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COMPACT HYPERBOLIC COXETER THIN CUBES

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Abstract: Andreev's Theorem provides a complete characterization of 3-dimensional compact hyperbolic combinatorial polytope having non-obtuse dihedral angles. Cube is one of such polytope. In this article, with the help of Andreev's Theorem, a special type of compact hyperbolic coxeter cube, called thin cube has been studied. Using graph theory and combinatorics, it has been found that there are exactly 3 such cubes in hyperbolic space upto symmetry.

Keywords: planar graph; thin cube; dihedral angles; coxeter polytope.

2010 AMS Subject Classification: 52B10, 20F55, 52A55.

1. Introduction

The angle between two faces of a polytope, measured from perpendiculars to the edge created by the intersection of the planes is called a *dihedral angle*. For a combinatorial polyhedron P with E edges, the space of dihedral angles D_P of all compact hyperbolic polyhedrons that realize P is generally not a convex subset of $\mathbb{R}^E$ [19]. If P has more than four faces, Andreev's Theorem states that the corresponding space D_P obtained by restricting to non-obtuse angles is a convex polytope. Tetrahedron, the only polyhedron with

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four faces and the space of dihedral angles of compact hyperbolic tetrahedron after restricting to non-obtuse angles is non-convex. Therefore, except tetrahedron, Andreev's Theorem [22] provides a complete characterization of 3-dimensional compact hyperbolic combinatorial polytope having non-obtuse dihedral angles. On the other hand, Roland K. W. Roeder's Theorem [11] provides the classification of compact hyperbolic tetrahedron by restricting to non-obtuse dihedral angles.

A simple polytope P in n-dimensional hyperbolic space H^n is said to be *coxeter*, if the dihedral angles of P are of the form $\frac{\pi}{n}$ where, n is a positive integer ≥ 2. There is no complete classification of hyperbolic coxeter polytopes. Vinberg proved in [21] that there are no compact hyperbolic coxeter polytopes in H^n when $n \geq 30$. Tumarkin classified the hyperbolic coxeter pyramids in terms of coxeter diagram and John Mcleod generalized it in his article [9]. P. Kalita and B. Kalita [1] found that there are exactly one, four and thirty coxeter Andreev's tetrahedrons having respectively two edges of order $n \geq 6$, one edge of order $n \geq 6$ and no edge of order $n \geq 6$, $n \in \mathbb{N}$ upto symmetry. Again using Roland K. W. Roeder's Theorem, P. Kalita and B. Kalita [2] proved that there are exactly 3 CHC (compact hyperbolic coxeter) tetrahedrons upto symmetry in real projective space. These 3 tetrahedrons can be realized uniquely [10] in Hyperbolic space and these are nothing but the 3 coxeter Andreev's tetrahedrons found by theorems 3.20 and 3.21 in [1].

Many communication networks can be viewed as graphs called *k*-ray *n*-cubes, whose special cases include rings, hypercubes, and toruses. A *k*-ray *n*-cube has k^n nodes. In this article, with the help of Andreev's Theorem, a special type of compact hyperbolic coxeter cube, called thin cube has been studied. Using graph theory and combinatorics, it has been found that there are exactly 3 such cubes in hyperbolic space upto symmetry.

The paper is organized as follows: The section 1 includes introduction. The section 2 includes some basic terminologies from graph theory and geometry. The section 3 focuses some already existed results on graph theory and geometry. Main results and conclusions are included in the sections 4 and 5 respectively.

2. Basic Terminologies

There is a strong link between graph theory and geometry. Graph theoretical concepts are used to understand the combinatorial structure of a polytope in geometry. Here we will mention some essential terminologies from graph theory and geometry.

Definition 2.1: Two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are said to be *isomorphic* if there exists a bijective function $f:V_1\to V_2$ such that $xy\in E(G_1)$ if and only if $f(x)f(y)\in E(G_2)$. The function f is said to an isomorphism. In case of weighted graphs, isomorphism preserves the weights as well. If G_1 and G_2 are identical, then f is said to an *automorphism*. An automorphism is therefore a permutation on the vertices of the graph that maps edges to edges and nonedges to nonedges. A permutation can be expressed as a product of disjoint cycle.

Definition 2.2: A *polytope* is a geometric object with surfaces enclosed by edges that exist in any number of dimensions. A polytope in 2D, 3D and 4D is said to be *polygon*, *polyhedron* (plural polyhedra or polyhedrons) and *polychoron* respectively. The enclosed surfaces are said to be faces. The line of intersection of any two faces is said to be an edge and a point of intersection of three or more edges is called a vertex.

Definition 2.3: Let P be a convex polyhedron. The abstract graph of P is denoted by $G(P)$ and is defined as $G(P)=(V(P),E(P))$, where $V(P)$ is the set of vertices of P and two vertices $x,y\in V(P)$ are adjacent if and only if (x,y) is an edge of P.

Definition 2.4: If the dihedral angle of an edge of a compact hyperbolic polytope is $\frac{\pi}{n},n$ is a positive number, then n is said to be the *order* of the edge. We define a trivalent vertex to be of order (l,m,n) if the three edges at that vertex are of order l,m,n.

Definition 2.5: A *coxeter* dihedral angle is a dihedral angle of the form $\frac{\pi}{n}$ where, n is a positive integer ≥ 2. A compact polytope in hyperbolic space with coxeter dihedral angles is called a *compact hyperbolic coxeter polytope*.

Definition 2.6: A cell complex C on S^2 is called trivalent if each vertex is the intersection of three faces.

Definition 2.7: A 3-dimensional combinatorial polytope is a cell complex C on S^2 that satisfies the following conditions:

(a) Each edge of C is the intersection of exactly two faces

(b) A nonempty intersection of two faces is either an edge or a vertex.

(c) Each face is enclosed by not less than 3 edges.

Any trivalent cell complex C on S^2 that satisfies the above three conditions is said to be *abstract polyhedron*.

Definition 2.8: A 3D polytope is called a *simple polytope* if each vertex is the intersection of exactly 3 faces. The 1-skeleton of a polytope is the set of vertices and edges of the polytope. The skeleton of any convex polyhedron is a planar graph and the skeleton of any $k-$dimensional convex polytope is a $k-$connected graph.

Definition 2.9: A prismatic $k-$circuit $\Gamma_p(k)$ is a $k-$circuit such that no two edges of C which correspond to edges traversed by $\Gamma_p(k)$ share a common vertex.

Hyperbolic geometry [2] is difficult to visualize as many of its theorems are contradictory to similar theorems of Euclidean geometry which are very similar to us. Therefore technology has been used in creation of geometric models in Euclidean space to visualize Hyperbolic geometry. Hyperbolic [11] planes in these models correspond to Euclidean hemispheres and Euclidean planes that intersect the boundary perpendicularly. Furthermore, these models are correct conformally. That is, the hyperbolic angle between a pair of such intersecting hyperbolic planes is exactly the Euclidean angle between the corresponding spares or planes.

We will concentrate on 3-dimensional compact hyperbolic orbifolds whose base spaces are homeomorphic to a convex polyhedron and whose sides are silvered. The compact hyperbolic polyhedron is simple, therefore, the combinatorial polyhedron of a compact hyperbolic polyhedron can be known from 3-connected planar graph of the polyhedron. In case of compact hyperbolic cube, the corresponding 3-connected planar graph has 8 vertices, 12 edges, 6 faces and exactly 4 disjoint edges upto symmetry. The following figure 2.1 and figure 2.2 shows the Euclidean cube and hyperbolic cube which are found in [8].

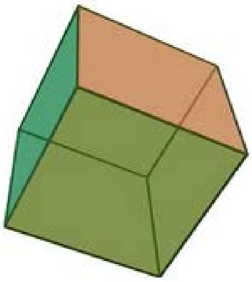


Figure 2.1

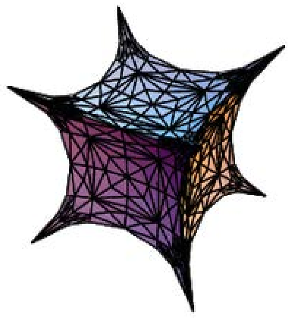


Figure 2.2

The figure 2.3 gives the Euclidean cube and the figure 2.4 gives an isomorphic graph of the Euclidean cube which is a 3-connected planar graph.

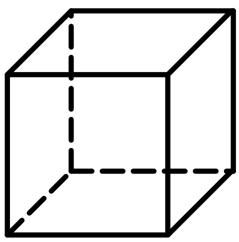


Figure 2.3

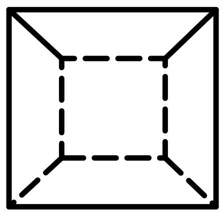


Figure 2.4

Definition 2.10: We define a thin cube to be a cube whose order of the opposite sides are same. Therefore the edges of a thin cube can be ordered by $n_1, n_2, n_3, n_4, n_5, n_6$ as shown in the figure 2.5.

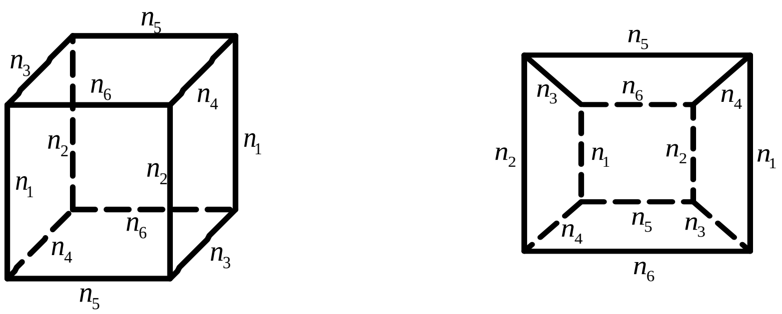


Figure 2.5

3. Known Results

E. M. Andreev provides a complete characterization of 3D compact hyperbolic polytope with non-obtuse dihedral angles in his article [22]. Therefore, Andreev's theorem is a fundamental tool for classification of 3D compact hyperbolic coxeter polytope. Before stating Andreev's Theorem, some already existed results [5] on graph and polytope have been mentioned.

Theorem 3.1: (Blind and Mani) *If P is a convex polyhedron, then the graph $G(P)$ determines the entire combinatorial structure of P. In other words, if two simple polyhedral have isomorphic graphs, then their combinatorial polyhedral are also isomorphic.*

Theorem 3.2: (Ernst Steinitz) *A graph $G(P)$ is a graph of a 3-dimensional polytope P if and only if it is simple, planar and 3-connected.*

Corollary 3.3: *Every 3-connected planar graph can be represented in a plane such that all the edges are straight lines, all the bounded regions determined by these and the union of all the bounded regions are convex polygons.*

Theorem 3.4: (Andreev Theorem, [14, 17]) *A combinatorial polyhedron P which is not isomorphic to a tetrahedron or a triangular prism has a geometric realization in H^3 with interior dihedral angle measures $0 < \theta_i \leq \frac{\pi}{2}$ at edge e_i if and only if:*

(1) The 1-skeleton of P is trivalent.

(2) If e_i, e_j, e_k are distinct edges which meet at a vertex, then $\theta_i + \theta_j + \theta_k > \pi$.

(3) If e_i, e_j, e_k form a prismatic 3-circuit $\Gamma_p(3)$, then $\theta_i + \theta_j + \theta_k < \pi$.

(4) If e_i, e_j, e_k, e_l form a prismatic 4-circuit $\Gamma_p(4)$, then $\theta_i + \theta_j + \theta_k + \theta_l < 2\pi$.

Furthermore, this polyhedron is unique up to isometrics of hyperbolic H^3.

Andreev's restriction to non-obtuse dihedral angles is necessary to ensure that P be convex. A compact hyperbolic polyhedral realizing a given abstract polyhedron may not be convex without the restriction of non-obtuse dihedral angle [17]. Since Coxeter polyhedrons have non-obtuse dihedral angles, Andreev's Theorem provides a complete characterization of 3-dimensional compact hyperbolic Coxeter polyhedra.

Corollary 3.5: *For compact hyperbolic coxeter cube, the condition (3) of Andreev's theorem is vacuous.*

Figure 3.1

Proof: A compact hyperbolic cube has no prismatic 3-circuit. Therefore the condition (3) of Andreev's theorem is vacuous. It has 3 numbers of prismatic 4-circuits. In figure 3.1, these are shown in red, blue and green colors.

4. Main Results

The main results are here presented below:

Theorem 4.1: *In a 3-dimensional compact hyperbolic coxeter polytope* T*, the order of the edges at one vertex is one of the forms:* $(2,2,n\geq 2),(2,3,3),(2,3,4),(2,3,5)$.

Proof: Suppose the three distinct edges e_i, e_j, e_k of T with respective orders n_i, n_j, n_k meet at one vertex v. Then the coxeter dihedral angles of e_i, e_j, e_k at v are $\frac{\pi}{n_i}, \frac{\pi}{n_j}, \frac{\pi}{n_k}$, with positive integers $n_i, n_j, n_k \geq 2$. By Second condition of Andreev's Theorem:

$$\alpha_i + \alpha_j + \alpha_k > \pi \Rightarrow \frac{\pi}{n_i} + \frac{\pi}{n_j} + \frac{\pi}{n_k} > \pi \Rightarrow \frac{1}{n_i} + \frac{1}{n_j} + \frac{1}{n_k} > 1$$

So upto permutations, the triples (n_i, n_j, n_k) are respectively $(2,2,n\geq 2),(2,3,3),(2,3,4)$, $(2,3,5)$. Hence the order of the edges at one vertex is one of the forms: $(2,2,n\geq 2),(2,3,3)$, $(2,3,4),(2,3,5)$.

Theorem 4.2: *In a 3-dimensional compact hyperbolic coxeter polytope* T*, if an edge at one vertex is of order* $n \geq 6$*, then the other two edges must be of order 2.*

Proof: Let e_i, e_j, e_k be three edges at one vertex v of T with orders $n_i \geq 6, n_j, n_k$ respectively.

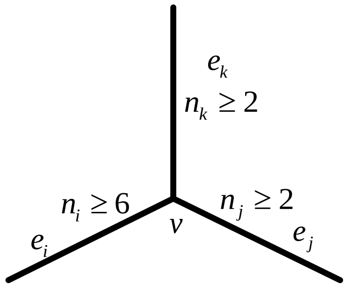


Figure 4.1

By Second condition of Andreev's Theorem:

$$\frac{\pi}{n_i}+\frac{\pi}{n_j}+\frac{\pi}{n_k}>\pi$$

Since $n_i \geq 6$, therefore:

$$\pi<\frac{\pi}{n_i}+\frac{\pi}{n_j}+\frac{\pi}{n_k} \Rightarrow \pi<\frac{\pi}{6}+\frac{\pi}{n_j}+\frac{\pi}{n_k} \Rightarrow \frac{5\pi}{6}<\frac{\pi}{n_j}+\frac{\pi}{n_k} \tag{3}$$

Since, n_j, n_k are positive integers, therefore, the inequality (3) has only the solutions $n_j = n_k = 2$.

Corollary 4.3: *In a 3-dimensional compact hyperbolic coxeter polytope T, the number of edges of order 2 at one vertex is at least 1 and at most 3.*

Proof: From Theorem 4.1, in a 3-dimensional compact hyperbolic coxeter polytope T, the order of the edges at one vertex is one of the forms: $(2,2,n\geq 2),(2,3,3),(2,3,4),(2,3,5)$. Therefore, it is clear that at one vertex, the number of edges of order 2 is at least 1 and at most 3.

Theorem 4.4: *Suppose C is a compact hyperbolic coxeter thin cube. If n is the order of an edge of C then $n<6$.*

Proof: Suppose n is the order of an edge of a compact hyperbolic coxeter thin cube C and suppose $n \geq 6$. Without loss of generality, let the order of the edge AD be $n \geq 6$ and hence the order of the edge FG be $n \geq 6$. By Theorem 4.2, other two edges adjacent to the edge of order $n \geq 6$ must be of order 2 as shown in the following figure 4.2.

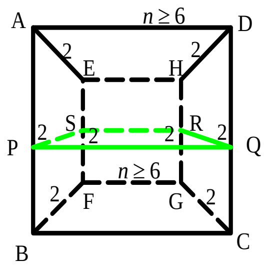


Figure 4.2

In figure 4.2, PQRS is a prismatic 4-circuit and hence by Andreev's 4th condition:

$$\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}<2\pi \Rightarrow 2\pi<2\pi$$

And this is not possible. Therefore if n is the order of an edge of a compact hyperbolic coxeter thin cube C then $n<6$.

Corollary 4.5: *Suppose C is a compact hyperbolic coxeter thin cube. If n is the order of an edge of C then* $2 \leq n \leq 5$.

Proof: By Theorem 4.1, in a 3-dimensional compact hyperbolic coxeter polytope, the order of the edges at one vertex is one of the forms: $(2,2,n\geq 2),(2,3,3),(2,3,4),(2,3,5)$. Also by Theorem 4.4, if n is the order of an edge of compact hyperbolic coxeter thin cube C then $n<6$. Thus by Theorem 4.1 and Theorem 4.4, we have the conclusion $2 \leq n \leq 5$.

Theorem 4.6: *Suppose C is a compact hyperbolic coxeter thin cube with edges ordered by n_1,n_2,n_3,n_4,n_5,n_6. Then at most three out of n_1,n_2,n_3,n_4,n_5,n_6 are 2.*

Proof: Suppose the edges of the compact hyperbolic coxeter thin cube C are ordered as shown in the following figure 4.3.

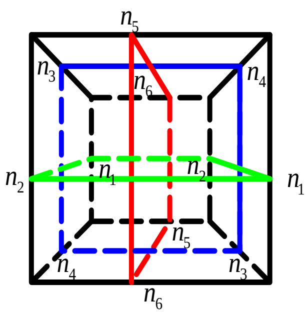


Figure 4.3

There are 3 numbers of prismatic 4-circuits and hence by Andreev's 4th condition:

$$\frac{\pi}{n_1}+\frac{\pi}{n_2}+\frac{\pi}{n_1}+\frac{\pi}{n_2}<2\pi,\frac{\pi}{n_3}+\frac{\pi}{n_4}+\frac{\pi}{n_3}+\frac{\pi}{n_4}<2\pi,\frac{\pi}{n_5}+\frac{\pi}{n_6}+\frac{\pi}{n_5}+\frac{\pi}{n_6}<2\pi$$
$$\Rightarrow 2\left(\frac{\pi}{n_1}+\frac{\pi}{n_2}\right)<2\pi,2\left(\frac{\pi}{n_3}+\frac{\pi}{n_4}\right)<2\pi,2\left(\frac{\pi}{n_5}+\frac{\pi}{n_6}\right)<2\pi$$
$$\Rightarrow \frac{1}{n_1}+\frac{1}{n_2}<1,\frac{1}{n_3}+\frac{1}{n_4}<1,\frac{1}{n_5}+\frac{1}{n_6}<1$$

Therefore $(n_1,n_2)\neq(2,2),(n_3,n_4)\neq(2,2),(n_5,n_6)\neq(2,2)$. Clearly at most one n_i of each pair $(n_1,n_2),(n_3,n_4),(n_5,n_6)$ is 2 and hence at most three out of n_1,n_2,n_3,n_4,n_5,n_6 are 2.

Theorem 4.7: *Suppose C is a compact hyperbolic coxeter thin cube with edges ordered by n_1,n_2,n_3,n_4,n_5,n_6. Then at least three out of n_1,n_2,n_3,n_4,n_5,n_6 are 2.*

Proof: Suppose the edges of the compact hyperbolic coxeter thin cube C are ordered as shown in the following figure 4.4.

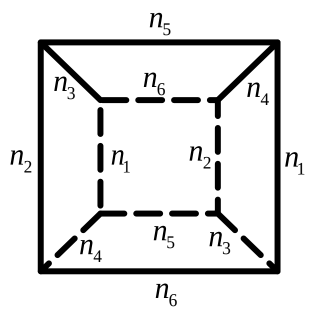


Figure 4.4

There are 8 vertices in the cube (thin). By corollary 4.3, in a 3-dimensional compact hyperbolic coxeter polytope, the number of edges of order 2 at one vertex is at least one. Also each edge is adjacent with two vertices. Therefore there are at least four edges are of order 2. Since C is a thin cube therefore at least two out of n_1,n_2,n_3,n_4,n_5,n_6 are 2. Suppose exactly two out of n_1,n_2,n_3,n_4,n_5,n_6 are 2. Now all the edges of order 2 must be disjoint (otherwise, it will violate the corollary 4.3) and hence we have only the cases as shown in the following figure 4.5.

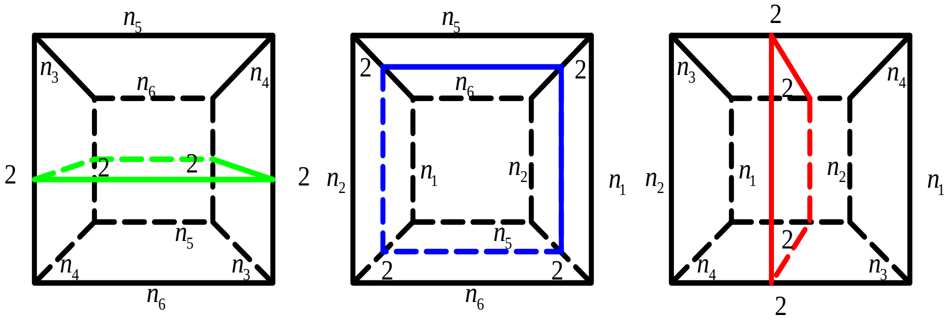


Figure 4.5

There are 3 numbers of prismatic 4-circuits in cube (thin) and in each case, due to prismatic 4-circuit by Andreev's 4th condition, we must have

$$\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}<2\pi \Rightarrow 2\pi<2\pi$$

This is not possible. Therefore, there cannot be exactly two out of n_1,n_2,n_3,n_4,n_5,n_6 are 2 and hence at least three out of n_1,n_2,n_3,n_4,n_5,n_6 are 2.

Corollary 4.8: *Suppose C is a compact hyperbolic coxeter thin cube with edges ordered by n_1,n_2,n_3,n_4,n_5,n_6. Then there are exactly three out of n_1,n_2,n_3,n_4,n_5,n_6 are 2. In other words, exactly one n_i of each pair $(n_1,n_2),(n_3,n_4),(n_5,n_6)$ is 2*

Proof: Clear from Theorem 4.6 and 4.7.

Theorem 4.9: *Suppose C is a compact hyperbolic coxeter thin cube, then there are exactly 3 such cubes in hyperbolic space upto symmetry.*

Proof: Suppose C is a compact hyperbolic coxeter thin cube with edges ordered by n_1,n_2,n_3,n_4,n_5,n_6 as shown in the figure 4.6.

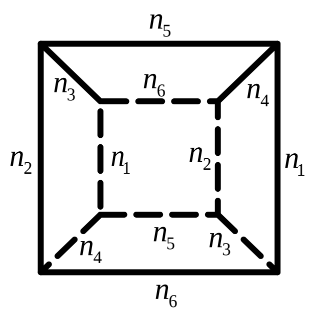


Figure 4.6

From corollary 4.8, there are exactly three out of $n_1, n_2, n_3, n_4, n_5, n_6$ are 2 or in other words, exactly one n_i of each pair $(n_1, n_2), (n_3, n_4), (n_5, n_6)$ is 2. Therefore we have the only 8 cases obtained from the following tree diagram in figure 4.7.

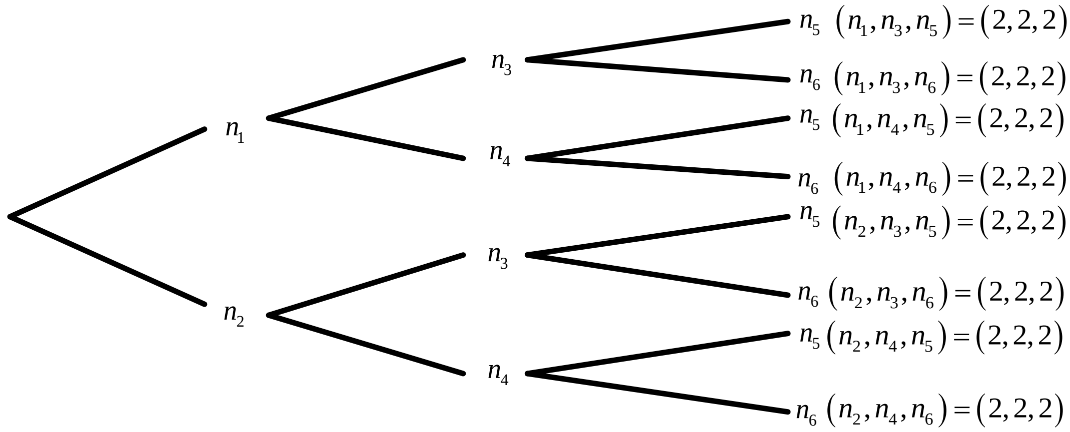


Figure 4.7

For $(n_1, n_3, n_5) = (2,2,2), (n_1, n_4, n_6) = (2,2,2), (n_2, n_3, n_6) = (2,2,2)$ and $(n_2, n_4, n_5) = (2,2,2)$; the cubes are respectively C-1, C-2, C-3 and C-4 as shown in figure 4.8.

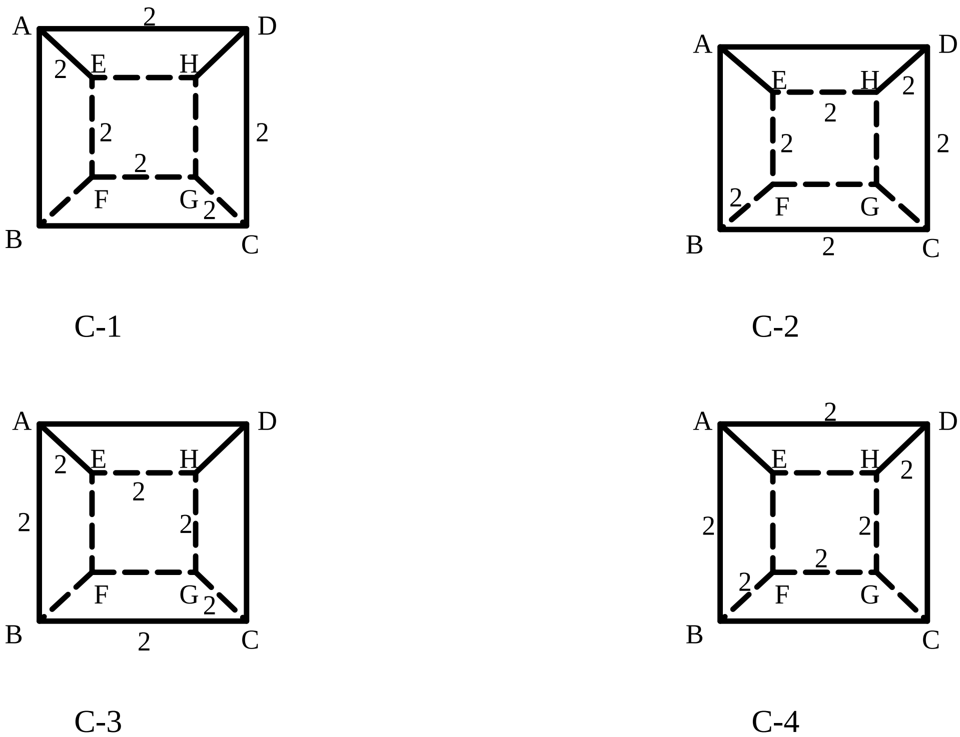


Figure 4.8

Now we treat these graphs C-1, C-2, C-3 and C-4 as weighted graphs, where the weights are the orders of the edges. Although the graphs C-1, C-2, C-3 and C-4 are different due to their weights, they are automorphic image of each other. For example, the graph C-1 is an automorphic image of the graphs C-2, C-3 and C-4 under the automorphisms $(\mathrm{ABCD})(\mathrm{EFGH})$, $(\mathrm{AC})(\mathrm{BD})(\mathrm{EG})(\mathrm{FH})$ and $(\mathrm{ADCB})(\mathrm{EHGF})$ (expressed as a product of disjoint cycles) respectively. Therefore these graphs C-1, C-2, C-3 and C-4 are same upto automorphism. Let us take one of these graphs and name it C shown in the figure 4.9.

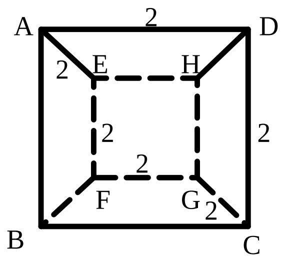
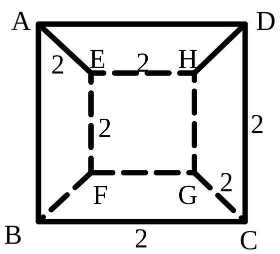


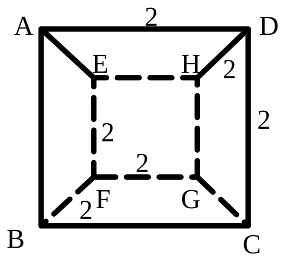
Figure 4.9

Now, there exists one vertex B (in figure 4.9) at which there is no edge of order 2. But by corollary 4.3, in a 3-dimensional compact hyperbolic coxeter polytope, the number of edges of order 2 at one vertex is at least one. Therefore the graph shown in figure 4.9 is not possible.

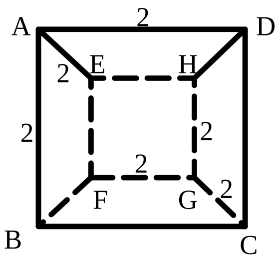
For $(n_1, n_3, n_6) = (2,2,2), (n_1, n_4, n_5) = (2,2,2), (n_2, n_3, n_5) = (2,2,2)$ and $(n_2, n_4, n_6) = (2,2,2)$; the cubes are respectively C-5, C-6, C-7 and C-8 as shown in figure 4.10.



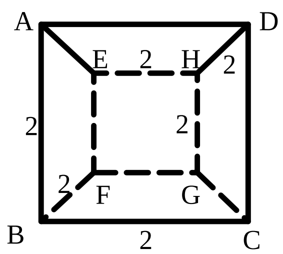
C-5



C-6



C-7



C-8

Figure 4.10

Again we treat these graphs C-5, C-6, C-7 and C-8 as weighted graphs and they are automorphic image of each other. For example, the graph C-5 is an automorphic image of the graphs C-6, C-7 and C-8 under the automorphisms $(\mathrm{ADCB})(\mathrm{EHGF})$, $(\mathrm{AC})(\mathrm{BD})(\mathrm{EG})(\mathrm{FH})$ and $(\mathrm{ABCD})(\mathrm{EFGH})$ (expressed as a product of disjoint cycles) respectively. Therefore these graphs C-5, C-6, C-7 and C-8 are same upto automorphism. Let us take one of these graphs and name it TC shown in the figure 4.11.

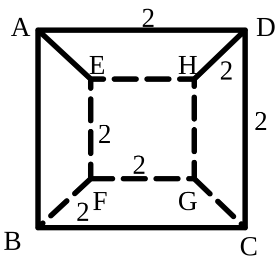


Figure 4.11

Next we order the remaining edges of the thin cube shown in figure 4.11. By corollary 4.5, the order of each vertex is one of the forms: $(2,2,2 \le n \le 5),(2,3,3),(2,3,4),(2,3,5)$. The form $(2,2,n=2)$ has been discussed in figure 4.9 and in figure 4.11. By corollary 4.8, there are exactly 6 edges of order 2 which are already assigned in figure 4.11. Therefore we cannot assign any more 2 to the remaining edges and hence the form $(2,2,3\le n\le 5)$ cannot be took place. Now we are left with the forms: $(2,3,3),(2,3,4),(2,3,5)$. Therefore at least one edge at one vertex must be of order 3. Let us take one vertex H. At H, we can take $\mathrm{HG}=2$ or $\mathrm{HE}=2$ and these two are same under automorphism. Therefore suppose $\mathrm{HG}=3$ and hence $\mathrm{AB}=3$ (due to thin cube) as shown in figure 4.12.

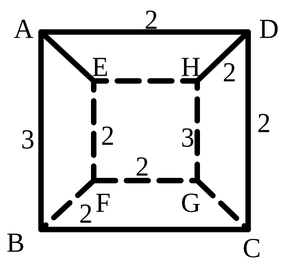


Figure 4.12

Now the only possibility for EH (and BC) is 3 and the only possibilities for AE (and GC) are 3, 4, 5. Therefore we have exactly 3 thin cubes in hyperbolic space upto symmetry shown in figure 4.13.

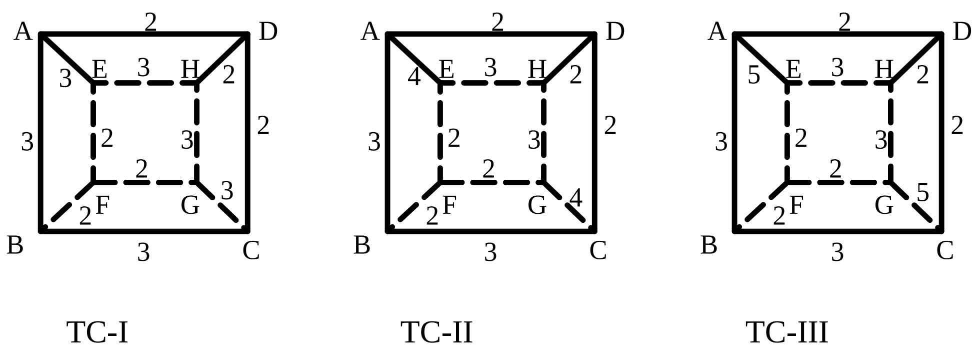


Figure 4.13

These 3 compact hyperbolic coxeter thin cubes can be realized [22] uniquely in Hyperbolic space upto symmetry.

5. Conclusions

We found all the compact hyperbolic coxeter thin cubes in this article. Using Andreev's theorem, we proved that there exist exactly 3 such cubes in hyperbolic space and these are unique [22] up to symmetry. This research can be extended to other compact as well as non-compact hyperbolic polytopes in spaces of different dimensions.

Conflict of Interests

The author declares that there is no conflict of interests.

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