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EQUILIBRIUM PROBLEM UNDER VARIOUS TYPES OF CONVEXITIES IN BANACH SPACE

NIHAR KUMAR MAHATO AND CHANDAL NAHAK*

Department of Mathematics, Indian Institute of Technology Kharagpur
Kharagpur-721302, India

Abstract. In this paper we establish the existence and uniqueness solution of the dual equilibrium problem on closed convex set. We give a link between the solutions set of equilibrium problem and its dual. The same link have also been established between invex equilibrium problem and its dual. For invex equilibrium problem, we have shown that the nonempty and boundedness of the solution sets imply the coercivity conditions of the bi-function. We have shown, how an equilibrium problem can be cast to a corresponding variational inequality problem for some special class of bi-functions.

Keywords: equilibrium problem; invex equilibrium problem; coercivity conditions; sesquilinear form.

2000 AMS Subject Classification: 90C33; 90C26

1. Introduction

Let X be a real Banach space and K be a nonempty closed convex subset of X. Consider the function $f : K \times K \to \mathbb{R}$ with $f(x, x) = 0$, for all $x \in K$. Then the equilibrium problem (in short, EP) is to find $\overline{x} \in K$, such that

$$f(\overline{x}, y) \geq 0, \text{ for all } y \in K. \tag{1}$$

*Corresponding author

E-mail addresses: nihariitkgp@gmail.com(N. K. Mahato), cnahak@maths.iitkgp.ernet.in (C. Nahak)

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Equilibrium problem is relatively new. This kind of inequalities was first considered and introduced by Ky Fan [6]. But (EP) appeared with this name in the seminal paper of Blum and Oettli in 1994 [2]. After which, many researchers made many contributions in (EP) [4, 8, 10]. (EP) plays an important role in nonlinear analysis, optimization, game theory. The (EP) includes many mathematical problems as a particular cases for example, mathematical programming problems, complementary problems, variational inequality problems, Nash equilibrium problems in noncooperative games, minimax inequality problems and fixed point problems. The (EP) relates the above problems in a very convenient way. It is well known that the (EP) is closely related to the dual equilibrium problem (in short, DEP), which states that find $u \in K$, such that

$$f(y, u) \leq 0, \text{ for all } y \in K. \tag{2}$$

This type of dual can be obtained by interchanging the arguments of the bifunction and change the sign on the left hand side of the inequality. It is obvious that such a dual satisfies the fundamental duality property "the dual of the dual is primal".

In literature, when we deal with (EP) along with the existence of its solution, the most common assumptions are the convexity of the domain and the generalized convexity and monotonicity together with some weak continuity assumptions of the function. To the best of our knowledge Karamardian and Schaible [7] introduced the concept of generalized monotonicity maps. After that Yang et al. [12] studied generalized invexity and generalized invariant monotonicity. Konnov and Schaible proposed various types of duals and primal-dual relationships which were established under certain generalized convexity and generalized monotonicity assumptions. Bianchi and Pini [4] studied equilibrium problems and its dual in a topological space. They also defined the concept of proper quasi-monotonicity for the bi-functions and proved that proper quasi-monotonicity is sharp in order to solve the dual equilibrium problem. Bianchi and Pini [2] established an equivalence between the non-empty and boundedness of the solution set and coercivity conditions under the setting of both pseudo-monotonicity and quasi-monotonicity. In pseudo-monotone case, they compared their coercivity conditions with existing conditions that appeared in the literature. The invex equilibrium problem was first studied by

Noor [10] in the setting of invexity and proved that invex equilibrium problem includes variational-like inequality problem, equilibrium problem and variational inequality problem as special cases. Hence collectively, the invex equilibrium problem covers a vast range of applications.

We study the existence and uniqueness solution of the dual equilibrium problem for the function of type $f(x, y) = g(x, y) + h(x, y)$ on closed convex set. We consider some weaker continuity conditions for the functions g and h. Here g is monotone, whereas h is not necessarily monotone, but has to satisfy upper semicontinuity (in short, u.s.c.) condition in the first argument. For invex equilibrium problem, we show that the nonempty and boundedness of the solution sets imply a coercivity condition of the bi-function. We establish a relation between (EP) and (VIP) for bounded sesquilinear functional on a Hilbert space.

2. Definitions and examples

Definition 1. Let X be a vector space and $K \subseteq X$ be a closed convex set. A function $f : K \times K \to \mathbb{R}$ is said to be monotone if $f(x, y) + f(y, x) \leq 0,$ for all $x, y \in K$.

Definition 2. A function $f : K \times K \to \mathbb{R}$ is said to be pseudomonotone if $f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0,$ for all $x, y \in K$.

Definition 3. A function $f(., y) : K \times K \to \mathbb{R}$ is said to be hemicontinuous, if for all $x \in K$ and $t \in [0, 1]$, the mapping $t \to f(ty + (1 - t)x, y)$ is continuous for all $y \in K$ (i.e. continuous on any line segment in K).

Definition 4. Let K, C are two sets with $C \subset K$. Then $core_K C = \{x \in C : C \cap (x, y) \neq \Phi,$ for all $y \in K \setminus C\}$, where (x, y) denotes the line between the points x and y.

Definition 5. Let Y be an arbitrary set in a topological vector space X and $F : Y \to X$ is a set valued mapping. If for every $y \in Y$, $F(y)$ is closed in X, then F is said to be KKM-map if convex hull of any finite set $\{y_1, y_2, ..., y_n\}$ of Y is contained in $\bigcup_{i=1}^{n} F(y_i)$.

Lemma 1. (Fan-KKM lemma) Let Y be an arbitrary set in a topological vector space X and $F : Y \to X$ is KKM-map. If $F(y_0)$ is compact for some $y_0 \in Y$. Then $\bigcap_{y \in Y} F(y) \neq \phi$.

We describe next three particular cases of (EP) which were discussed in [1].

Example 1. Optimization problem : Let $\phi : K \to \mathbb{R}$. It is requested to find $\overline{x} \in K$ such that $\phi(\overline{x}) \leq \phi(y)$, for all $y \in K$. Set $f(x, y) = \phi(y) - \phi(x)$. Then by (EP) we have to find out $\overline{x} \in K$ such that $f(\overline{x}, y) \geq 0$, for all $y \in K$. Therefore the optimization problem coincides with equilibrium problem. Also the function f is monotone in this case.

Example 2. Variational inequality problem (VIP): If we define $f(x, y) = \langle Tx, y - x\rangle$ where $T : K \to X^*$ be a given mapping, where X^* denotes the space of all continuous linear maps on X. Then (EP) collapses into the classical (VIP) which states that, find $\overline{x} \in X$ such that $\overline{x} \in K$, with $\langle T\overline{x}, y - \overline{x}\rangle \geq 0$, for all $y \in K$.

Example 3. Fixed point problem (FPP): Let X be a Hilbert space and K is a nonempty closed convex subset of X. Let $T : K \to K$ be a given mapping. (FPP) states that find $\overline{x} \in K$ such that $\overline{x} = T\overline{x}$. Set $f(x, y) = \langle x - Tx, y - x\rangle$. Then $\overline{x}$ solves (EP) if and only if $\overline{x}$ is a solution of (FPP).

3. Main results

We study the existence and uniqueness solution of the dual equilibrium problem on closed convex set. Let $f : K \times K \to \mathbb{R}$, such that $f(x, y) = g(x, y) + h(x, y)$, where

(i) $g(x, x) = 0$ and $h(x, x) = 0$, $\forall x \in K$.

(ii) $g(., x)$ and $h(., x)$ are concave $\forall x \in K$.

(iii) $g(x, .)$ and $h(x, .)$ are u.s.c $\forall x \in K$.

(iv) $-g$ is monotone and $g(., x)$ is l.s.c $\forall x \in K$.

(v) There exists a compact, convex subset C of K such that for every $x \in C \setminus core_K C$, $\exists\, a \in core_K C$, such that $f(a, x) \geq 0$.

Lemma 2. *Let us assume that* $f : K \times K \to \mathbb{R}$ *satisfies the conditions (i)-(v). Then* $\exists\, \overline{x} \in C$ *such that* $g(\overline{x}, y) \geq h(y, \overline{x})$, *for all* $y \in C$.

Proof : Consider the sets $S(y) = \{x \in C : g(x,y) \geq h(y,x),\ y \in C\}$, which are closed sets as g is l.s.c. in first argument and h is u.s.c. in second argument.

It is enough to show that $\bigcap_{y \in C} S(y) \neq \phi$.

Let $\{y_i\}$ be a finite subset of C where $i \in I \subset \mathbb{N}$ and let $\xi \in conv\ \{y_i : i \in I\}$, where $conv\ \{y_i : i \in I\}$ stands for convex hull of $\{y_i : i \in I\}$.

Then $\xi = \sum_{i \in I} \mu_i y_i$ with $\mu_i \geq 0$ and $\sum_{i \in I} \mu_i = 1$. Let us assume that $g(\xi, y_i) < h(y_i, \xi),$ for all $i \in I$. Since not all $\mu_i = 0$ together, we have

$$\sum_{i \in I} \mu_i g(\xi, y_i) < \sum_{i \in I} \mu_i h(y_i, \xi). \tag{3}$$

$$\begin{aligned} \text{Now, } \sum_{i \in I} \mu_i g(\xi, y_i) &\geq \sum_{i \in I} \sum_{j \in I} \mu_i \mu_j g(y_j, y_i) \\ &= \frac{1}{2} \sum_{i,j} \mu_i \mu_j \{g(y_i, y_j) + g(y_j, y_i)\} \geq 0. \end{aligned}$$

Again, $0 = h(\xi, \xi) \geq \sum_{i \in I} \mu_i h(y_i, \xi)$. Therefore we have $\sum_{i \in I} \mu_i h(y_i, \xi) \leq 0 \leq \sum_{i \in I} \mu_i g(\xi, y_i)$, which is a contradiction to equation (3). Hence $g(\xi, y_i) \geq h(y_i, \xi),$ for some $i \in I$.

Thus $\xi \in S(y_i)$ for some $i \in I$. So

$$conv\ \{y_i : i \in I\} \subseteq \cup \{S(y_i) : i \in I\}.$$

Since this holds for any subset $I \subset \mathbb{N}$ and $S(y)$ are compact subsets of C. It follows from KKM-lemma that $\bigcap_{y \in C} S(y) \neq \phi$. □

Lemma 3. Let $f : K \times K \to \mathbb{R}$ be a function that satisfies the conditions (i)-(v). Then $\exists\ \overline{x} \in C$ such that $g(y, \overline{x}) + h(y, \overline{x}) \leq 0,$ for all $y \in C$.

Proof : Let $y \in C$ be arbitrary and let $x_t = ty + (1-t)\overline{x},\ 0 < t \leq 1$. Then $x_t \in C$ and hence from Lemma 2, we have $g(\overline{x}, x_t) \geq h(x_t, \overline{x})$. Now

$$\begin{aligned} 0 = g(x_t, x_t) &\geq tg(y, x_t) + (1-t)g(\overline{x}, x_t) \\ &\geq tg(y, x_t) + (1-t)h(x_t, \overline{x}) \\ &\geq tg(y, x_t) + (1-t)\{th(y, \overline{x}) + (1-t)h(\overline{x}, \overline{x})\} \\ &= tg(y, x_t) + t(1-t)h(y, \overline{x}). \end{aligned}$$

Dividing by t we have, $g(y, x_t) + (1-t)h(y, \overline{x}) \leq 0$. Letting $t \to 0$ and using the hemicontinuity of g we have, $g(y, \overline{x}) + h(y, \overline{x}) \leq 0,$ for all $y \in C$. □

Lemma 4. *Suppose* $\psi : K \to R$ *is concave,* $z \in core_K C$, *such that* $\psi(z) \geq 0$, *and* $\psi(y) \leq 0,$ for all $y \in C$, *then* $\psi(y) \leq 0,$ for all $y \in K$.

Proof : Let us assume that $\psi(y) > 0$ for some $y \in K \backslash C$. Let $z_t = tz + (1-t)y, 0 < t < 1$. Then $\psi(z_t) \geq t\psi(z) + (1-t)\psi(y) > 0$. Therefore $\psi(z_t) > 0$ for all $z_t \in (z, y)$. Since $C \cap (z, y) \neq \phi$, therefore there exists $z_t \in C$ with $\psi(z_t) > 0$. Which is a contradiction as $\psi(y) \leq 0,$ for all $y \in C$. Hence $\psi(y) \leq 0,$ for all $y \in K$. □

Theorem 1. *Let us assume that* $f : K \times K \to \mathbb{R}$ *satisfies the conditions (i)-(v). Then there exists* $\overline{x} \in C$, *such that* $g(y, \overline{x}) + h(y, \overline{x}) \leq 0,$ for all $y \in K$.

Proof: By Lemma 3, $\exists\ \overline{x} \in C$ such that $g(y, \overline{x}) + h(y, \overline{x}) \leq 0,$ for all $y \in C$. Set $\psi(.) = g(., \overline{x}) + h(., \overline{x})$. Then $\psi(.)$ is concave and $\psi(y) \leq 0,$ for all $y \in C$. If $\overline{x} \in core_K C$, then in Lemma 4 choose $z = \overline{x}$ and if $\overline{x} \in C \setminus core_K C$ then set $z = a$ in assumption (iv). In both of the cases $\psi(z) \geq 0$ and $z \in core_K C$. Hence from Lemma (4), $\psi(y) \leq 0,$ for all $y \in K$. □

Corollary 1. *If* K *is compact then the coercivity assumption (iv) is satisfied vacuously with* $C = K$, *since then* $C \setminus core_K C = \phi$.

We denote by $S_k,\ S_k^D,\ S_{k,loc}$ and $S_{k,loc}^D$ the solutions set of (EP), (DEP), local solution of the (EP) and local solution of (DEP), respectively. Where $S_{k,loc}$ define by
$S_{k,loc} = \{x \in K : \exists\ r > 0, f(x, y) \geq 0, \forall y \in K, \|y - x\| < r\}$.

Theorem 2. *Let K be a closed and convex set and let $f : K \times K \to \mathbb{R}$ be a function satisfying the following conditions:*

(i) $f(x,x) = 0$, *for all* $x \in K$;

(ii) $-f(y,.)$ *is upper sign continuous, for all* $y \in K$;

(iii) for $x, y, z \in K$, *if* $f(y,x) = 0$ *and* $f(z,x) > 0$, *then* $f((1-t)y + tz, x) > 0$, *for all* $t \in (0,1)$;

(iv) $f(.,x)$ *is quasiconcave, for all* $x \in K$.

Then $S_{k,loc} \subseteq S_k^D$.

Proof: Let $\overline{x} \in S_{K,loc}$. Then there exists $r > 0$ such that $f(\overline{x}, y) \geq 0, \forall y \in K, \|\overline{x}-y\| < r$. Take any $z \in K$ and choose $\overline{z} = (1-t)\overline{x} + tz$, $\forall t \in (0,1)$ such that $\|\overline{z} - \overline{x}\| < r$.
Let $z_t = (1-t)\overline{x} + t\overline{z}$, $t \in (0,1)$. Now (i) and (iv) implies that $0 = f(z_t, z_t) \geq \min\{f(\overline{z}, z_t), f(\overline{x}, z_t)\}$. If possible, let $f(\overline{x}, z_{t'}) < f(\overline{z}, z_{t'})$ for some $t' \in (0,1)$. Then, $0 \leq f(\overline{x}, z_{t'}) \leq 0 \Rightarrow f(\overline{x}, z_{t'}) = 0$, so $f(\overline{z}, z_{t'}) > 0$.
Therefore from (iii), $f((1-t')\overline{x} + t'\overline{z}, z_{t'}) > 0 \Rightarrow f(z_{t'}, z_{t'}) > 0$, a contradiction to (i).
So we have $f(\overline{x}, z_t) \geq f(\overline{z}, z_t)$, $\forall t \in (0,1)$ and $f(\overline{z}, z_t) \leq 0$,$\forall t \in (0,1)$. Therefore from (ii), $f(\overline{z}, \overline{x}) \leq 0$. Next to show that $f(z, \overline{x}) \leq 0$, assume by contradictory that $f(z, \overline{x}) > 0$. Since $f(\overline{x}, \overline{x}) = 0$, assumption (iii) implies $f((1-t)\overline{x} + tz, \overline{x}) > 0 \Rightarrow f(\overline{z}, \overline{x}) > 0$, which is a contradiction. Therefore $f(z, \overline{x}) \leq 0$, $\forall z \in K$. So $\overline{x} \in S_K^D$. □

Denote K_r the subset of K defined as $K_r = \{x \in K : \|x\| \leq r\}$.

Theorem 3. *Assume that* $f : K \times K \to \mathbb{R}$ *satisfies condition (iii) of Theorem (2). If* $\overline{x} \in S_{k_r}^D$ *and if* $\exists$ $z \in K$ *with* $\|z\| < r$, *such that* $f(z, \overline{x}) \geq 0$. *Then* $\overline{x} \in S_k^D$.

Proof: From the assumption $f(z, \overline{x}) = 0$. If possible let there exists $y \in K \backslash K_r$ such that $f(y, \overline{x}) > 0$. Therefore $f((1-t)z + ty, \overline{x}) > 0 \ \Rightarrow f(z_t, \overline{x}) > 0$ where $z_t = (1-t)z + ty, \forall t \in (0,1)$. Since $\|z\| < r$, if $t \to 0$, $z_t \in K_r$ and we have $f(z_t, \overline{x}) > 0$, which is a contradiction as $\overline{x} \in S_{k_r}^D$. That is our assumption is wrong. Hence $f(y, \overline{x}) \leq 0$, $\forall y \in K$. □

- **The Invex Equilibrium Problems:**

Let X be a Banach space and K be a nonempty closed subset of X. Let $f : K \to X$ and $\eta : K \times K \to X$.

Definition 6. [11] Let $x \in K$. Then the set K is said to be invex at x with respect to η, if for every $x, y \in K$, $t \in [0, 1]$, we have $x + t\eta(y, x) \in K$. K is said to be an invex set with respect to η, if K is invex at each $x \in K$.

Definition 7. Let $f : K \to \mathbb{R}$. f is prequasiinvex on K if $f(y) \leq f(x) \Rightarrow f(y+t\eta(x, y)) \leq f(x)$.

From now onwards let us assume that K is a nonempty closed invex subset in X with respect to η, unless otherwise specified. For a given bi-function $f : K \times K \to \mathbb{R}$ with $f(x, x) = 0$, for all $x \in K$, the invex equilibrium problem (in short, IEP) states that to find $\overline{x} \in K$, such that

$$f(\overline{x}, y) \geq 0, \text{ for all } y \in K. \tag{4}$$

The dual invex equilibrium problem (in short, DIEP), which states that to find $\overline{x} \in K$, such that

$$f(y, \overline{x}) \leq 0, \text{ for all } y \in K. \tag{5}$$

Lemma 5. [10] Let the function $f : K \times K \to \mathbb{R}$ be pseudomonotone and hemicontinuous. If the function $f(x, .)$ is preinvex in the second argument, then the problem (4) is equivalent to problem (5).

We denote I_K and I_k^D are the solution sets of (IEP) and (DIEP), respectively. Let $I_{K,loc}$ and $I_{k,loc}^D$ are the local solution sets of (IEP) and (DIEP), respectively. Obviously, $I_k^D \subseteq I_{k,loc}^D$.

Definition 8. The function $f(., y)$ is said to be η-upper sign continuity if $f(x+t\eta(y, x), y) \geq 0 \Rightarrow f(x, y) \geq 0$, $\forall x \in K$ and $t \in (0, 1)$.

The following lemma provides a link between $I_{k,loc}$ and I_k^D. We make use here η-upper sign continuity of the function $-f(y, .)$.

Lemma 6. *Let $f : K \times K \to \mathbb{R}$ be a function satisfying the following conditions:*

(i) $f(x,x) = 0$, $\forall x \in K$;

(ii) $-f(y,.)$ *is η-upper sign continuous;*

(iii) for $x, y, z \in K$, *if* $f(y,x) = 0$ *and* $f(z,x) > 0$, *then* $f(y+t\eta(z,y),x) > 0$, $\forall t \in (0,1)$;

(iv) $-f(.,x)$ *is quasiinvex,* $\forall x \in K$.

(v) if for $r > 0$, $\|\eta(x,y)\| < r$ *implies* $\|\eta(y+t\eta(x,y),y)\| < r$.

Then, $I_{K,loc} \subseteq I^D_K$.

Proof: Let $\overline{x} \in I_{K,loc}$. Then there exists $r > 0$, such that $f(\overline{x},y) \geq 0$, $\forall y \in K$, $\|\overline{x}-y\| < r$. Take any $z \in K$ and choose $\overline{z} = \overline{x} + t\eta(z,\overline{x})$, $\forall t \in (0,1)$ such that $\|\overline{z}-\overline{x}\| < r$.
Let $z_t = \overline{x} + t\eta(\overline{z},\overline{x})$, $t \in (0,1)$. Now assumptions (i) and (iv) imply that $0 = f(z_t,z_t) \geq \min\{f(\overline{z},z_t), f(\overline{x},z_t)\}$. If possible, let $f(\overline{x},z_{t'}) < f(\overline{z},z_{t'})$ for some $t' \in (0,1)$. Then from assumption (v), $0 \leq f(\overline{x},z_{t'}) \leq 0 \Rightarrow f(\overline{x},z_{t'}) = 0$, so $f(\overline{z},z_{t'}) > 0$.
Therefore from (iii), $f(\overline{x}+t'\eta(\overline{z},\overline{x}),z_{t'}) > 0 \Rightarrow f(z_{t'},z_{t'}) > 0$, a contradiction to (i). So we have $f(\overline{x},z_t) \geq f(\overline{z},z_t)$, $\forall t \in (0,1)$ and $f(\overline{z},z_t) \leq 0$,$\forall t \in (0,1)$. Therefore from (ii), $f(\overline{z},\overline{x}) \leq 0$. Next to show that $f(z,\overline{x}) \leq 0$, assume by contradictory that $f(z,\overline{x}) > 0$. Since $f(\overline{x},\overline{x}) = 0$, assumption (iii) implies $f(\overline{x}+t\eta(z,\overline{x}),\overline{x}) > 0 \Rightarrow f(\overline{z},\overline{x}) > 0$, which is a contradiction. Therefore $f(z,\overline{x}) \leq 0$, $\forall z \in K$. So $\overline{x} \in I^D_K$. □

Lemma 7. Let K be a closed invex subset of X and $f : K \times K \to \mathbb{R}$ be such that $f(u,x) = 0$, and $f(z,x) > 0$ imply $f(u+t\eta(z,u),x) > 0, \forall t \in (0,1)$. If $\overline{x} \in I^D_{K_r}$ and $\exists\ w \in K$ with $\|w\| < r$ such that $f(w,\overline{x}) = 0$, then $\overline{x} \in I^D_K$.

Proof: From the given assumption $f(w,\overline{x}) = 0$. If possible suppose there exists $y \in K \backslash K_r$ such that $f(y,\overline{x}) > 0$. Therefore $f(w+t\eta(y,w),\overline{x}) > 0 \ \Rightarrow f(w_t,\overline{x}) > 0$ where $w_t = w+t\eta(y,w), \forall t \in (0,1)$. Since $\|w\| < r$, if $t \to 0, w_t \in K_r$ and we have $f(w_t,\overline{x}) > 0$, which is a contradiction as $\overline{x} \in I^D_K$. That is our assumption is wrong. Hence $f(y,\overline{x}) \leq 0$, $\forall y \in K$. □

Let us introduce the following coercivity property:

(C): There exists $r > 0$, such that $\forall x \in K \backslash K_r$, $\exists\ y \in K_r$, such that $f(y,x) > 0$.

Theorem 4. *Let $f : K \times K \to \mathbb{R}$ be such that*

(i) $f(x,x) = 0,\ \forall x \in K$;

(ii) for $x, y, z \in K$, *if* $f(y,x) = 0$ *and* $f(z,x) > 0$, *then* $f(y+t\eta(z,y),x) > 0$, $\forall t \in (0,1)$;

(iii) $f(x,.)$ *is prequasiinvex,* $\forall x \in K$.

If the set I_k^D is nonempty and bounded, then condition (C) holds.

Proof: Assume that condition (C) does not hold, that is for every $r > 0$, there exist $x_r \in K\backslash K_r$ such that $f(y,x_r) \leq 0, \forall y \in K_r$. Let $z \in I_K^D$, so $f(y,z) \leq 0, \forall y \in K$. Choose $z_r = x_r + t\eta(z,x_r)$, a point such that $r-1 < \|z_r\| < r$. Now from (iii),
$f(y,z_r) \leq \max\{f(y,x_r), f(y,z)\} \leq 0$. So $f(y,z_r) \leq 0, \forall y \in K_r$. Therefore, $z_r \in I_{K_r}^D$ and $\|z_r\| < r$. Hence by Lemma 7, we have $z_r \in I_K^D$. Taking z_r big enough, that means I_K^D is unbounded. Which contradicts the boundedness of I_K^D. So our assumption is wrong, which proves the theorem. $\square$

In the next theorem, the generalized monotonicity properties of f allows us to replace the prequasiinvexity of f. The pseudo-monotone case has been studied extensively, in both variational inequality and equilibrium problems setting. Our first result generalizes the Theorem 3.1 in [2].

Theorem 5. *Let $f : K \times K \to \mathbb{R}$ be a function such that*

(i) $-f$ *is pseudomonotone;*

(ii) $f(y,.)$ *is η-upper sign continuous for all* $y \in K$;

(iii) for $x, y, z \in K$, *if* $f(y,x) = 0$ *and* $f(z,x) > 0$ *implies* $f((y+t\eta(z,y),x) > 0$, $\forall t \in (0,1)$;

(iv) $-f(.,x)$ *is prequasiinvex for all* $x \in K$.

If the set I_K^D is nonempty and bounded, then the coercivity condition (C) holds.

Proof: As I_K^D is bounded, we have $I_K^D \subseteq I_{K_{r_0-1}}$ for some $r_0 > 0$. Suppose that condition (C) does not hold. Then for every $r > r_0$, there exist $x_r \in K\backslash K_r$ such that $f(y,x_r) \leq 0,\ \forall y \in K_r$. Since $-f$ is pseudomonotone, $f(x_r,y) \geq 0,\ \forall y \in K_r$. Now, let $z \in I_K^D$ and choose $z_r = x_r + t\eta(z,x_r)$ such that $r-1 < \|z_r\| < r$.

Since $-f$ is pseudomonotone $f(y,z) \leq 0 \Rightarrow f(z,y) \geq 0, \forall y \in K$. Since $-f$ is prequasiinvex, $f(z_r, y) \geq \min\{f(x_r, y), f(z,y)\} \geq 0, \forall y \in K_r$. So, $z_r \in I_{K_r} \Rightarrow z_r \in I_K^D$ (by Lemma 7). Now, we can take z_r big enough. Which contradicts the boundedness of I_K^D. $\square$

2. Relation between (EP) and (VIP)

In this section we will discuss how to cast an (EP) into a (VIP). We have considered the equilibrium bi-function f which is bounded sesquilinear functional over Hilbert space H. Let $f : K \times K \to \mathbb{R}$ be a bounded sesquilinear functional, where K be a closed subspace of the Hilbert space H. Then by Riesz representation theorem $f(x,y) = \langle Sx, y\rangle$, where $S : K \to K$ is a bounded linear operator. Since $f(x,x) = 0$, for all $x \in K$, then we have

$$f(x,y) = \langle Sx, y\rangle = \langle Sx, y\rangle - \langle Sx, x\rangle = \langle Sx, y - x\rangle. \tag{6}$$

So if the solution of the (VIP) associated with the operator S exists, then the solution of (EP) also exists. Another class of functional, namely closed linear functional also belongs to the above category. Which can be shown from the following theorem by replacing Banach space Y with the real line $\mathbb{R}$.

Theorem 6. [9] *Let X_1, X_2, Y be Banach spaces and $T : D(T) \subseteq X_1 \times X_2 \to Y$ a closed linear operator, where D is closed in $X_1 \times X_2$, then the operator T is bounded.*

When dealing with (EP) and existence of their solutions, the most common assumptions are the convexity of domain and the generalized convexity and monotonicity of the function. We construct the following example to show that the monotonicity condition of the defining bi-functions is not enough for the existence solutions of the equilibrium problem. We have the following counterexample to support our work.

Example 4. Let f be the standard inner product in $\mathbb{R}^2$. Here S will be the identity operator I. Obviously, I is a monotone operator. But f is not monotone, as if we take $x = (1,2)$ and $y = (3,4)$ then $f(x,y) + f(y,x) = \langle x, y\rangle + \langle y, x\rangle = 11 + 11 = 22 \geq 0$.

Theorem 7. *Let f be a sesquilinear and x_1, x_2 are two solutions of (EP) for the function f, then $\alpha x_1 + \beta x_2$ $(\alpha \geq 0, \beta \geq 0)$ is also a solution of the same (EP).*

Proof: The result will follow since f is linear in first argument. □

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