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PARAMETER ESTIMATION FOR A THIN LAYER BY MEASURING TEMPERATURE INDUCED BY A HEAT SOURCE

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Abstract. We study the temperature profile for a semi-infinite thin layer due to a Gaussian heat source. We follow the time-dependent temperature profile at the center of the top surface of the layer, deriving the asymptotic behavior and both the short time and long time behaviors. We also show that one can estimate thermal properites from various time measurements of the temperature at the center of the top surface. Finally, we investigate the statistical uncertainty in estimating parameter values from temperature measurements.

Keywords: Parameter estimation; heat equation; temperature profile.

2000 AMS Subject Classification: 34E05; 35K05

1. Introduction

Laser heating is an important technology in materials processing [14]. Since 1970s there are many studies on the temperature profiles induced by laser radiation. A few examples include the spatial distribution of the temperature rise due to a stationary Gaussain laser beam in a solid [9, 10, 11], time-dependent solution for a scanning Gaussin beam [7, 15], laser-melted front [4],

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continuous wave (cw) laser annealing of heterogeneous multilayer structure [6], temperature profiles due to a moving ellptical cw laser beam [12], the solution of the transient heat equation with mixed boundary conditions [1], the Green's function solution for a two-layer structure with scanning circular beams [5], solution for moving semi-infinite medium under the effect of a moving laser heat source [2], time-dependent heat conduction in a thin metal film [17], solution of the temperature distribution in a finite solid caused by a moving heat source [3], analysis of short-pulse laser heating of metals [13], temperature rise induced by a rotating or dithering laser beam in various solid structures [16, 18, 19, 20].

Despite all these advances in the studies of laser heating, it is still desirable to obtain analytical results even for simple cases. In this paper we consider the temperature profile of an infinite thin layer produced by a heat source in the form of a stationary Gaussian beam and reveal how to estimate thermal properties from the temperature measurements. We first give the mathematical formulation and then obtain analytical solution, paying special attention to the time-dependent temperature at the center of the heat source at the top surface. We further discuss the asymptotic behaviors of the temperature at the center of the top surface for several cases. In addition, we show how to derive thermal properties from the measurements of the temperature profile at the center of the top surface, including short time data, long time data or a full range of time data. We then evaluate the statistical error in the parameter estimations. Finally, we summarize our results.

2. Mathematical formulation and analytical solution

Consider a thin layer, which is infinite in both the x- and y- dimensions, with a thickness ε in the z-dimension. We establish the coordinate system such that the top surface of the layer is at $z = 0$ and the bottom surface is at $z = -\varepsilon$.

We assume that in the z- direction, the heat source is concentrated at a level immediately below the top surface ($z = 0$), as illustrated in Figure 1.

In the x-y plane, the heat source has a Gaussian distribution centered at the origin with standard deviation σ (we shall call σ the radius of heat source). Thus, the heat source has the

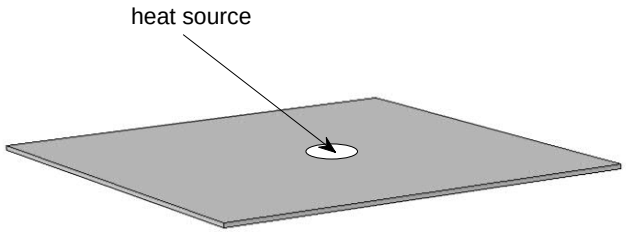

FIGURE 1. A heat source is concentrated at a level immediately below the top surface of a thin layer where $z = 0$.

distribution defined by

$$\phi(x,y,z) = \phi_{total} \cdot f(x,\sigma^2) f(y,\sigma^2) \delta(z - 0_-) \tag{1}$$

where δ is the Dirac Delta function and

$$f(x,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right) \tag{2}$$

is the probability density function of a Gaussian distribution and ϕ_{total} is the total heat flux of the heat source (total heat input per unit time) given by

$$\phi_{total} = \int \phi(x,y,z) dx dy dz. \tag{3}$$

Let us consider the situation where the thin layer is insulated at both the top surface and the bottom surface. Let $u(x,y,z,t)$ denote the temperature at position (x,y,z) at time t. Then u is governed by the heat equation with boundary and initial conditions:

$$\frac{\partial u}{\partial t} = \frac{\kappa}{\rho c_p}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{1}{\rho c_p}\phi(x,y,z) \tag{4}$$

$$\left.\frac{\partial u}{\partial z}\right|_{z=0} = 0, \qquad \left.\frac{\partial u}{\partial z}\right|_{z=-\varepsilon} = 0$$

$$u|_{t=0} = u_0,$$

where

- κ is the thermal conductivity,
- ρ is the density, and
- c_p is the specific heat.

The boundary conditions of zero heat flux at $z=0$ and at $z=-\varepsilon$ can be enforced by extending function u in the z- direction. We first extend u to $z\in[-\varepsilon,\varepsilon]$ by mirror reflection with respect to $z=0$.

$$u(z)=u(-z) \quad \text{for } z\in[0,\varepsilon].$$

Then we extend u to $z\in(-\infty,\infty)$ by periodical extension with period $=2\varepsilon$:

$$u(z+2\varepsilon)=u(z).$$

After the extension, u satisfies the initial value problem (IVP):

$$\frac{\partial u}{\partial t}=\frac{\kappa}{\rho c_p}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\frac{2\phi_{total}}{\rho c_p}f(x,\sigma^2)f(y,\sigma^2)\sum_{j=-\infty}^{+\infty}\delta(z-2j\varepsilon) \tag{5}$$

$$u|_{t=0}=u_0.$$

To solve the IVP (5), we first carry out non-dimensionalization. For mathematical convenience, we shall denote the physical variable before non-dimensionalization by x_{phy} and denote the variable after non-dimensionalization simply as x without a subscript. We nondimensionalize the physical variables as follows.

$$x=x_{phy}\frac{1}{\sigma}, \qquad y=y_{phy}\frac{1}{\sigma}, \qquad z=z_{phy}\frac{1}{\sigma},$$

$$t=t_{phy}\frac{\kappa}{\rho c_p\sigma^2},$$

$$u=(u_{phy}-u_0)\frac{\kappa\sigma}{2\phi_{total}}. \tag{6}$$

After non-dimensionalization, the initial value problem (5) becomes

$$\frac{\partial u}{\partial t}=\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+f(x,1)f(y,1)\sum_{j=-\infty}^{+\infty}\delta(z-2j\frac{\varepsilon}{\sigma}) \tag{7}$$

$$u|_{t=0}=0.$$

The fundamental solution of the one-dimensional heat equation (also called the heat kernel) can be expressed in terms of the Gaussian density as [8]

$$\frac{1}{\sqrt{4\pi t}}\exp\left(\frac{-x^2}{4t}\right) = f(x,2t). \tag{8}$$

The solution of (7) is then given by

$$\begin{aligned} u(x,y,z,t) &= \int_0^t \int_{(x',y',z')} f(x-x',2(t-t'))f(y-y',2(t-t'))f(z-z',2(t-t')) \\ &\times f(x',1)f(y',1)\sum_{j=-\infty}^{+\infty}\delta(z'-2j\frac{\varepsilon}{\sigma})dx'dy'dz'dt'. \end{aligned} \tag{9}$$

Recall the fact that the convolution of two Gaussian densities is still a Gaussian density:

$$\int f(x-x',\sigma_1^2)f(x',\sigma_2^2)dx' = f(x,\sigma_1^2+\sigma_2^2).$$

Using this fact and making a change of variables $\tau = t-t'$, we can rewrite the solution as

$$u(x,y,z,t) = \int_0^t f(x,2\tau+1)f(y,2\tau+1)\sum_{j=-\infty}^{+\infty} f\left(z-2j\frac{\varepsilon}{\sigma},2\tau\right)d\tau. \tag{10}$$

From an experimental point of view, we are interested in the temperature profile at the origin $(x,y,z) = (0,0,0)$, which is exactly the center of the heat source at the top surface. The temperature at the origin has the expression:

$$u(0,0,0,t) = \int_0^t \frac{1}{2\pi(2\tau+1)}\frac{1}{\sqrt{4\pi\tau}}h\left(\frac{\sigma^2\tau}{\varepsilon^2}\right)d\tau, \tag{11}$$

where function $h(\eta)$ is defined as

$$h(\eta) \equiv \sum_{j=-\infty}^{+\infty}\exp(-j^2/\eta). \tag{12}$$

We apply another change of variables $s = \sqrt{\tau}$ to obtain the analytical form of the temperature at the center of the top surface:

$$u(0,0,0,t) = \frac{1}{2\pi^{\frac{3}{2}}}\int_0^{\sqrt{t}}\frac{1}{(2s^2+1)}h\left(\frac{\sigma^2 s^2}{\varepsilon^2}\right)ds. \tag{13}$$

Now we study the properties of function $h(\eta)$ defined in (12).

For small η, $h(\eta)$ is very close to 1 since

$$\begin{aligned} h(\eta) &= 1+2\exp(-1/\eta)+2\exp(-4/\eta)+\cdots \\ &= 1+\text{T.S.T. (Transcendentally Small Term)} \end{aligned}$$

For large η, let $\Delta x = 1/\sqrt{\eta}$, we have

$$\begin{aligned} h(\eta) &= \sum_{j=-\infty}^{+\infty} \exp(-j^2/\eta) \\ &= \sqrt{\eta} \sum_{j=-\infty}^{+\infty} \exp(-(j\Delta x)^2)\Delta x \\ &\approx \sqrt{\eta} \int_{-\infty}^{+\infty} \exp(-x^2)dx \\ &= \sqrt{\eta}\sqrt{\pi}. \end{aligned}$$

Actually, for large η, we have

$$h(\eta) = \sqrt{\eta}\sqrt{\pi} + \text{T.S.T. (Transcendentally Small Term)}$$

The plot of $h(\eta)$, together with 1 and $\sqrt{\eta\pi}$, is shown in Figure 2. Figure 2 confirms the above asymptotic results. Namely, for small values of η, $h(\eta)$ can be closely approximated by the constant function 1; for large values of η, $h(\eta)$ can be well approximated by $\sqrt{\eta\pi}$.

3. Asymptotic solutions at the center of the top surface

Having obtained the analytical form of the time-dependent temperature at the center of the top surface, namely, $u(0,0,0,t)$, we now turn our attention to its asymptotic behaviors.

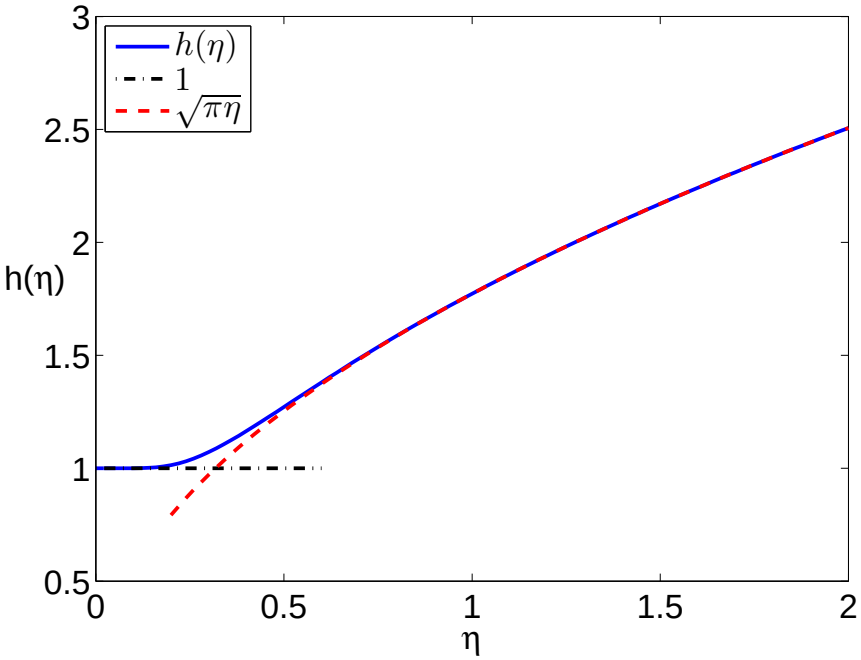

FIGURE 2. The plot of functions $h(\eta)$, 1 and $\sqrt{\eta\pi}$.

For small t (when $t\sigma^2/\varepsilon^2 << 1$), the temperature at the origin can be written as

$$\begin{aligned} u(0,0,0,t) &= \frac{1}{2\pi^{\frac{3}{2}}}\int_0^{\sqrt{t}} \frac{1}{(2s^2+1)} h\left(\frac{\sigma^2 s^2}{\varepsilon^2}\right) ds \\ &= \frac{1}{2\pi^{\frac{3}{2}}}\int_0^{\sqrt{t}} \frac{1}{(2s^2+1)} ds + \text{T.S.T} \\ &= \frac{1}{2\pi^{\frac{3}{2}}}\frac{1}{\sqrt{2}}\arctan(\sqrt{2t}) + \text{T.S.T} \\ &= \frac{1}{2\pi^{\frac{3}{2}}}\sqrt{t}\left(1-\frac{2}{3}t+O(t^2)\right). \end{aligned}$$

For large t (when $t\sigma^2/\varepsilon^2 >> 1$), the temperature at the origin is given by

$$\begin{aligned} u(0,0,0,t) &= \frac{1}{2\pi^{\frac{3}{2}}}\int_0^{\sqrt{t}} \frac{1}{(2s^2+1)} h\left(\frac{\sigma^2 s^2}{\varepsilon^2}\right) ds \\ &= \frac{1}{2\pi^{\frac{3}{2}}}\int_0^{\sqrt{t}} \frac{1}{(2s^2+1)} \left(\sqrt{\pi}\frac{\sigma}{\varepsilon}s + h\left(\frac{\sigma^2 s^2}{\varepsilon^2}\right) - \sqrt{\pi}\frac{\sigma}{\varepsilon}s\right) ds \\ &= \frac{\sigma}{2\pi\varepsilon}\int_0^{\sqrt{t}} \frac{s}{2s^2+1} ds + \frac{1}{2\pi^{\frac{3}{2}}}\int_0^{\infty} \frac{1}{2s^2+1}\left(h\left(\frac{\sigma^2 s^2}{\varepsilon^2}\right) - \sqrt{\pi}\frac{\sigma}{\varepsilon}s\right) ds + \text{T.S.T} \\ &= \frac{\sigma}{8\pi\varepsilon}\log(2t+1) + \frac{1}{2\pi^{\frac{3}{2}}} q\left(\frac{\sigma}{\varepsilon}\right) + \text{T.S.T} \end{aligned} \tag{14}$$

where function $q(\theta)$ is defined as

$$q(\xi) \equiv \int_0^{\infty} \frac{1}{2s^2+1}\left(h(\xi^2 s^2) - \sqrt{\pi}\,\xi s\right) ds. \tag{15}$$

Expanding the term $\log(2t+1)$ on the righ-hand side of (14), we arrive at

$$u(0,0,0,t) = \frac{\sigma}{8\pi\varepsilon}\log(t) + \left(\frac{1}{2\pi^{\frac{3}{2}}} q\left(\frac{\sigma}{\varepsilon}\right) + \frac{\sigma}{8\pi\varepsilon}\log(2)\right) + O(t^{-1}). \tag{16}$$

Note that $q(\sigma/\varepsilon)$ is a constant, independent of t.

We now go back to the physical variables t_{phy} and u_{phy}. Recalling (6), we then find that

$$u_{phy}(0,0,0,t_{phy}) = u_0 + \frac{2\phi_{total}}{\kappa\sigma} u\left(0,0,0,\frac{\kappa}{\rho c_p \sigma^2} t_{phy}\right). \tag{17}$$

For small values of t_{phy}, we obtain

$$\begin{aligned} u_{phy}(0,0,0,t_{phy}) &= u_0 + \frac{2\phi_{total}}{\kappa\sigma}\frac{1}{2\pi^{\frac{3}{2}}}\sqrt{t}\left(1 - \frac{2}{3}t + O(t^2)\right) \\ &= u_0 + \frac{\phi_{total}}{\pi^{\frac{3}{2}}\kappa^{\frac{1}{2}}(\rho c_p)^{\frac{1}{2}}\sigma^2}\sqrt{t_{phy}}\left(1 - \frac{2}{3}\frac{\kappa}{(\rho c_p)\sigma^2} t_{phy} + O(t_{phy}^2)\right). \end{aligned} \tag{18}$$

For large values of t_{phy}, it follows that

$$\begin{aligned} u_{phy}(0,0,0,t_{phy}) &= u_0 + \frac{2\phi_{total}}{\kappa\sigma}\frac{\sigma}{8\pi\varepsilon}\log(t) \\ &\quad + \frac{2\phi_{total}}{\kappa\sigma}\left(\frac{1}{2\pi^{\frac{3}{2}}}q\left(\frac{\sigma}{\varepsilon}\right) + \frac{\sigma}{8\pi\varepsilon}\log(2)\right) + O(t^{-1}) \\ &= u_0 + \frac{\phi_{total}}{4\pi\kappa\varepsilon}\log(t_{phy}) \\ &\quad + \frac{\phi_{total}}{4\pi\kappa\varepsilon}\left(\log\left(\frac{2\kappa}{(\rho c_p)\sigma^2}\right) + \frac{4}{\sqrt{\pi}}\frac{\varepsilon}{\sigma}q\left(\frac{\sigma}{\varepsilon}\right)\right) + O(t_{phy}^{-1}). \end{aligned} \tag{19}$$

In the following we discuss how to estimate thermal properties from the measurements of the time-dependent temperature at the center of the top surface and then give both short and long time behaviors of the temperature increase at the center of the top surface of a thin layer. Statistical errors associated with parameter estimations will be evaluated as well.

3.1 Estimation of thermal properties from short or long time measurements of temperature at the center of the top surface

We consider the situation where ϕ_{total} (the total heat flux of the heat source), σ (the Gaussian radius of the heat source) and ε (the thickness of the plate) are known, and $u_{phy}(0,0,0,t_{phy})$ is measured as a function of t_{phy}. We want to estimate two thermal parameters: ρc_p (specific heat capacity per volume) and κ (the thermal conductivity).

The first step is to fit the measured values of $(u_{phy}(0,0,0,t_{phy}) - u_0)/\sqrt{t_{phy}}$ vs t_{phy} to a linear function of t_{phy} for small t_{phy}.

$$\frac{u_{phy}(0,0,0,t_{phy}) - u_0}{\sqrt{t_{phy}}} = b_0 - b_1 \cdot t_{phy}.$$

The coefficients b_0 and b_1 are determined from measured data for small t_{phy} using the linear least squares fitting method. From the analysis above, we know that b_0 and b_1 are related to the

unknown parameters as

$$\begin{aligned} b_0 &= \frac{\phi_{total}}{\pi^{\frac{3}{2}}\kappa^{\frac{1}{2}}(\rho c_p)^{\frac{1}{2}}\sigma^2} \\ b_1 &= \frac{2}{3}\frac{\phi_{total}}{\pi^{\frac{3}{2}}\kappa^{\frac{1}{2}}(\rho c_p)^{\frac{1}{2}}\sigma^2}\cdot\frac{\kappa}{(\rho c_p)\sigma^2} \end{aligned}$$

Next in a similar manner we fit the measured values of $(u_{phy}(0,0,0,t_{phy})-u_0)$ vs $\log(t_{phy})$ to a linear function of $\log(t_{phy})$ for large t_{phy} by the relationship

$$u_{phy}(0,0,0,t_{phy})-u_0 = c_0 + c_1\cdot\log(t_{phy}).$$

The coefficients c_0 and c_1 are determined from the measured data for large t_{phy} in a linear least squares fitting where we treat $\log(t_{phy})$ as an independent variable. Coefficients c_0 and c_1 are related to the unknown parameters by

$$\begin{aligned} c_0 &= \frac{\phi_{total}}{4\pi\kappa\varepsilon} \\ c_1 &= \frac{\phi_{total}}{4\pi\kappa\varepsilon}\left(\log\left(\frac{2\kappa}{(\rho c_p)\sigma^2}\right)+\frac{4}{\sqrt{\pi}}\frac{\varepsilon}{\sigma}q\left(\frac{\sigma}{\varepsilon}\right)\right). \end{aligned}$$

In turn, the unknown parameters $(\rho c_p,\kappa)$ can be estimated from either (b_0,b_1) or (c_0,c_1). Suppose coefficients (b_0,b_1) have been calculated from measured temperature at short times. Parameters $(\rho c_p,\kappa)$ are expressed in terms of (b_0,b_1) as

$$\kappa = \sqrt{\frac{3}{2}}\cdot\frac{\phi_{total}}{\pi^{\frac{3}{2}}\sigma}\cdot b_0^{\frac{-3}{2}}b_1^{\frac{1}{2}} \tag{20}$$

$$\rho c_p = \sqrt{\frac{2}{3}}\cdot\frac{\phi_{total}}{\pi^{\frac{3}{2}}\sigma^3}\cdot(b_0b_1)^{\frac{-1}{2}}. \tag{21}$$

Similarly, based on the temperature data at long times, we can calculate coefficients (c_0,c_1). Parameters $(\rho c_p,\kappa)$ can be written in terms of (c_0,c_1) as

$$\kappa = \frac{\phi_{total}}{4\pi\varepsilon}\cdot\frac{1}{c_0} \tag{22}$$

$$\rho c_p = \frac{\phi_{total}}{2\pi\varepsilon\sigma^2}\cdot\frac{1}{c_0}\exp\left[\frac{4}{\sqrt{\pi}}\frac{\varepsilon}{\sigma}q\left(\frac{\sigma}{\varepsilon}\right)-\frac{c_1}{c_0}\right]. \tag{23}$$

It is clear that if we have temperature data either at short times or at long times, we can estimate the thermal parameters ρc_p and κ from either (20)-(21) or (22)-(23), repectively.

3.2 Number of unknown parameters that can be estimated from short and long time measurements of temperature at the center of the top surface

One may wonder that if we have temperature data at both short times and long times, would it be possible to estimate more than 2 unknown parameters from the 4 coefficients (b_0, b_1, c_0, c_1)? Below we show that it is not possible to determine the parameters if all three of $(\rho c_p, \kappa, \phi_{total})$ are unknown.

Specifically, we demonstrate that there are two different sets of $(\rho c_p, \kappa, \phi_{total})$ that produce the same value of (b_0, b_1, c_0, c_1). Suppose we multiply $(\rho c_p, \kappa, \phi_{total})$ by a factor λ

$$(\rho c_p)_{new} = \lambda(\rho c_p)$$

$$\kappa_{new} = \lambda \kappa$$

$$(\phi_{total})_{new} = \lambda \phi_{total}.$$

It is straightforward to verify that

$$(b_0)_{new} = b_0,$$

$$(b_1)_{new} = b_1,$$

$$(c_0)_{new} = c_0,$$

$$(c_1)_{new} = c_1.$$

It becomes apparent that one can estimate only two unknown parameters from the short time or long time temperature profile at the center of the top surface.

3.3 Short time behavior of the temperature increase at the center of the top surface

We study the temperature increase (over the initial temperature u_0) at the origin for short time. For small values of t_{phy}, the leading term of the temperature increase takes the form

$$u_{phy}(0,0,0,t_{phy}) - u_0 = \frac{\phi_{total}}{\pi^{\frac{3}{2}} \kappa^{\frac{1}{2}} (\rho c_p)^{\frac{1}{2}} \sigma^2} \sqrt{t_{phy}} \left(1 - \frac{2}{3} \frac{\kappa}{(\rho c_p)\sigma^2} t_{phy} + O(t_{phy}^2) \right).$$

Thus, the short time temperature increase at the origin satisfies the following properties:

- It is proportional to the total heat flux of the heat source.
- The leading term is proportional to the square root of time.
- The leading term is inversely proportional to the square root of the thermal conductivity.
- The leading term is inversely proportional to the square root of the specific heat capacity per volume.
- The leading term is inversely proportional to the square of the Gaussian radius of the heat source.
- It is independent of the thickness of the layer.

3.4 Long time behavior of the temperature increase at the center of the top surface

Now we assume a thin layer and study the temperature increase (over the initial temperature u_0) at the origin for long time. By "thin layer" we mean $\varepsilon << \sigma$. That is, the thickness of the layer is much smaller than the Gaussian radius of the heat source. We first examine the asymptotic behavior of function $q(\xi)$ for large ξ. In the definition of $q(\xi)$ given in (15), we apply a change of variables $w = \xi s$ to rewrite $q(\xi)$ as

$$q(\xi) = \frac{1}{\xi} \int_0^\infty \frac{1}{2\frac{w^2}{\xi^2} + 1} \left(h(w^2) - \sqrt{\pi} w \right) dw. \tag{24}$$

When w is large, $(h(w^2) - \sqrt{\pi} w) =$ T.S.T. Thus, in the integral above, the contribution from the region of large w is negligible. It follows that for large ξ, in the region of dominant contribution,

$\frac{w^2}{\xi^2}$ is small and we can expand in terms of $\frac{w^2}{\xi^2}$.

$$q(\xi) = \frac{1}{\xi}\int_0^\infty \left(1-2\frac{w^2}{\xi^2}\right)(h(w^2)-\sqrt{\pi}w)dw+\cdots$$

$$\equiv \frac{q_1}{\xi}+\frac{q_3}{\xi^3}+\cdots, \tag{25}$$

where coefficients q_1 and q_3 are given by

$$q_1 = \int_0^\infty (h(w^2)-\sqrt{\pi}w)dw, \tag{26}$$

$$q_3 = -\int_0^\infty 2w^2(h(w^2)-\sqrt{\pi}w)dw. \tag{27}$$

In Figure 3 we plot the function $q(\xi)$ and its approximation $\frac{q_1}{\xi}+\frac{q_3}{\xi^3}$. As shown in Figure 3, the two curves are getting closer to each other when ξ becomes bigger.

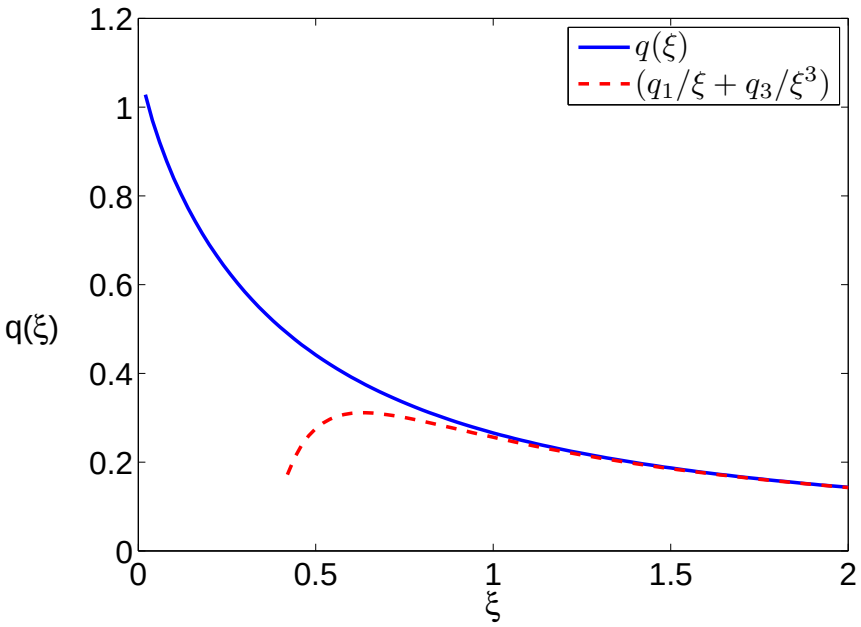

FIGURE 3. The plot of $q(\xi)$ and its approximation.

Substituting the expansion of $q(\xi)$ into (14), we obtain that for a thin layer and for large values of t the temperature increase at the origin is described by

$$\begin{aligned} & u_{phy}(0,0,0,t_{phy}) - u_0 \\ & = \frac{\phi_{total}}{4\pi\kappa\varepsilon}\left(\log(t_{phy}) + \log\left(\frac{2\kappa}{(\rho c_p)\sigma^2}\right) + \frac{4q_1}{\sqrt{\pi}}\frac{\varepsilon^2}{\sigma^2} + \frac{4q_3}{\sqrt{\pi}}\frac{\varepsilon^4}{\sigma^4} + \cdots\right). \end{aligned} \tag{28}$$

For a thin layer, the long time temperature increase at the origin possesses all of the following:

- It is proportional to the total heat flux of the heat source;
- The leading term increases logarithmically with the time;
- The leading term is inversely proportional to the thermal conductivity;
- It is inversely proportional to the thickness of the layer;
- The leading term is independent of the specific heat capacity per volume (ρc_p) and is independent of the Gaussian radius of the heat source (σ); more specifically, at long time t_{phy}, the leading term of the temperature increase is of the order $O(\log(t_{phy}))$ while the effect of ρc_p and σ on the temperature increase is of the order $O(1)$.

3.5 Fitting analytical solution to measured temperature profile over a full range of time

We consider the problem of fitting the analytical solution to the measured temperature over the full range of time. That is, instead of fitting to measured temperature for either short time or for long time, we illustrate how to use all measured temperature data points in fitting. As we will see below, by using the temperature data over the full range of time, we can determine three coefficients.

By combining equations (13) and (17), we write the physical temperature as

$$\begin{aligned} u_{phy}(0,0,0,t_{phy}) - u_0 &= \frac{2\phi_{total}}{\kappa\sigma} u\left(0,0,0,\frac{\kappa}{\rho c_p \sigma^2} t_{phy}\right) \\ &\equiv \alpha_1\, g(\sqrt{\alpha_2 t_{phy}};\, \alpha_3), \end{aligned} \tag{29}$$

where coefficients α_1, α_2 and α_3 are given by

$$\begin{aligned} \alpha_1 &= \frac{2\phi_{total}}{\kappa\sigma} \\ \alpha_2 &= \frac{\kappa}{\rho c_p \sigma^2} \\ \alpha_3 &= \frac{\sigma}{\varepsilon} \end{aligned} \tag{30}$$

and function $g(\cdot)$ is defined as

$$g(w;\alpha_3) = \frac{1}{2\pi^{\frac{3}{2}}} \int_0^w \frac{1}{(2s^2+1)} h\left(\alpha_3^2 s^2\right) ds.$$

From measured data points of $u_{phy}(0,0,0,t_{phy}) - u_0$ at various values of t_{phy}, we can estimate coefficients α_1, α_2 and α_3 by minimizing the distance between the right hand side (the fitting function) and the left hand side (the data). The theoretical expressions of α_1, α_2 and α_3 contain 5 parameters:

$$\phi_{total},\ \kappa,\ (\rho c_p),\ \sigma,\ \varepsilon.$$

Once we know the values of α_1, α_2, and α_3, we can express κ, (ρc_p) and ε in terms of ϕ_{total} and σ.

$$\begin{aligned} \kappa &= \frac{2\phi_{total}}{\alpha_1 \sigma} \\ (\rho c_p) &= \frac{2\phi_{total}}{\alpha_1 \alpha_2 \sigma^3} \\ \varepsilon &= \frac{\sigma}{\alpha_3} \end{aligned}$$

Therefore, the temperature data over a full range of time allows us to estimate three parameters.

3.6 Statistical errors in parameter estimations

Lastly, we turn to the problem of estimating coefficients $(\alpha_1, \alpha_2, \alpha_3)$ from the measured data of $u_{phy}(0,0,0,t_{phy})$ at various values of t_{phy}. We consider the case of noisy data where each measured value of $u_{phy}(0,0,0,t_{phy})$ is

$$u_{phy}^{(measured)}(0,0,0,t_{phy}) = u_{phy}(0,0,0,t_{phy}) + 0.01 \cdot N(0,1).$$

Here $N(0,1)$ denotes a random variable of normal (Gaussian) distribution with mean = 0 and variance = 1. The measurement noise in data will lead to statistical error in the estimated coefficients $(\alpha_1, \alpha_2, \alpha_3)$. We study how the statistical error varies with the distribution of time instances at which temperatures are measured. The result will serve as a guide in selecting time instances for measuring temperatures in real experiments.

In our study of statistical errors, the true values of the 3 coefficients are

$$\alpha_1 = 3, \qquad \alpha_2 = 1.5, \qquad \alpha_3 = 2.5.$$

The coefficients $(\alpha_1, \alpha_2, \alpha_3)$ are related to $u_{phy}(0,0,0,t_{phy})$ via function $g(w;\alpha_3)$, as given in equation (29). We first examine the behavior of function $g(w;\alpha_3)$. Shown in Figure 4 are plots of $g(w;\alpha_3)$ (A) as a function of w^2, which is proportional to t_{phy}; (B) as a function of w, which is proportional to $t_{phy}^{\frac{1}{2}}$; and (C) as a function of $\sqrt{w}$, which is proportional to $t_{phy}^{\frac{1}{4}}$. It appears that the best way to robustly capture all features of function $g(w;\alpha_3)$ from noisy data is to use a uniform grid in $t_{phy}^{\frac{1}{4}}$.

In our numerical simulations, we use true values of $(\alpha_1, \alpha_2, \alpha_3)$ to calculate $u_{phy}(0,0,0,t_{phy})$ at various values of t_{phy}, and then add Gaussian noise to generate a numerical data set of measured temperatures. Each numerical data set contains measured temperatures at 1000 time instances. We compare statistical errors for 3 distributions of the time instances.

- Distribution 1: time instances uniformly distributed in t_{phy} between 0 and t_{final}
- Distribution 2: time instances uniformly distributed in $t_{phy}^{\frac{1}{2}}$ between 0 and t_{final}
- Distribution 3: time instances uniformly distributed in $t_{phy}^{\frac{1}{4}}$ between 0 and t_{final}

We measure the statistical error in $(\alpha_1, \alpha_2, \alpha_3)$ as follows.

$$Err_{stats} = E\left\{\sqrt{\left(\frac{\alpha_1^{(est)} - \alpha_1}{\alpha_1}\right)^2 + \left(\frac{\alpha_2^{(est)} - \alpha_2}{\alpha_2}\right)^2 + \left(\frac{\alpha_3^{(est)} - \alpha_3}{\alpha_3}\right)^2}\right\},$$

where $\alpha_j^{(est)}$ is the estimated value of α_j from a data set and the average $E\{\cdot\}$ is calculated over 5000 independent data sets. In Figure 5, the statistical error is plotted as a function of t_{final}, respectively for each of the 3 distributions of the time instances. When the time instances are uniformly distributed in t_{phy}, the statistical error is the largest and it is sensitive to the value of

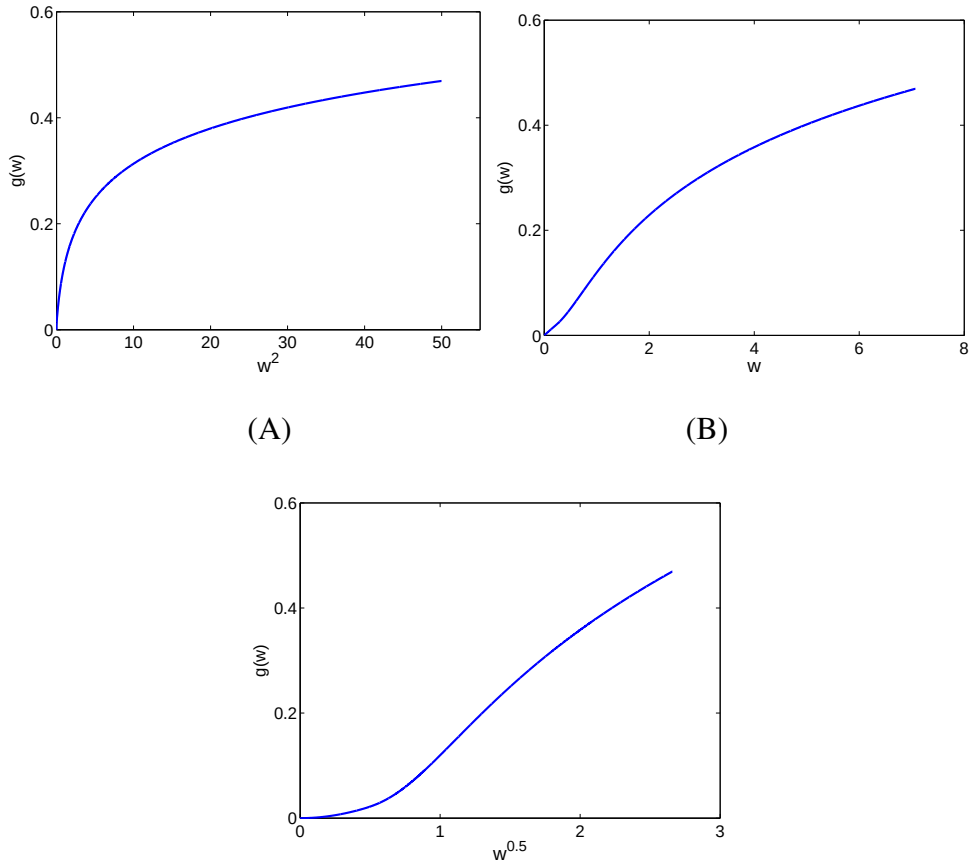

FIGURE 4. Plots of $g(w;\alpha_3)$ (A) as a function of w^2, which is proportional to t_{phy}; (B) as a function of w, which is proportional to $t_{phy}^{\frac{1}{2}}$; and (C) as a function of $\sqrt{w}$, which is proportional to $t_{phy}^{\frac{1}{4}}$.

t_{final}. When the time instances are uniformly distributed in $t_{phy}^{\frac{1}{4}}$, the statistical error is the lowest and it is also least sensitive to the value of t_{final}.

Figure 6 shows the statistical error vs t_{final} for a large range of t_{final}. It is clear that when time instances are uniformly distributed in $t_{phy}^{\frac{1}{4}}$, the parameter estimation is the most robust: it has the lowest statistical error and is least sensitive to the value of t_{final}.

4. Conclusions

We have considered the temperature distribution for a semi-infinite thin layer induced by a Gaussian heat source. In particular, we have studied the asymptotic behavior of the temperature

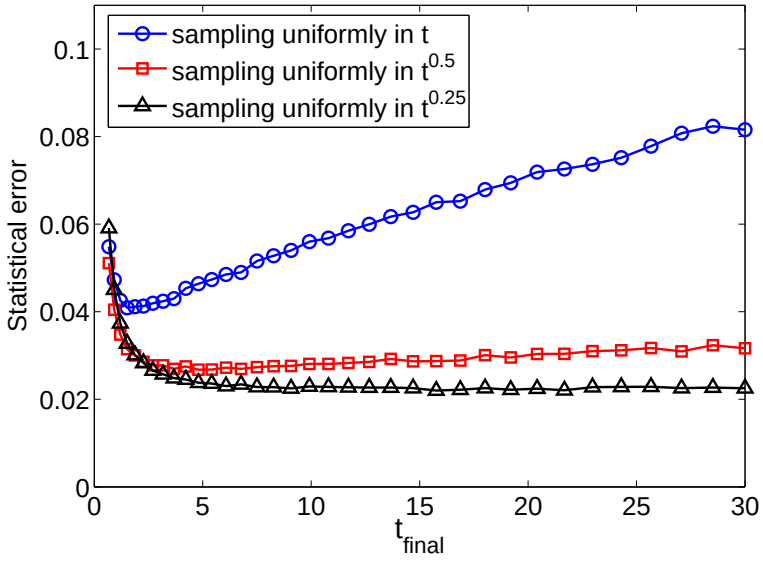

FIGURE 5. Statistical error as a function of t_{final} for each of the 3 distributions of the time instances.

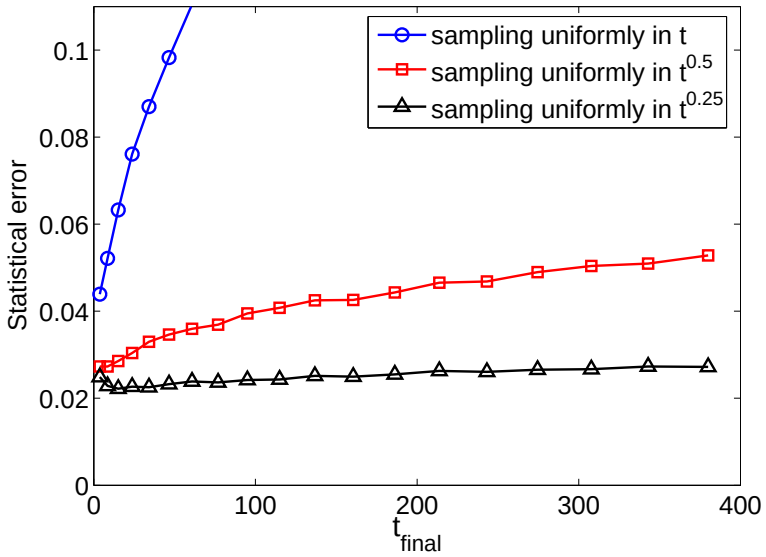

FIGURE 6. Statistical error, for each of the 3 distributions of the time instances, as a function of t_{final} for a large range of t_{final}.

at the center of the top surface of the layer. We have shown that one can estimate thermal properites from various time measurements of the temperature at the center of the top surface. We have computed the statistical errors assciated with the parameter estimations and found that the parameter estimation is the most robust if time instances for collecting temperature data are

uniformly distributed in $t^{0.25}$. As a future work, we would like to validate our predictions with experimental data.

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Conflict of Interests

The authors declare that there is no conflict of interests.

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