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ON THE PORTFOLIO STRATEGY WITH THE MEIXNER-EXPONENTIAL DISTRIBUTIONAL RELATIONSHIP

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Abstract: The Meixner process as a special type of the Levy process is related to the Meixner-Pollaczek polynomial by a Martingale relation. Since discovered, the Meixner process has been applied to financial data to show that the Normal distribution is a very poor model to fit log-returns of financial assets like stocks and indices.

In this paper, we derived a distribution which is related to Meixner and Exponential distribution. We call this distributional relationship 'the Meixner-Exponential distribution' and fit it to financial data to show how good it fits. Furthermore, we apply the distribution to determine the expected value (wealth) of an investor whose initial wealth is W_0 and whose returns is R_τ.

Keywords: Meixner-Pollaczek polynomial, Exponential distribution, Expected value, Pricing derivative.

2000 AMS Subject Classification: 91B28; 91G10, 33C45.

1. Introduction

The Meixner-Pollaczek Polynomials first discovered by [1] are known in the literature as the Meixner Polynomials of the second kind [2]. These polynomials were later studied by [3].

The polynomials are denoted by $p_n^d(x,\phi)$, with a hypergeometric representation

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$$P_n^d(x,\phi) = \frac{(2d)n}{n!} e_2^{in\phi} 2F_1\left(\begin{matrix}-n, d+ix\\ 2d\end{matrix}\middle| 1-e^{-iz\phi}\right), d>0, 0<\phi, \tag{1.1}$$

where

$$_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\middle| \partial\right) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k x^k}{(l)_k k!},$$

and

$$(a)_k := a(a+1)\ldots(a+k-1).$$

The Polynomials are completely described by the recurrence formula [4]

$$p_{-1}^d(x,\emptyset) = u, p_0^d(x,\emptyset) = 1,$$

$$(n+1)p_{n+1}^d(x,\emptyset) - 2[x\sin\emptyset + (n-d)\cos\emptyset]p_n^d(x,\emptyset) + (n+2d-1)p_{n-1}^d$$

$(x,\emptyset) = 0$, for $n \geq 1$,

and have a generating function

$$G_d(x,t) = (i-e^{i\theta}t)^{-d+ix}(1-e^{-i\theta}t)^{-d-ix} = \sum_{n=0}^{\infty} p_n^d(x,\emptyset)t^n. \tag{1. 2}$$

They are orthogonal on the real line with respect to the weight function

$$w(x;d,\rho) = |\Gamma(d+ix)|^2 exp\{(\pi-2\phi)x\}, \tag{1.3}$$

and the orthogonality is given as

$$\int_{-\infty}^{+\infty} p_n^{(d)}(x;\emptyset)p_n^{(d)}(x:\emptyset)w(x;d,\emptyset)dx = \frac{\Gamma(n+2d)}{(2\sin\phi)^{2d}n!}\delta_{mm}. \tag{1.4}$$

The Meixner process is related to the Meixner-Pollaczek polynomials by a martingale relation. It is a special type of levy process which originates from the theory of orthogonal polynomials. Its distribution belongs to the class of the infinitely divisible distribution and as such gives rise to a Levy process (The Meixner Process). The Meixner process is very flexible, has a simple structure and leads to analytically and numerically tractable formulas.

While [5] introduced the process, [6] proposed it to serve as a model in financial data. The Meixner distribution is a special case of the generalized z-distributions (GZ), defined through the characteristic function [7]

$$\emptyset_{GX}(X,a,b_1,b_2,d,m) = \left(\frac{B\left(b_1+\frac{iax}{2\pi}, b_2-\frac{iax}{2\pi}\right)}{B(b_1,b_2)}\right)^{2d} \exp(imx), \tag{1.5}$$

where $a, b_1, b_2, d > 0$ and $m \in R$.

Moreover, the monic Meixner – Pollaczek Polynomials $\{\tilde{P}_m(x;d,ç), m = 0,1,\ldots\}$

are martingales for the Meixner process $\left(a = 1, m = 0, d = 1, ç = \left(\frac{b+\pi}{2}\right)\right)$ such that;

$$E[\tilde{P}_m(M_t; ç|M_s] = \tilde{P}_m(M_s; s, ç). \tag{1.6}$$

The similarity with classical Martingale relation between standard Brownian motion $\{B_t, t \geq 0\}$ and the Hermite polynomials$\{H_m(x;\ \sigma), m = 0, 1, \dots\}$ is:

$$E\left[\widetilde{H}_m(B_t; t|B_s\right] = \widetilde{H}_m(B_s; s,). \tag{1.7}$$

The Meixner$(x;\ \alpha, \beta, \delta, m)$ is self-decomposable [8]. Therefore we have

$$w(x) = \delta\lambda(\pi - \beta)exp\left(\left(\frac{\beta+\pi}{\alpha}\right)x\right) + (\pi + \beta)exp\left(\left(\frac{\beta-\pi}{\alpha}\right)x\right)\left(sinh\left(\frac{\pi}{\alpha}x\right)\right)^{-2}, \tag{1.8}$$

with cumulant function of the self- decomposable law given as;

$$K(u) = \alpha\delta\lambda u\ tan\left(\left(\frac{\alpha u-\beta}{2}\right)\right) - \lambda m. \tag{1.9}$$

The decomposed Meixner $(x;\ \alpha, \beta, \delta, m)$ with the application of the Esscher transform to obtain the optimal option hedging strategy was done in [9], where the option price by solving the parabolic partial differential equation which arises from the Meixner-OU process was obtained.

Instead in this paper, we derive a distribution which is related to the Meixner distribution but has the distributional properties of the exponential distribution. We call this distribution the Meixner-Exponential (M-E) distribution and further fit it into some financial data, and hence obtain the future wealth of an investor via his investment using the M-E distribution.

2. Distributional Relationship

2.1 Meixner-Pollaczek-Exponential- Relationship.

In this section, we obtain the distributional relationship between the Meixner distribution and the exponential distribution in which we shall call the Meixner-Exponential distribution.

Describe for each $d > 0$ the Symmetric Meixner-Pollaczek polynomials $P_n^{(d)}(x)$ by the following recurrence relation [4]:

$$P_{-1}^{(d)}(x) = 0,\ P_0^{(d)}(x) = 1$$

and

$$(n + 1)P_{n+1}^{(d)}(x) - xP_n^{(d)}(x) + (n - 1 + 2d)P_{n-1}^{(d)}(x) = 0, n = 1,2, \dots. \tag{2.1}$$

This sequence of polynomials has the generating function

$$G_d(x,t) = \frac{e^{xarctant}}{(1+t^2)^d} = \sum_{n=0}^{\infty} P_n^{(d)}(x) t^n \quad , \tag{2.2}$$

with a hypergeometric representation given by

$$P_n^{(d)}(x) = \frac{(2d)^n}{n!} i^n 2F_1\left(\begin{matrix} -n, d + i\, {}^{x}/_{2} \\ 2d \end{matrix}\middle| 2\right), \tag{2.3}$$

and a weight function [5]

$$W_d(x) = \frac{\left|\Gamma\left(d+\frac{ix}{2}\right)\right|^2}{2\pi}. \tag{2.4}$$

The density of the Meixner distribution related to the Symmetric Meixner-Pollaczek polynomials $P_n^{(d)}(x)$ is given by [6] as;

$$\mathrm{f}(x; a, b, d, m) = \frac{(2\cos(\frac{b}{2}))^{2d}}{2a\pi\Gamma(2d)} \exp\left(\frac{b(x-m)}{a}\right) \Gamma(d + \frac{i(x-m)}{a})|^2, \tag{2.5}$$

where $a > 0, -\pi < b < \pi, d > 0 \; and \; m \in \mathbb{R}$, and weight function

$$W_d(x) = \left|\Gamma\left(d + i\left(\frac{x-m}{a}\right)\right)\right|^2 \tag{2.6}$$

The moments of this distribution are outlined in[9]. To this end we state;

Theorem 1: Given (2.5), the Meixner-Exponential distribution is:

$$f(x; a, b, d, m) = \frac{2^{(2-(a+m))d}\left(Cos\left(\frac{b}{2}\right)\right)^{2d}}{2Cosh^{(a+m)d}(t_n)} e^{-\frac{b}{a}(m-x)} \tag{2.7}$$

The proof of theorem1is a consequence of the proposition below.

Proposition 1: Let $f(t) = e^{i\omega_0 t}$ in the sense of distribution with known facts

1. $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt = 2\pi\delta(\omega - \omega_0) = 2\pi\delta_{\omega_0}(\omega.)$.

2. From $\Gamma(z) = \int_0^{\infty} e^{-s} s^z \frac{ds}{s}$, put $s = e^u$, then, $\Gamma(z) = \int_0^{\infty} e^{-(e^u)} e^{zu} du.$

3. $\cosh x = \frac{1+e^{-2x}}{2e^{-x}}$.

Then the Fourier transform of the weight function (2.6) is given by:

$$\mathfrak{F}\big(W_d(x)(t)\big) = \frac{a\pi 2^{-(a+m)d}\Gamma(2d)}{Cosh^{(a+m)d}(t)} \tag{2.8}$$

Proof: By definition

$$\mathfrak{F}\big(W_d(x)(t)\big) = \int_{-\infty}^{\infty} \left|\Gamma\left(d + i\left(\frac{x-m}{a}\right)\right)\right|^2 e^{-itx} dx$$

$$= \int_{-\infty}^{\infty} e^{-itx}\, \Gamma\left(d + i\left(\frac{x-m}{a}\right)\right) \Gamma\left(d - i\left(\frac{x-m}{a}\right)\right) dx,$$

and by fact 2

$$= \int_{-\infty}^{\infty} e^{-itx} \left(\int_{-\infty}^{\infty} e^{-e^u} e^{\left(d+i\left(\frac{x-m}{a}\right)\right)} du\right)\left(\int_{-\infty}^{\infty} e^{-e^v} e^{\left(d-i\left(\frac{x-m}{a}\right)\right)} dv\right) dx.$$

Let $y = \frac{x-m}{a}$ so that $dx = ady$, then

$$\mathfrak{F}\big(W_d(x)(t)\big) = a\int_{-\infty}^{\infty} e^{-(e^u)}e^{du}\left(e^{-(e^v)}e^{dv}\left(\int_{-\infty}^{\infty} e^{-i(-mt+(at-u+v)y)}dy\right)dv\right)du.$$

By fact 1;

$$= 2a\pi\int_{-\infty}^{\infty} e^{-(e^u)}e^{du}\left(\int_{-\infty}^{\infty} e^{-(e^v)}e^{dv}\left(\delta_{u-(a+m)t}(V)\right)dv\right)du$$

$$= 2a\pi\int_{-\infty}^{\infty} e^{-(e^u)}e^{du}e^{-(e^{u-(a+m)t})}e^{d(u-(a+m)t)}du.$$

Substituting back the variable $s = e^u$, gives

$$\mathfrak{F}\big(W_d(x)(t)\big) = 2a\pi\int_0^{\infty} e^{-s}s^d e^{-se^{-(a+m)t}}s^d e^{-(a+m)dt}\frac{ds}{s}$$

$$= 2a\pi e^{-(a+m)dt}\int_0^{\infty} s^{2d}e^{-\left(1+e^{-(a+m)t}\right)s}\frac{ds}{s}.$$

Again let $Z = \left(1+e^{-(a+m)t}\right)s$ =>$dZ = \left(1+e^{-(a+m)t}\right)ds$,such that

$$\mathfrak{F}\big(W_d(x)(t)\big) = \frac{2a\pi e^{-(a+m)dt}}{\left(1+e^{-(a+m)t}\right)2d}\int_0^{\infty} e^{-Z}Z^{2d}\frac{dZ}{Z}$$

$$= \frac{2a\pi e^{-(a+m)dt}}{\left(1+e^{-(a+m)t}\right)^{2d}}\Gamma(2d).$$

By fact 3, we have;

$$\mathfrak{F}\big(W_d(x)(t)\big) = a\pi\left(\frac{2^{-1}2e^{-t}}{1+e^{-2t}}\right)^{(a+m)d}\Gamma(2d)$$

$$= \frac{a\pi 2^{-(a+m)d}\Gamma(2d)}{Cosh^{(a+m)d}(t)} \text{ (as required).}$$

Substituting the Fourier transformed weight function (2.8) into (2.5) gives (2.7).
The moments of this distribution are straight forward and may be derived as follows:

$E(X) = \left(\frac{a}{b}\right)^2 K$, $E(X^2) = \left(\frac{a}{b}\right)^4 2K$ and $Var(X) = (2-K)\left(\frac{a}{b}\right)^2 K$, with

$$K = \frac{2^{(2-(a+m))d}\left(Cos\left(\frac{b}{2}\right)\right)^{2d}}{2Cosh^{(a+m)d}(t_n)}.$$

Notice that for $K = 1$, $E(X)$=$Var(X)$.

Corollary 1: Using the moment generating function, we obtain the Meixner-exponential distribution for $\omega = \frac{b}{2}$ and $a+m = 2$ as

$$f(x,a,b,d,m) = \frac{1}{2}e^{-\frac{b}{a}(m-x)}. \tag{2.9}$$

Proof: Replacing t by ω (where $\omega = \sum_{v\in(r,s,t)} arc\tan v$ [5]), we get

$$\int_{-\infty}^{\infty} e^{\omega x}W_d(x)dx = \frac{a\pi 2^{(2-(a+m))d}\Gamma(2d)}{(1-ts-tr-sr)^{2d}}u,$$

where $u = \{(1+t^2)(1+s^2)(1+r^2)\}^d$.
By proposition 1

$$\int_{-\infty}^{\infty} e^{\omega x} W_d(x)dx = \frac{a\pi 2^{(2-(a+m))d}\Gamma(2d)}{cos^{(a+m)d}(arctant+arctans+arctanr)}.$$

But

$$\cos(arctant + arctans + arctanr) = \frac{1-ts-tr-sr}{(1+t^2)^{0.5}(1+s^2)^{0.5}(1+r^2)^{0.5}} \;.$$

So that

$$\int_{-\infty}^{\infty} e^{\omega x} W_d(x)dx = \frac{a\pi 2^{(2-(a+m))d}\Gamma(2d)}{2cos^{(a+m)d}(\omega)} \qquad (2.10)$$

Substituting the Fourier transformed weight function (2.10) into (2.5) gives

$$\mathrm{f}(x;a,b,d,m) = \frac{(2\cos(\frac{b}{2}))^{2d}}{2cos^{(a+m)d}(\omega)} \exp\left(\frac{b(x-m)}{a}\right), \qquad (2.11)$$

which gives (2.9) for $a + m = 2$ and $\omega = \frac{b}{2}$.

It follows, therefore that the graph of the curve $x = \ln f(x,a,b,m)$ (or equivalently) $f(x,a,b,m) = \frac{b}{a}e^{(x-m)}$ is the reflection of the curve $f(x,a,b,m) = \ln x$, about the line $f = x$ as seen in figure 1 below.

The following properties are evident from the graph of

$$\frac{b}{a}e^{(x-m)}: \quad e^{(x-m)} > 0 \quad for\ all\ x,$$

$$\lim_{x\to+\infty} e^{(x-m)} = +\infty,$$

$$\lim_{x\to-\infty} e^{(x-m)} = 0.$$

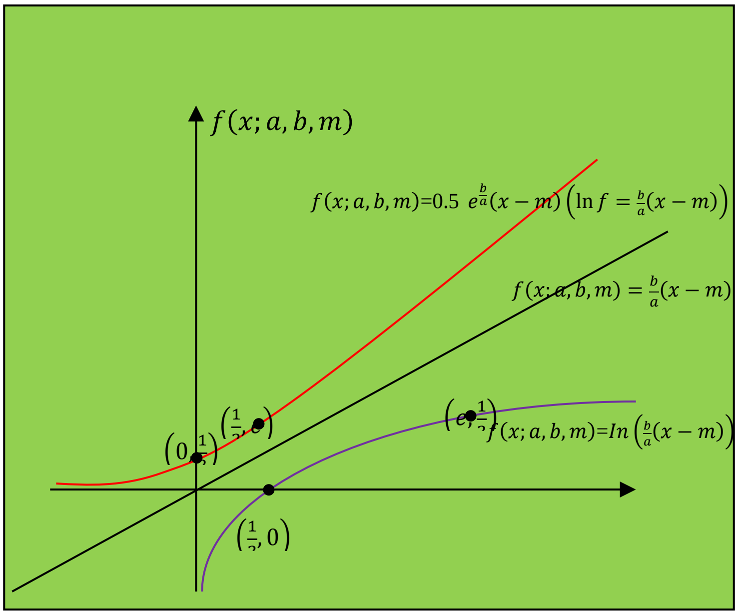


Figure 1:The graph of the curve $x = \ln f(x,a,b,m)$ (or equivalently) $f(x,a,b,m) = \frac{b}{a}e^{(x-m)}$ for $a + m = 2$ and $\omega = \frac{b}{2}$.

Lemma 1: Suppose that $i = \frac{a}{2(x-m)}$, then our Meixner-Exponential distributional relationship becomes

$$f(x,a,b,m)=\frac{1}{2}\sum_{d=0}^{\infty}\frac{(-1)^{d}\left(\frac{b}{2}\right)^{2d}}{2a(d!)^{2}}exp\left\{b\left(\frac{x-m}{a}\right)\right\} \qquad (2.12)$$

Proof: Given $i=\frac{a}{2(x-m)}$, the weight function (2.6) becomes;

$$W_d(x)=\left|\Gamma\left(d+\frac{1}{2}\right)\right|^{2}.$$

By the Legendre duplication formula, we have

$$\left(\Gamma\left(d+\frac{1}{2}\right)\right)^{2}=\left(\frac{\Gamma(\pi)\Gamma(2d)}{2^{k-1}\Gamma(d)}\right)^{2}=\pi\left(\frac{(2d!)}{2^{2d}d!}\right)^{2} \qquad (2.13)$$

Substituting (2.13) into (2.5) we have;

$$f(x,a,b,m)=\frac{\left(\cos\left(\frac{d}{2}\right)\right)^{2d}(2d!)}{2a(d!)^{2}}exp\left\{b\left(\frac{x-m}{a}\right)\right\}$$

$$=\frac{1}{2}\sum_{d=0}^{\infty}\frac{(-1)^{2d}\left(\frac{b}{2}\right)^{2d}}{2a(d!)^{2}}exp\left\{b\left(\frac{x-m}{a}\right)\right\},$$

given that $\cos\left(\frac{d}{2}\right)=\sum_{d=0}^{\infty}\frac{(-1)^{d}\left(\frac{b}{2}\right)^{2d}}{(2d!)}$.

3. Investor's Portfolio and Empirical Data Fitting to Distribution

A portfolio composition is predicted to change with investment time horizon (i.e., the time scale) in a way that can be fully determined once an adequate measure of risk is chosen. The portfolio optimization problem consists in finding the optimal diversification on a set of possibly dependent assets in order to maximize return and minimize risk. Let $x_i(t)$ be the price of asset i at time t, where time is counted for trading days in multiples of a fundamental unit (days, say).

With the notation

$$\mathfrak{z}_i(t,\tau)=\ln\left(\frac{x_i(t)}{x_i(t-\tau)}\right), \qquad (3.1)$$

the return $\mathfrak{r}_i(t,\tau)$ between time $t-\tau$ and t of asset i is defined as

$$\mathfrak{r}_i(t,\tau)=\frac{x_i(t)-x_i(t-\tau)}{x_i(t-\tau)}=e^{\mathfrak{z}_i(t,\tau)}-1. \qquad (3.2)$$

While (3.1) is the continuous return, (3.2) is the discrete return. Consider a portfolio with n_i shares of asset $i$whose initial wealth is

$$\mathrm{W}(0)=\sum_{i=1}^{N}n_i x_i(0) \qquad (3.3)$$

At time τ later, the wealth has become $\mathrm{W}(\tau)=\sum_{i=1}^{n}n_i x_i(\tau)$ and the wealth variation is ([10])

$$\delta_\tau \mathrm{W}=\mathrm{W}(\tau)-\mathrm{W}(0)=\sum_{i=1}^{n}n_i p_i(0)\frac{x_i(\tau)-x_i(0)}{x_i(0)}=\mathrm{W}(0)\sum_{i=1}^{N}\mathrm{w}_i\left(e^{\mathfrak{z}_i(\tau)}-1\right), \qquad (3.4)$$

where

$$\mathrm{w}_i = \frac{n_i x_i(0)}{\sum_{i=1}^{N} n_i x_i(0)} \tag{3.5}$$

is the fraction in capital invested in the ith asset at time 0. By (3.2), it is justifiable to write the R_τ of the portfolio over a time interval τ as the weight sum of the returns $\mathfrak{r}_i(\tau)$ of the assets $i = 1,2,\ldots,N$ over the time interval τ:

$$R_\tau = \frac{\delta_\tau \mathrm{W}}{\mathrm{W}(0)} = \sum_{i=1}^{N} \mathrm{w}_i \mathfrak{r}_i(\tau) \tag{3.6}$$

Now consider R_τ as an autonomous non-stationary τ-dimensional Bessel process govern by the scalar equation

$$dR_\tau\big(\mathfrak{r}_i(t)\big) = \frac{\delta-1}{2R_\tau(\mathfrak{z}_i)} d\mathfrak{r}_i(t) + dB\big(\mathfrak{r}_i(t)\big) \tag{3.7}$$

with $R_\tau(0) = r_0 \geq 0$.

The Bessel process R_τ is transformed into an autonomous process with finite speed measure (i.e, a process that possesses a limiting distribution) to which the Motoos theorem can be applied. More precisely, if [11]

$$x_\tau\big(\mathfrak{r}_i(t)\big) = e^{-\mathfrak{r}_i(t)}(t) R_\tau^2\big(e^{\mathfrak{r}_i(t)} - 1\big), \tag{3.8}$$

then

$$dx_\tau\big(\mathfrak{r}_i(t)\big) = (\tau - x_\tau\big(\mathfrak{r}_i(t)\big) d\mathfrak{r}_i(t) + 2\sqrt{S x_\tau(\mathfrak{r}_i, t)}\, dB\big(\mathfrak{r}_i(t)\big) \tag{3.9}$$

On the other hand, let π represent the investment policy. Under this policy the portfolio process $\{R_\tau^\pi\big(\mathfrak{r}_i(t)\big)\}$ of the investor over a time τ evolves according to the stochastic differential equation

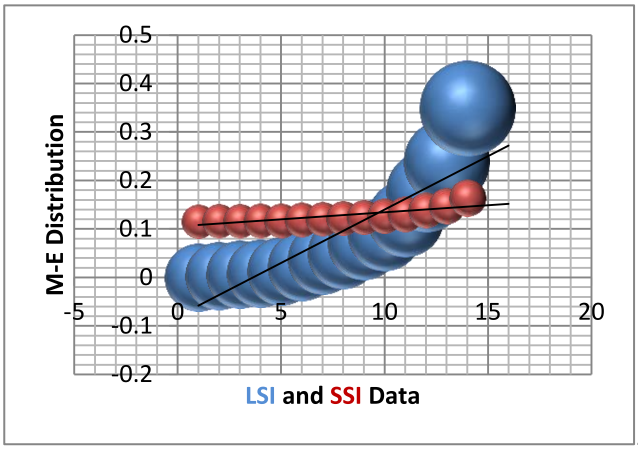
$$dR_\tau^\pi = [(\mathfrak{r} + \pi\mu)R_\tau^\pi - R_\tau^2]d\tau + \pi\sigma R_\tau^\pi dB(\tau) \tag{3.10}$$

This implies that $\{R_\tau^\pi\big(\mathfrak{r}_i(t)\big)\}$ is a temporally homogeneous diffusion process with drift function $b(x) = (\mathfrak{r} + \pi\mu)x - x^2$ and diffusion function $a^2(x) = \pi^2\sigma^2 x^2$ [12]. μ and σ are constants and $B(t)$ denotes the standard Brownian motion process.

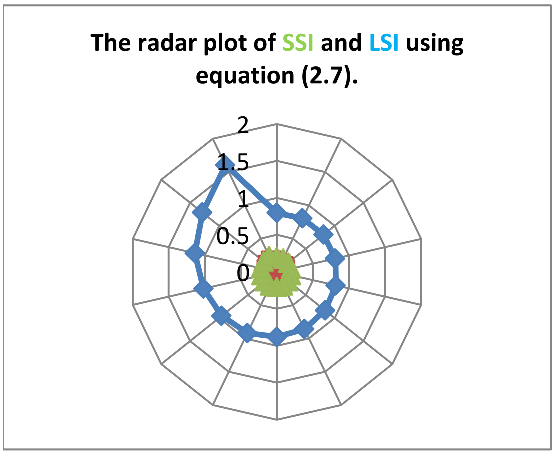
As a first application of the Meixner-Exponential process, we try to find how good our model fits to empirical entreprenual financial data. The data sets obtained consist of the closing prices for the shares denoted by $(x_i)_{1 \leq i < n}$.

The series of log returns is obtained by (3.2). We measure the portfolio for two sets data, namely, Small Scale Investment (SSI) and Large Scale Investment (LSI) from December, 2008 and 13 trading months ahead. Figures 2a and 2b below show the performance of (2.7) with the SSI and LSI respectively. To estimate the Meixner-Exponential distribution we assume independent observations as in the parent Meixner distribution and use moments. In particular case of the SSI and LSI financial data the result of the estimation procedure is given by $\hat{a} = 1.00025,$

$\hat{b} = 2.00121, \hat{d} = 2.00012, \hat{m} = 0.000156$ for the SSI and $\hat{a} = 2.00102, \hat{b} = 1.00011, \hat{d} = 1.00651, \hat{m} = 1.000102$ for the LSI.



2a



2b

Figure 2: The LSI and SSI financial data images using equation (2.7).

4.Pricing of Derivatives; The Expected Utility Approach

Starting the period with initial capital $W_0 > 0$,the investor is assumed to have preferences that are rational with respect to the end- of –period distribution of wealth $W_0 + x_\tau$. The preferences are therefore represented by the a utility function $u(W_0 + x_\tau)$ determined by the wealth variation $x_\tau = \delta x$ at the end- of-period τ. The expected utility theorem states that the investor's problem is to maximize $E^\tau[u(W_0 + x_\tau)]$, where $E(.)$ denotes the expectation operator:

$$E^\tau[u(W_0 + x_\tau)] = \int_{W_0}^{\infty} dxu(W_0 + x_\tau)G_{x_\tau}^\tau(x_\tau). \qquad (4.1)$$

The utility function $u(W)$ has a positive first derivative (wealth) and a negative second derivative (risk aversion)[11].

Let $\frac{u''}{u'} = a$, (a constant risk measure of risk aversion). It is easy to see that

$$u(W) = -exp\{-W\}. \qquad (4.2)$$

For large initial wealth, we that

$$E^\tau[u(W_0 + x_\tau)] = e^{-aW_0} \int_{-\infty}^{+\infty} e^{-ax_\tau} dx G_{x_\tau}^\tau(x_\tau) \qquad (4.3)$$

According to the fundamental theorem of asset pricing, the arbitrage free price V_t of the derivative at time $t \in [0, T]$ is given by

$$V_t = E_Q\left[e^{-r(T-t)} G(\{x_\tau, 0 \le \tau \le T\}) \middle| \mathcal{F}_t\right],$$

where Q is an equivalent Martingale measure ,$\{f_t\}_t$ is the natural filtration of ,$\{X_t\}_t$.

Let $G_{x_\tau}^\tau(x_\tau)$ be as in (2.7), then

$$V_\tau = E^\tau[u(W_0 + x_\tau)] = \frac{e^{-aW_0} 2^{(2-(a+d))d}\left(cos\left(\frac{b}{2}\right)\right)^{2d}}{cosh^{(a+m)d}(\tau_n)} \int_0^{+\infty} e^{-ax_\tau} e^{-\frac{b}{a}(m-x_\tau)} dx$$

$$= \frac{e^{-\left(aW_0 - \frac{b}{a}m\right)} 2^{(2-(a+d))d}\left(cos\left(\frac{b}{2}\right)\right)^{2d}}{cosh^{(a+m)d}(\tau_n)}$$

$$= e^{-\left(aW_0 - \frac{b}{a}m\right)} \sum_{d=0}^{\infty} \frac{(-1)^d\left(\frac{b}{2}\right)^{2d}}{(2d!)} \frac{2^{(2-(a+d))d}}{cosh^{(a+m)d}(\tau_n)}, \quad (4.4)$$

in particular. An indefinite solution is given as

$$E^\tau[u(W_0 + x_\tau)] = \frac{e^{-\left(aW_0 - \frac{b}{a}m\right)} 2^{(2-(a+d))d}\left(cos\left(\frac{b}{2}\right)\right)^{2d}}{cosh^{(a+m)d}(\tau_n)} \left[\left(\frac{e^{-\left(\frac{a^2-b}{a}\right)}}{b-a^2}\right) ax_\tau\right]$$

$$= e^{-\left(aW_0 - \frac{b}{a}m\right)} \sum_{d=0}^{\infty} \frac{(-1)^d\left(\frac{b}{2}\right)^{2d}}{(2d!)} \frac{2^{(2-(a+d))d}}{cosh^{(a+m)d}(\tau_n)} \left[\left(\frac{e^{-\left(\frac{a^2-b}{a}\right)}}{b-a^2}\right) ax_\tau\right]. \quad (4.5)$$

5. Conclusion

Equation (4.4) shows an asset allocation strategy that is continuously rebalanced so as to always keep a fixed constant proportion of wealth investor at each point in time. The fixed investment policy $\boldsymbol{\pi}$ is the process $\{\mathfrak{r}_\pi(\tau), \tau \geq 0\}$ .For such investment strategies in continuous time, the rate of return on investment is defined as the net gain in wealth divided by the cumulative investment. For the policy under which total wealth is always invested in the risk-free asset, we have by [12]

$$\mathfrak{r} = \frac{\boldsymbol{e^{\mathfrak{z}(t)} - 1}}{\boldsymbol{\int_0^t e^{\mathfrak{z}s} ds}} \ .$$

The mean return on investment is maximized by the same strategy that maximizes logarithmic utility, which is also known to maximize the exponential rate at which wealth grows.

Notice also that in (2.7), $f(x, a, b, d, m)$ will optimally decrease over time if $m > x$, but increase over time if $m < x$.

Conflict of Interests

The author declares that there is no conflict of interests.

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