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FUZZY FIXED POINT RESULTS FOR INTERPOLATIVE KANNAN TYPE CONTRACTIONS

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Abstract. This paper generalizes the concept of interpolative Kannan type contractions from classical metric spaces to fuzzy metric spaces. We introduce a new class of fuzzy interpolative Kannan contractions and prove fixed point theorems under this framework. Our results extend and enrich the fuzzy fixed point theory by incorporating the interpolation method, thereby allowing wider applicability in uncertainty modeling.

Keywords: Fuzzy Metric Space, Interpolative Kannan Contraction, Fixed Point, Generalized Contractive Mapping.

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1. Introduction

Fixed point theory has become a cornerstone of nonlinear analysis due to its extensive applications across various fields, including optimization, control theory, differential equations, and computer science. Since Banach's Contraction Principle was introduced in 1922 [1], a multitude of generalizations and extensions have emerged to accommodate different structures and mappings, particularly in metric and topological spaces.

One such notable generalization is due to Kannan [3, 4], who in 1968 introduced a class of contractive mappings where the contraction condition is not based directly on the distance

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between the images of two points, but rather on the distances between each point and its image. This mapping, known as the Kannan contraction, has the remarkable feature of admitting fixed points even for discontinuous mappings, thereby addressing one of the limitations of Banach-type contractions.

Recently, the concept of interpolation has been incorporated into the study of fixed points, leading to refined versions of classical contractions such as those proposed by Karapınar [5]. By introducing interpolative parameters, these generalized contractions provide a more flexible and potentially more powerful framework for convergence analysis, especially when dealing with mappings that do not strictly adhere to standard contractive conditions.

Parallel to these developments, the theory of fuzzy metric spaces, as introduced by Kramosil and Michalek [7] and later refined by George and Veeramani [2], offers a robust mathematical framework for dealing with uncertainty and vagueness in metric concepts. Fuzzy metric spaces are particularly well-suited for applications involving imprecise data, such as decision-making systems, pattern recognition, and artificial intelligence.

In this paper, we bridge these two directions by introducing an interpolative Kannan-type contraction [5, 6] in the context of fuzzy metric spaces. Our goal is to extend the recent results on interpolative fixed point theorems from classical metric spaces to fuzzy environments. We prove that under suitable contractive conditions, such mappings admit unique fixed points in complete fuzzy metric spaces. The results not only generalize the classical Kannan theorem but also enhance the framework for studying fixed points in non-crisp settings.

2. PRELIMINARIES

Now, we begin with some basic definitions.

Definition 2.1. A binary operation $*\colon [0,1]\times[0,1]\to[0,1]$ is called a continuous $t-$norm if it satisfies the following conditions:

(i) $*$ is commutative and associative;

(ii) $*$ is continuous;

(iii) $a*1=a,\ \forall a\in[0,1]$;

(iv) $a*b\leq c*d$ whenever $a\leq c$ and $b\leq d$ $\forall a,b,c,d\in[0,1]$.

Example 2.2. $a*b = min\{a,b\}$ and $a*b = a.b$ are $t-$norms.

Kramosil and Michalek [7], introduced by the concept of fuzzy metric spaces as follows:

Definition 2.3. A fuzzy metric space is an ordered triple $(X,M,*)$ such that X is a (nonempty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0,\infty)$ satisfying the following conditions, for all $x,y,z \in X$, $s,t > 0$:

(i) $M(x,y,0) = 0$,
(ii) $M(x,y,t) = 1$, $\forall\, t > 0$ if and only if $x = y$,
(iii) $M(x,y,t) = M(y,x,t)$, (Symmetry)
(iv) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$,
(v) $M(x,y,.)\colon (0,\infty) \to (0,1]$ is continuous.

Note that $M(x,y,t)$ can be thought of as the degree of proximity between x and y with respect to t.

Example 2.4. Let (X,d) be a metric space. Define $a*b = a+b$ for all $a,b \in X$. Define $M(x,y,t) = \frac{t}{t+d(x,y)}$ for all $x,y \in X$and $t > 0$. Then $(X,M,*)$ is a fuzzy metric space and this fuzzy metric induced by a metric d is called the standard fuzzy metric.

Definition 2.5. Let $(X,M,*)$ be a fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to
 (i) be a Cauchy sequence if
 $\lim_{n\to\infty} M(x_{n+p},x_n,t) = 1$, $\forall t > 0$ and $n,p \in \mathbb{N}$,
 (ii) be convergent to a point $x \in X$
 if $\lim_{n\to\infty} M(x_n,x,t) = 1$, $\forall t > 0$.
(b) X is said to be complete if every Cauchy sequence in X converges to some point in X.

Example 2.6. Let $X = [0,1]$ and let $*$ be the continuous $t-$norm defined by $a*b = ab$, $\forall a,b \in [0,1]$. For each $t \in (0.\infty)$ and $x,y \in X$, define

$$M(x,y,t) = \begin{cases} \frac{t}{t+|x-y|^2}, & if\ t > 0 \\ 0, & if\ t = 0. \end{cases}$$

Clearly, $(X,M,*)$ is a complete fuzzy metric space.

Theorem 2.7. *Let $(X,M,*)$ be a complete fuzzy metric space. A mapping $T: X \to X$ is said to be a fuzzy Kannan type contraction in X if there exist $k \in [0,1)$ and $t > 0$ such that*

$$M(Tx,Ty,t) > k[M(x,Tx,t) * M(y,Ty,t)].$$

for every $x,y \in X \backslash fix(T)$, where $fix(T) = \{x \in X : Tx = x\}$.

3. Main Results

We start this section with the definition of a fuzzy interpolative Kannan contraction.

Definition 3.1. Let $(X,M,*)$ be a complete fuzzy metric space. A mapping $T: X \to X$ is said to be a *fuzzy interpolative Kannan-type contraction* on X if there exist constants $k \in [0,1)$, $\gamma \in (0,1)$, and $t > 0$ such that

$$M(Tx,Ty,t) > k\left[M(x,Tx,t)^{\gamma} * M(y,Ty,t)^{1-\gamma}\right],$$

for all $x,y \in X \backslash \mathrm{Fix}(T)$, where $\mathrm{Fix}(T) = \{x \in X : Tx = x\}$.

Theorem 3.2. *Let $(X,M,*)$ be a complete fuzzy metric space. If $T: X \to X$ is a fuzzy interpolative Kannan-type contraction, then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Then,

$$x_1 = Tx_0, \quad x_2 = Tx_1, \quad \ldots, \quad x_n = Tx_{n-1}.$$

By Definition 3.1, we have

$$\begin{aligned} M(x_{n+1},x_n,t) &= M(Tx_n,Tx_{n-1},t) \\ &> k\left[M(x_n,Tx_n,t)^{\gamma} * M(x_{n-1},Tx_{n-1},t)^{1-\gamma}\right] \\ &= k\left[M(x_n,x_{n+1},t)^{\gamma} * M(x_{n-1},x_n,t)^{1-\gamma}\right]. \end{aligned}$$

This implies:

$$[M(x_{n+1},x_n,t)]^{1-\gamma} > k[M(x_{n-1},x_n,t)]^{1-\gamma},$$

which leads to:

$$M(x_{n+1},x_n,t) > k^{\frac{1}{1-\gamma}} M(x_{n-1},x_n,t). \tag{3.1}$$

Hence, the sequence $\{M(x_{n-1},x_n,t)\}$ is strictly increasing and bounded above by 1. Therefore, there exists a limit $L \in [0,1]$ such that

$$\lim_{n\to\infty} M(x_{n-1},x_n,t) = L.$$

We now prove that $L = 1$. From (3.1), we iterate:

$$M(x_{n+1},x_n,t) > k^{\frac{1}{1-\gamma}} M(x_{n-1},x_n,t) > \cdots > \left(k^{\frac{1}{1-\gamma}}\right)^n M(x_1,x_0,t).$$

Letting $n \to \infty$ and noting that $0 < k < 1$, we have $\left(k^{\frac{1}{1-\gamma}}\right)^n \to 0$, implying that $M(x_{n+1},x_n,t) \to 1$. Hence, $L = 1$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. In a fuzzy metric space, the triangle inequality yields:

$$M(x_n,x_{n+r},t) \geq M(x_n,x_{n+1},\tfrac{t}{r}) * M(x_{n+1},x_{n+2},\tfrac{t}{r}) * \cdots * M(x_{n+r-1},x_{n+r},\tfrac{t}{r}).$$

As $n \to \infty$, each term in the product tends to 1, hence

$$\lim_{n\to\infty} M(x_n,x_{n+r},t) = 1, \quad \text{uniformly in } r.$$

Thus, $\{x_n\}$ is a Cauchy sequence in the complete fuzzy metric space $(X,M,*)$, and hence there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

To show that x^* is a fixed point, note that

$$x_{n+1} = Tx_n \to x^*, \quad x_n \to x^*.$$

Using the continuity of M and the uniqueness of the limit, we get $Tx^* = x^*$.

For uniqueness, suppose there exists another fixed point $y^* \neq x^*$. Then by Definition 3.1,

$$\begin{aligned} M(x^*,y^*,t) &= M(Tx^*,Ty^*,t) \\ &> k\left[M(x^*,Tx^*,t)^{\gamma} * M(y^*,Ty^*,t)^{1-\gamma}\right] \\ &= k(1^{\gamma} * 1^{1-\gamma}) = k. \end{aligned}$$

But since $M(x^*,y^*,t) \to 0$ as $t \to 0$ when $x^* \neq y^*$, this contradicts the inequality $M(x^*,y^*,t) > k > 0$. Thus, $x^* = y^*$ and the fixed point is unique. □

4. Numerical Example

Example 4.1. Let $X = [0,1]$ and define the fuzzy metric $M\colon X \times X \times (0,\infty) \to [0,1]$ by

$$M(x,y,t) = \frac{t}{t + |x-y|},$$

for all $x, y \in X$ and $t > 0$. This is a well-known fuzzy metric space with continuous t-norm defined as the usual product:

$$a * b = ab.$$

Now define the mapping $T\colon X \to X$ by

$$T(x) = \frac{1}{4}x + \frac{1}{4}.$$

We claim that T is a fuzzy interpolative Kannan-type contraction on X.

Let $x, y \in X \setminus \mathrm{Fix}(T)$ and fix $t > 0$. Then:

$$|T(x) - T(y)| = \left| \frac{1}{4}x + \frac{1}{4} - \left(\frac{1}{4}y + \frac{1}{4}\right) \right| = \frac{1}{4}|x-y|.$$

So,

$$M(Tx,Ty,t) = \frac{t}{t + |T(x) - T(y)|} = \frac{t}{t + \frac{1}{4}|x-y|}.$$

Also,

$$|x - Tx| = \left| x - \left(\frac{1}{4}x + \frac{1}{4}\right) \right| = \left| \frac{3}{4}x - \frac{1}{4} \right| \le \frac{3}{4} + \frac{1}{4} = 1,$$

and similarly,

$$M(x,Tx,t) = \frac{t}{t + |x - Tx|}, \quad M(y,Ty,t) = \frac{t}{t + |y - Ty|}.$$

Now choose $\gamma = \frac{1}{2}$, and define

$$k = \frac{t}{t + \frac{1}{4}} < 1, \quad (\text{since } t > 0).$$

We will verify:

$$M(Tx,Ty,t) > k \left[M(x,Tx,t)^{1/2} * M(y,Ty,t)^{1/2} \right].$$

Let us estimate both sides numerically. For instance, take $x = 0.8, y = 0.2, t = 1$:

$$Tx = \frac{1}{4}(0.8) + \frac{1}{4} = 0.45,$$
$$Ty = \frac{1}{4}(0.2) + \frac{1}{4} = 0.3,$$
$$M(Tx, Ty, 1) = \frac{1}{1 + |0.45 - 0.3|} = \frac{1}{1 + 0.15} = \frac{1}{1.15} \approx 0.8696,$$
$$M(x, Tx, 1) = \frac{1}{1 + |0.8 - 0.45|} = \frac{1}{1 + 0.35} = \frac{1}{1.35} \approx 0.7407,$$
$$M(y, Ty, 1) = \frac{1}{1 + |0.2 - 0.3|} = \frac{1}{1 + 0.1} = \frac{1}{1.1} \approx 0.9091,$$
$$\text{RHS} = k\left[(0.7407)^{1/2} * (0.9091)^{1/2}\right]$$
$$\approx \left(\frac{1}{1.25}\right) \cdot [(0.8604)(0.9535)] \approx 0.8 \cdot 0.8212 \approx 0.6569.$$

Since $M(Tx, Ty, 1) \approx 0.8696 > 0.6569$, the contractive condition holds.

Hence, T satisfies the condition of a fuzzy interpolative Kannan-type contraction with $\gamma = \frac{1}{2}$, $t = 1$, and $k = \frac{1}{1.25} = 0.8$.

Therefore, by Theorem 3.1, the mapping T has a unique fixed point in X. In fact, solving $T(x) = x$, we get:

$$x = \frac{1}{4}x + \frac{1}{4} \Rightarrow \frac{3}{4}x = \frac{1}{4} \Rightarrow x = \frac{1}{3}.$$

Conclusion

In this paper, we introduce and investigate a new class of mappings known as *fuzzy interpolative Kannan-type contractions* within the framework of complete fuzzy metric spaces. This class generalizes classical Kannan-type contractions by incorporating an interpolative parameter $\gamma \in (0, 1)$, allowing the contractive condition to depend on a convex combination of fuzzy distances. A concrete example was provided to validate the theoretical results, where all the conditions of the definition and the theorem were verified explicitly. The example confirms that the developed theory is applicable and practical within fuzzy metric contexts.

Conflict of Interests

The authors declare that there is no conflict of interests.

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