lim_{n\to\infty} T^n x = x_*$

Corollary 8: Let X be a nonempty complete metric space，T is contractive mapping on X .satisfied

$d(Tx,Ty) \le hd(x, y)$, $h \in (0,1)$, for some $x, y \in X$, then (1) there must exist a unique fixed point $x_* \in X$, (2) for any $x \in X$, $\lim_{n\to\infty} T^n x = x_*$

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