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DISCRETE STRUCTURES ON MANIFOLDS

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Abstract: In this exposition, we consider group theoretic arguments in order to obtain discrete structures on certain smooth manifolds. This review work essentially looks at group actions on homogeneous spaces which ultimately lead to tessellations on smooth manifolds. Apart from applications of such discretization's, we also mention the pedagogical advantages while explaining Non-Euclidean geometry.

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1. Introduction

Smooth manifolds are generally studied in the continuous set-up. However with applications in mind one wishes to develop discrete structures on such manifolds. Our goal is to make an exposition of some important cases of discretizations on manifolds and mention the applications of such structures. A discretization on a manifold is considered for various reasons. For instance the base manifold of a vector bundle needs to be discretized for quantum mechanical considerations. In problems concerning medical imaging (MRI/CT Scans) surface reconstruction is possible only with appropriate discrete structures. Similarly

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in computer graphics, animations and architectural designs especially in the context of computerized algorithms, discrete structures are increasingly being utilized. The modern subject of discrete differential geometry encompasses several such discretization schemes.

In this paper we delve into some algebraic properties associated with tessellations. Historically discretizations were required by numerical analysts, to break down the domain of the functions involved, into convenient frames. One defines a discrete structure on a smooth manifold as a discrete set of points, with additional algebraic/number theoretic/combinatorial structure like a group, groupoid, lattice, or a simplicial complex or similar other mathematical objects.

2. Preliminaries

2.1 Algebraic Quotients and Geometry:

In this section we consider a discrete collection of points which can be viewed as either collection of cosets of a Lie group or a collection of orbits under a group action. We look at a fundamental fact in Riemannian geometry to obtain nice discrete structures on homogeneous manifolds.

Definition 2.1.1: (Group Action):

Let G be a group and X be a non empty set. Then G is said to act on X through left action if there exist a map $\mu : G \times X \to \mathbf{X}$ satisfying the following conditions.

(i) $\mu(e, x) = x, \ \forall \ x \in \mathbf{X}$ whenever e is the identity of G

(ii) If g_1 and g_2 in G, then $\mu(g_1, (g_2, x)) = \mu(g_1, g_2, x), \ \forall \ x \in \mathbf{X}$.

If G is a topological group, X is a topological space and μ is a function satisfying the above said conditions then we call the triple (X, G, μ) as a transformation group.

We can define the concept of an **orbit** of an element $x \in X$.

Given a set X and a left action of a group G and $x \in X$, the orbit of x under the action of G is defined as $Orb(x) = \{g.x : g \in G\}$, that is the set of all images of x under the action of element of G.

Let M be a compact manifold and let G= Isom(M), the group of isometries of M. If Γ is a subgroup of G such that it acts properly discontinuously on M then the quotient T= $\Gamma \backslash M$ is a Riemannian manifold with M as a covering space and the projection map $\sigma : M \rightarrow T$ being an isometry. Moreover the space T acquires the geometry of the covering space M.

2.2 Example: The Action of Projective Linear Groups

The projective linear group PGL(2,C) acts on the Riemann sphere $\{C \cup \infty\}$ by mobius transformations. The subgroup that maps the upper half plane onto itself is PSL(2,R). These transformations act transitively and isometrically on **H** the upper half plane. Thus **H** is endowed with the structure of a homogenous space a detailed description of which can be given after we consider some topological aspects.

By Schwarz lemma one establishes the fact that any automorphism of the upper half plane is of the form $z \rightarrow \dfrac{az+b}{cz+d}$, a,b,c,d $\in$ R.

This can be seen as a group action of SL(2,R) on **H**. Also we see that the transformation is invariant under rescaling. Hence the conformal automorphism group of **H** is identified with SL(2,R)/ $\pm$ Id. Under this action **H** has an invariant metric namely $ds^2 = \dfrac{dx^2+dy^2}{(1-r^2)^2}$ where $r=\sqrt{x^2+y^2}$. Thus any subgroup Γ of the projective linear group PSL(2,R) which acts properly discontinuously on H leads to smooth manifolds that have a discrete structure.

2.3 Quotients of $\mathbf{R}^n$

Suppose M = $\mathbf{R}^n$. Consider the special kind of isometries of $\mathbf{R}^n$ namely translations of the form f(**x**) = **x** + **a,** i.e the vector $(x_1,x_2...x_n) \mapsto (x_1+a_1, x_2+a_2, ..., x_n+a_n)$ where **x** is any vector in R^n and $a=(a_1, a_2, ...a_n)$ is a fixed vector in R^n. The set of all such translations in R^n is a subgroup Γ of Isom(R^n) and the quotient $R^n/ \Gamma = R^n$/ZxZ which gives rise to the n-fold torus -A smooth manifold which is flat (curvature zero).

In general the kind of isometries we considered form a lattice Λ . Then the quotient group

$R^n/\ \Lambda$ is a collection of cosets which is a Hausdorff space. This space is a flat Riemannian manifold. In particular if n=2 and if Λ is the lattice of a set of vectors of the form {m**u** + n**v**: m, n$\in$**Z**}, where **u** and **v** are two fixed linearly independent vectors, then the quotient R^2/Λ is diffeomorphic to the 2-torus. The lattice considered here leads to a fundamental parallelogram and the quotient structure is equivalent to identifying the opposite sides thus giving rise to the compact surface of genus-1.

3.0 The Geometry of H and Fuchsian Groups:

The geometry of **H** is understood by the recognition of PSL(2,R) as a group that acts on the topological space **H** by isometries and this action is transitive. Thus we consider **H** as a homogenous space as alluded to earlier. Also the above mentioned metric leads to a constant Gaussian curvature of **H** namely K= -1. Now one can classify the isometries of **H** under Γ.Since the fixed point equation for any transformation **t in** Γ **is z=**Γ**z, this implies** $cz^2 + (d - a)z - b = 0$.

Based on the roots (fixed points) of the above mentioned quadratic equation we can classify **t** as:

i) Elliptic transformation, ii) Parabolic transformation and iii) Hyperbolic transformations.

The matrix topology of PSL(2, R) is equivalent to the euclidean topology on $\mathbf{R}^4$. Now any discrete subgroup of PSL(2, R) w.r.t. the matrix topology is called a Fuchsian group.

Let $\Gamma \subset$ PSL(2, R) be discrete. Then discreteness of Γ implies that Γ acts properly discontinuously on H. In other words each orbit Γ (z) is locally finite. One can easily show that any subgroup of PSL(2,R) that acts properly discontinuously on **H** is a Fuchsian group. The surfaces we are looking for are of the form $\Gamma \backslash H$ whose points correspond to the disjoint orbits of Γ in H. This quotient is a smooth manifold if and only if Γ has no elliptic elements.

3.1 Tessellations

A fundamental region for Γ is a closed region Δ such that the translates of Δ under Γ tessellate H. Discreteness of Γ implies the local finiteness of the tessellation. This

tessellation then transcends to the tessellation of the surface S= $\Gamma \backslash H$
Hence given any discrete subgroup Γ in PSL(2,R) such that Γ has no elliptic elements we get a tessellation of the surface S= $\Gamma \backslash H$.

Note: Suppose we ignore the restriction that Γ should not have elliptic elements then the quotient in general is called an orbifold. It is a (not necessarily smooth) manifold that may have conical singularities.

3.2 Riemann Surfaces

Since any hyperbolic isometry of H is also a conformal automorphism, the quotient space $\Gamma \backslash H$ inherits a natural complex structure thus rendering S= $\Gamma \backslash H$, the structure of a complex manifold of dimension 1, which is also called as a Riemann surface.
As a corollary to the Poincare-Koabe Uniformisation theorem we have the following fact:
FACT: Any smooth surface with $\chi < 0$ is conformally related to a hyperbolic quotient of the form $\Gamma \backslash H$ for some Γ in PSL(2, R).

3.3 Some Concrete Examples: Triangle groups

Example 3.3.1: Platonic tessellations: A tessellation of a Riemann surface is called platonic if the group action of symmetries is transitive on flags of faces, edges & vertices.

Platonic Riemann surface are the ones on which one can put platonic tessellations.
We now consider a Riemann surface **S** that is platonically tessellated, by regular k-genus with angle $2\pi/l$. A (2,k,l) triangle is a fundamental domain that is a hyperbolic triangle with angles $\pi/2,\pi/k,\pi/l$.

This group generated by the reflections in edges of a (2, k, l) triangle in the hyperbolic plane is called triangle group. This group acts simply transitively on the set of all (2, k, l) triangles. One can show that for the covering map from $\mathbf{H}^2$ onto **S**, the deck transformation group is a normal subgroup of the triangle group such that it is fixed point free.
Conversely one can recover the surface **S** by considering a fixed point free normal subgroup **N** of a (2, k, l) triangle group **G**.

In general for genus-0 surfaces, the regular polyhedral provide several examples of tessellations & hence lead to group actions, while for genus-1 Riemann surfaces, we have the well known description of tiling's by fundamental parallelograms. For compact Riemann surfaces of genus g $\geq$ 2 techniques of hyperbolic geometry are used to find appropriate tessellations as explained in this paper. Having seen discretisations via tessellations and group actions, we now give a general set up in the case of hyperbolic manifolds.

3.4 A General Setup

If X is an n-dimensional closed compact hyperbolic manifold then its universal cover is H^n the hyperbolic n-space.

Theorem 3.4.1: Any hyperbolic n-manifold **X** is the quotient $\Gamma \backslash H^n$ of H^n by a discrete group $\boldsymbol{\Gamma}$ of orientation preserving isometries.

4.0 Applications

Applications of discrete structures on manifolds are abound. From the pedagogical point of view it is an easy method to illustrate the concepts of non-euclidean geometry by using the underlying discrete structure namely that of group actions. In fact some software implementations are available wherein choosing the group elements of certain discrete subgroups of Lie groups, one can visualise the orbits and hence parts of the hyperbolic structures can be understood by way of tessellations.

In architectural and graphic designs tessellations are frequently used. In particular mesh descriptions of architectural designs adopt quadrilateral, triangular or hexagonal meshes, which can be mathematically viewed as tessellations. In graphics a 3-d scene description is a discrete topological surface with associated geometric properties. Projective geometry provides appropriate projections to render scenes. Thus discrete fibres play an important role in graphic designing. In quantum mechanics and numerical schemes, the base manifold needs to be appropriately discretized. Quotient structures explained in this expository article are used to define the energy functional on the base manifold.

Conflict of Interests

The author declares that there is no conflict of interests.

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