Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 3, 593-603
ISSN: 1927-5307

STABILITY OF WAVELET FRAMES AND RIESZ BASIS, WITH RESPECT TO TRANSLATIONS IN $L^p(R^n), 1 \leq p < \infty$

DEVENDRA KUMAR*

Department of Mathematics, M.M.H.College,Model Town,Ghaziabad-201001, U.P., India

Abstract. We consider the perturbation problem of wavelet frame (Riesz basis) $\{\varphi_{j,k,a_0,b_0}\} = \{a_0^{nj/2}\varphi(a_0^j x - kb_0)\}$ about translation parameter b_0 in $L^p(R^n)-$norm. If $\varphi \in S(R^n)$ (Schwartz space), we can estimate the frame bounds about the perturbation of translation parameter b_0. Our results generalize the Balan's [1] results. Our method and approach are different from Balan's [1].

Keywords: frames; stability; wavelets; Schwartz space.

2000 AMS Subject Classification: 42C15; 41A30

1. Introduction

A set of vectors $\{f_j\}_{j\in N}$ in a separable Hilbert space H is called a frame if there are constants A and B such that for every $f \in H$

$$A\|f\|^2 \leq \sum_{j\in N} | < f, f_j > |^2 \leq B\|f\|^2.$$

*Corresponding author

E-mail addresses: d_kumar001@rediffmail.com (D. Kumar)

This work was done in the memory of Prof. H.S.Kasana, Senior Associate, ICTP, Trieste, Italy.

Received December 24, 2011

The constants A and B are called frame bounds. If only the right hand side inequality is satisfied for all $f \in H$, then $\{f_j\}$ is called a Bessel sequence with bound B. Many people considered the stability of frames. The concept of frames was introduced in 1952 in the paper [5]. Favier and Zalik [6] and Zhang [9] considered the wavelet frames $\{a^{nj/2}\varphi(a^j x - kb)\}_{j\in z, k\in z^n}$ and Gabor frames $\{e^{i(jb.x)}\varphi(x - ka)\}_{j,k\in z^n}$, which are useful in many areas of mathematics, engineering, quantum mechanics, signal and image processing etc. Balan [1] studied the perturbation of translation parameter b. This problem was first considered by Daubechies and and Tehamichian for Meyer orthogonal wavelet basis in 1990 [4].

Now we mention the Balan's result as follows.

Theorem A [1]. Let $\{\varphi_{j,k,a_0,b_0}\}$ be a wavelet Riesz basis on $L^2(R)$ with frame bounds A and B. Furthermore suppose that $\widehat{\varphi}$, the Fourier transform of φ, satisfies the following requirements :

$\widehat{\varphi}$ is in the class C^1 on R and both $\widehat{\varphi}$ and $\widehat{\varphi}'$ are bounded by

$$|\widehat{\varphi}(\xi)|, |\widehat{\varphi}'(\xi)| \le C\frac{|\xi|^\alpha}{(1+|\xi|)^\gamma}$$

for some $C > 0, \gamma > 1 + \alpha > 1$. Then there exists on $\varepsilon > 0$ such that for any b with $|b - b_0| < \varepsilon$, the set $\{\varphi_{j,k,a_0,b_0}\}$ is a Riesz basis, where $\varphi_{j,k,a,b}(x) = a^{j/2}\varphi(a^j x - kb)$.

Various authors [2],[3],[8] studied the stability of frames in Hilbert space. Feichtinger and Gröchenig studied the stability theory for atoms in Banach spaces [7]. In this paper we study the stability problem, that is, the perturbation of translation parameter b in $L^p(R^n)$. Our methods and approach is different from those of authors mentioned above.

The text has been divided in three sections. Section 1, i.e., introduction incorporate the introductory exposition of the topic. Section 2 deals with some preliminary results. Finally in Section 3, we have proved our main theorem.

2. Preliminary Results

In this paper we use an abbreviation for the scaler product in the following way. For $x, s \in R^n, x = (\xi_1, \cdots, \xi_n), s = (\sigma_1, \cdots, \sigma_n)$ we define

$$xs = < x, s > = \sum_{i=1}^{n} \xi_i \sigma_i, x^2 = < x, x > = \sum_{i=1}^{n} \xi_i^2, |x| = \left(\sum_{i=1}^{n} \xi_i^2 \right)^{1/2}$$

Let $f \in L^1(\Omega), \Omega \subseteq R^n$ be an open set and

$$(\Im f)(s) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} f(x) e^{-ixs} dx, \quad s \in R^n.$$

The function $\Im f : R^n \to C$ is called the Fourier transform of f and the mapping $\Im : f \to \Im f$ is called Fourier transformation. Obviously, the functional $\Im f$ is well-defined, measurable and linear.

We define

$$C^\infty(R^n) = \bigcap_{m \in N} C^m(R^n)$$

where $C^m(R^n)$ is the set of all $m-$times continuously differentiable functions defined on R^n.

Definition 2.1. Let $f : R^n \to C$ and $x^\alpha = x_1^{\alpha_1}, \cdots, x_n^{\alpha_n}$. The function f is said to be quickly decreasing if and only if

$$\lim_{|x| \to \infty} x^\alpha f(x) = 0 \quad for \;\; all \quad \alpha \in N_0^n.$$

The space $S(R^n) = \{f \in C^\infty(R^n) | D^\beta f$ is quickly decreasing for all $\beta \in N_0^n\}$ is called Schwartz space. The elements of the Schwartz space are called Schwartz functions.

Example 2.1. The function $\gamma : R^n \to X$ defined by $\gamma(x) = e^{-x^2}, x \in R^n$, is a Schwartz function because

$$\begin{aligned}\lim_{|x|\to\infty} x^{\alpha}\gamma(x) &= \lim_{|x|\to\infty} x_1^{\alpha_1}.\cdots.x_n^{\alpha_n}.e^{-(x_1^2+\cdots+x_n^2)}\\ &= \lim_{|x|\to\infty} \frac{x_1^{\alpha_1}}{e^{x_1^2}}.\cdots.\frac{x_n^{\alpha_n}}{e^{x_n^2}} = 0 \quad for \quad all \quad \alpha\in N_0^n.\end{aligned}$$

Lemma 2.1. Let $f \in C^{\infty}(R^n)$. Then the function f is a Schwartz function if and only if

$$\sup_{x\in R^n}(1+|x|^m)|D^{\beta}f(x)| < \infty \quad for \quad all \quad m\in N_0, \beta\in N_0^n. \tag{2.1}$$

Proof. Let f be a Schwartz function. Then by definition we have

$$\lim_{|x|\to\infty} |x|^m D^{\beta}f(x) = 0 \quad for \quad all \quad m\in N_0, \beta\in N_0^n,$$

i.e.,

$$\lim_{|x|\to\infty} |x|^m |D^{\beta}f(x)| = 0 \quad for \quad all \quad m\in N_0, \beta\in N_0^n.$$

Since all the functions are continuous it follows

$$\sup_{x\in R^n} |x|^m |D^{\beta}f(x)| < \infty \quad for \quad all \quad m\in N_0, \beta\in N_0^n.$$

For $m = 0$ we get

$$\sup_{x\in R^n} |x|^m |D^{\beta}f(x)| < \infty \quad for \quad all \quad \beta\in N_0^n.$$

Together we get

$$\sup_{x\in R^n}(1+|x|^m)|D^{\beta}f(x)| \le \sup_{x\in R^n}|D^{\beta}f(x)| + \sup_{x\in R^n}|x|^m|D^{\beta}f(x)| < \infty$$

for all $m \in N_0$, for all $\beta \in N_0^n$ i.e., (2.1) holds.

Now let (2.1) be true. Then there are constants C_1 and C_2 such that

$$\left(|x|^m + |x|^{m+1}\right)\left|D^\beta f(x)\right| \leq \begin{cases} 2|x|^{m+1}|D^\beta f(x)| \leq 2(1+|x|^{m+1})|D^\beta f(x)| \leq C_1, if \quad |x| \geq 1 \\ 2|x|^m|D^\beta f(x)| \leq 2(1+|x|^m)|D^\beta f(x)| \leq C_2, if \quad |x| < 1. \end{cases}$$

Choosing $c = \max\{C_1, C_2\}$ we get

$$|x|^m|D^\beta f(x)| = \frac{(|x|^m + |x|^{m+1})|D^\beta f(x)|}{1+|x|} \leq \frac{C}{1+|x|} \to 0 \quad as \quad |x| \to \infty.$$

Hence the proof is completed.

Lemma 2.2. $S(R^n)$ is dense in $L^p(R^n)$ for all $p \geq 1$.

Proof. Let f be a Schwartz function, i.e., $f \in S(R^n)$. Then by (2.1) (setting $\beta = 0$) there is a bound C such that

$$(1+|x|^m)|f(x)| \leq C \quad for \quad all \quad x \in R^n,$$

or

$$|f(x)| \leq \frac{C}{1+|x|^m} \quad for \quad all \quad x \in R^n.$$

Now first we will prove $S(R^n) \subseteq L^p(R^n)$ for all $p \geq 1$. We can choose $m \in N$ sufficiently great such that $mp - (n-1) > 1$. Now we use polar coordinates

$$
\begin{aligned}
x_1 &= r.cos\phi_1 \\
x_2 &= r.sin\phi_1.cos\phi_2 \\
x_3 &= r.sin\phi_1.sin\phi_2.cos\phi_3 \\
&\vdots \\
x_{n-1} &= r.sin\phi_1.sin.\phi_2.sin\phi_3.\cdots.sinn\phi_{n-2}.cos\phi_{n-1}, \\
x_n &= r.sin\phi_1.sin\phi_2.sin\phi_3.\cdots.sin\phi_{n-2}.sin\phi_{n-1},
\end{aligned}
$$

where $0 \leq \phi \leq \pi$ for $i = \{1, \cdots, n-2\}$ and $0 \leq \phi_{n-1} \leq 2\pi$.

Then the corresponding Wronski-determinant is given by

$$D_\Phi(\phi_1, \cdots, \phi_{n-1}) = sin^{n-2}\phi.sin^{n-3}\phi_2.\cdots.sin^2\phi_{n-3}.sin\phi_{n-2}.$$

Then

$$\int_{R^n} |f(x)|^p dx \leq \int_{R^n} \left(\frac{C}{1+|x|^m}\right)^p dx$$

$$
\begin{aligned}
&= C^p \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \int_0^{\infty} \left(\frac{1}{1+r^m}\right)^p r^{n-1} D_\Phi dr d\phi_1 \cdots d\phi_{n-1}, \\
&= C^p \int_0^{\infty} \left(\frac{1}{1+r^m}\right)^p r^{n-1} dr \underbrace{\int_0^{2\pi} \int_0^{\infty} \cdots \int_0^{\infty} D_\Phi d\phi_1 \cdots d\phi_{n-1}} \\
&= C^p w_n \int_0^{\infty} \left(\frac{1}{1+r^m}\right)^p r^{n-1} dr, \left(\underbrace{\int_0^{2\pi} \int_0^{\infty} \cdots \int_0^{\infty} D_\Phi d\phi_1 \cdots d\phi_{n-1}}\right) = w_n \\
&= C^p w_n \int_0^{\infty} \frac{r^{n-1}}{(1+r^m)^p} dr = C^p w_n \int_0^{\infty} \frac{r^{n-1}}{r^{mp} + P(r)} dr, \\
&= C^p.w_n \frac{r^n}{n}.2F_1\left(\frac{n}{m}, p, 1+\frac{n}{m}, -r^m\right)|_0^{\infty} < \infty \quad for \quad \frac{n}{m} < p,
\end{aligned}
$$

where $2F_1$ is the hypergeometric function and $w_n = \frac{2\pi^{1/2}}{\Gamma\frac{n}{2}}$ is the surface area of unit sphere in R^n. This follows that $f \in L^p(R^n)$. Set

$$D(R^n) = \{\varphi \in C^\infty(R^n) | \sup(\varphi) \;\; is \;\; compact \;\; \} = C_0^\infty(R^n).$$

Obviously, $S(R^n)$ is a vector space and $D(R^n) \subseteq S(R^n)$. So we have

$$D(R^n) \subseteq S(R^n) \subseteq L^p(R^n), 1 \leq p < \infty. \tag{2.2}$$

Now we shall prove that $D(R^n)$ is dense in $L^p(R^n)$. For this set $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi(x/\varepsilon)$. Let $\Omega \subseteq R^n$ be an arbitrary open set and $f \in L^p(\Omega)$. We consider the set

$$K_m = \{x \in \Omega | \|x\| \leq m, d(x, \partial\Omega) \geq 2/m\}, m \in N.$$

The sets K_m are compact and $\bigcup_{m\in N} K_m = \Omega$. Hence

$$\int_{K_m} |f(x)|^p dx \to \int_\Omega |f(x)|^p dx \;\; as \;\; m \to \infty.$$

Now we set

$$f_m(x) = \int_{K_m} f(y)\varphi_{1/m}(x-y)dy$$

then $f_m \in D(\Omega)$ and it is $\|f - f_m\|_{L_p} \to 0$ as $m \to \infty$. Obviously this construction is a simultaneous approximation in all spaces L^p, i.e., if $f \in L^p(\Omega) \bigcap L^q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$, then it follows simultaneously both $\|f - f_m\|_{L^p} \to 0$ as $m \to \infty$ and $\|f - f_m\|_{L^q} \to 0$ as $m \to \infty$. Now from (2.2), we get $S(R^n)$ is dense in $L^p(R^n)$. Hence the proof is completed.

Definition 2.2. A bounded function $W : [0, \infty) \to R^+$ is a radial decreasing L^1-majorant of g if $|g(x)| \leq W(|x|)$ and W satisfying the following conditions :

(i) $W \in L^1([0, \infty))$,

(ii) W is decreasing

(iii) $W(0) < \infty$.

Remark 2.1. If $W \in S(R^n)$. Then W is also a radial decreasing L^1-majorant of g.

Proof. Since $S(R^n)$ is dense in $L^p(R^n), 1 \leq p < \infty$ and W satisfying (i),(ii) and (iii) easily. The proof follows.

Now we observe that

If $f \in L^\infty(R^n), g_{j,k,a,b} \in L^1(R^n)$ then

$$\begin{aligned}\sum_{j\in z}\sum_{k\in z^n} |< f, g_{j,k,a,b} >| &= \sum_{j\in z}\sum_{k\in z^n}\left|\int_{R^n} \overline{a^{nj/2}g(a^jk-kb)}f(y)dy\right| \\ &\leq \left|\int_{R^n}\sum_{j\in z}\sum_{k\in z^n} a^{nj/2}W\left(|a^jk-kb|\right)f(y)dy\right| \\ &\leq C\|f\|_\infty\|W\|_{L^1[0,\infty)},\end{aligned}$$

and if $f \in L^1(R^n), g_{j,k,a,b} \in L^1(R^n)$ then

$$\begin{aligned}\int_{R^n}\sum_{j\in z}\sum_{k\in z^n} | < f, g_{i,k,a,b} > |dx &\leq C\int_{R^n}\left[\sum_{j\in z}\sum_{k\in z^n} a^{nj/2}W(|a^jx-kb|)|f(y)|dy\right]dx \\ &\leq C\|f\|_{L^1(R^n)}\|W\|_{L^1[0,\infty)}.\end{aligned}$$

Hence by the application of Riesz Theorin Theorem, we get

$$\left\|\sum_{j\in z}\sum_{k\in z^n} | < f, g_{j,k,a,b} > |\right\|_{L^p(R^n)} \leq C\|f\|_{L^p(R^n)}\|W\|_{L^1[0,\infty)} \quad for \quad f \in L^p(R^n), g \in L^1(R^n).$$

Lemma 2.3. Let $\{f_j\}$ be a frame (Riesz basis) in Banach space M with frame bounds A and B. Assume $\{g_j\} \subset M$ and $\{f_j - g_j\}$ is a Bessel sequence with bound $\eta < A$. Then $\{g_j\}$ is a frame (Riesz basis) with frame bounds $A\left[1-(\eta/A)^{1/p}\right]^p$ and $B\left[1+(\eta/B)^{1/p}\right]^p, 1 \leq p < \infty$.

Proof. The proof follows after a simple manipulation in ([3],[6]).

3. Main Results

Theorem 3.1. If $\{\varphi_{j,k,a_0,b_0}\}$ is a frame (Riesz basis) on $L^p(R^n)$ with frame bounds A, B and $\varphi_{j,k,a_0,b_0} \in S(R^n)$. Then there exists a $\delta > 0$ such that for any b with $|b - b_0| < \delta, \{\varphi_{j,k,a_0,b_0}\}$ is a frame (Riesz basis) on $L^p(R^n)$.

Proof. Using Balan's idea to define a unitary operator we obtain

$$U_b : S(R^n) \to S(R^n), (U_b\varphi)(x) = (b/b_0)^{n/2}\varphi\left(\frac{b}{b_0}x\right) = \phi(x).$$

We observe that

$$U_b\varphi_{j,k,a_0,b} = \phi_{j,k,a_0,b_0}.$$

Therefore, $\{\varphi_{j,k,a_0,b}\}$ is a frame if and only if $\{\phi_{j,k,a_0,b_0}\}$ is a frame. If W is the L^1-majorant of φ and ϕ then φ and $\phi \in L^p(R^n)$. For all $f \in L^p(R^n), \phi$ and $\varphi \in L^1(R^n)$, we have

$$\begin{aligned}
&\sum_{j\in z}\sum_{k\in z^n} || < f, (\varphi - \phi)_{j,k,a_0,b_0}|^p \\
= &\sum_{j\in z}\sum_{k\in z^n}\left|\int_{R^n}\overline{\left[a_0^{nj/2}\varphi(a_0^j x - kb_0) - a_0^{nj/2}\phi(a_0^j x - kb_0)\right]}f(y)dy\right|^p \\
\le &\sum_{j\in z}\sum_{k\in z^n}\int_{R^n}\left|\overline{\left[a_0^{nj/2}\varphi(a_0^j x - kb_0) - a_0^{nj/2}\phi(a_0^j x - kb_0)\right]}f(y)\right|^p dy
\end{aligned}$$

We know that if μ be a finite positive Borel measure, then there is a sequence μ_n of atomic measure that converges to μ weakly or if φ and ϕ have compact support then

$$\int_{R^n} d\mu_n(\varphi(a_0^j x - kb_0) - \phi(a_0^j x - kb_0)) \to \int_{R^n}(\varphi(a_0^j x - kb_0) - \phi(a_0^j x - kb_0))d\mu$$

or $\mu_n \to \mu$ weakly.

If $\varphi - \phi \in L^1(R^n), d\mu = \left|\varphi(a_0^j x - kb_0) - \phi(a_0^j x - kb_0)\right| dx$ is a finite Borel measure, so we can find

$$\mu_N = \sum_{i=1}^{N} C_i^N \delta_{\lambda_i^N} \to \mu \quad weakly.$$

Consider

$$\begin{aligned}
&\sum_{j\in z}\sum_{k\in z^n}\int_{R^n}\left|\overline{a_0^{nj/2}\left[\varphi(a_0^j x - kb_0) - \phi(a_0^j x - kb_0)\right]} f(y)\right|^p dy \\
\leq\ &\sum_{j\in z}\sum_{k\in z^n}\int_{R^n}\left|a_0^{nj/2} d\mu_n(x) f(y)\right|^p dy. \\
\leq\ &\sum_{j\in z}\sum_{k\in z^n}\left\|a_0^{nj/2}\sum_{i=1}^{N} C_i^N f(\lambda_i^N)\right\|_p^p \\
\leq\ &\sum_{j\in z}\sum_{k\in z^n}\left[a_0^{nj/2}\sum_{i=1}^{N} C_i^N \|f\|_p^p\right] \\
=\ &\sum_{j\in z}\sum_{k\in z^n} a_0^{nj/2}\int_{R^n} d\mu_n(x)\|f\|_p^p \\
=\ &\sum_{j\in z}\sum_{k\in z^n} a_0^{nj/2}\int_{R^n}\left|\varphi(a_0^j x - kb_0) - \phi(a_0^j x - kb_0)\right| dx\|f\|_p^p.
\end{aligned}$$

Since φ and ϕ are in $S(R^n)$ it follows that

$$\begin{aligned}
\sup_{0\leq|x|<\infty}\sum_{j\in z}\sum_{k\in z^n}|\varphi_{j,k,a_0,b_0}| &\leq \frac{C}{1+|x|^m} \to 0 \quad as \quad |x| \to \infty, \\
\sup_{0\leq|x|<\infty}\sum_{j\in z}\sum_{k\in z^n}|\phi_{j,k,a_0,b_0}| &\leq \frac{C}{1+|x|^m} \to 0 \quad as \quad |x| \to \infty,
\end{aligned}$$

Hence we get

$$\left\|\sum_{j\in z}\sum_{k\in z^n} < f, (\varphi - \phi)_{j,k,a_0,b_0} >\right\|_p^p \leq \varepsilon\|f\|_p^p$$

which shows that $\{\phi_{j,k,a_0,b_0}\}$ is a frame (Riesz basis) in $L^p(R^n)$ for b sufficiently close to b_0 by Lemma 2.3. Hence the proof is completed.

References

[1] Balan, R., *Stability theorems for Fourier frames and wavelet Riesz basis*, The Journal of Fourier Analysis and Applications, 3(1997), 499-504.

[2] Christenson, O., *Frame Perturbation*, Proc. Amer. Math. Soc. 123 (1995), 1217-1220.

[3] Christenson, O. and Heil, C.E., *Perturbation of Banach frames and atomic decompositions*, Math. Nachr. 185(1997), 33-47.

[4] Daubechies, I., *The wavelet transform, time frequency localization and signal analysis*, IEEE Trans. Inform. Theory 36 (1990), 961-1005.

[5] Duffin, R.J. and Schaeffer, A.C., *A class of non-harmonic Fourier series*, Trans. Amer. Math. Soc. 72 (1952), 341-366.

[6] Favier, S.J. and Zalik, R.A., *On the stability of frames and Riesz basis*, Applied Comput. Harm. Anal. 2(1995), 160-173.

[7] Feichtinger, H.G. and Grähenig., *Banach spaces related to integrable group representations and their atomic decompositions*, I.J. Funct. Anal. 86 (1989), 307-340.

[8] Heil, C.E. and Walnut, D.F., *Continuous and discrete wavelet transforms*, SIAM Rev. 31 (1989), 628-666.

[9] Zhang Jing, *On the stability of wavelet and Gabor frames (Riesz basis)*, J. of Fourier Anal. 1(1999), 105-125.