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FIRST ORDER LINEAR NON HOMOGENEOUS ORDINARY DIFFERENTIAL EQUATION IN FUZZY ENVIRONMENT BASED ON LAPLACE TRANSFORM

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Abstract: In this paper the solution procedure of First Order Linear Non Homogeneous Ordinary Differential Equation (FOLNODE) is described in fuzzy environment. Here coefficients and /or initial condition of FOLNODE are considered as Generalized Triangular Fuzzy Numbers (GTFNs).The solution procedure of the FOLNODE is developed by Laplace transform. It is illustrated by numerical examples. Finally an imprecise concentration problem is described in fuzzy environment.

Keywords: Fuzzy Differential Equation, Generalized Triangular fuzzy number, 1st Order differential equation, Laplace transforms.

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1.Introduction:

In recent years it is seen that Fuzzy Differential Equation (FDE) has been emerging field among the researchers. From the theoretical point of view and as well as of their

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applications FDE is a proven important topic. For example, in HIV model [1], decay model [2],predator-prey model [3],population model [4],civil engineering [5] ,hydraulic models [6], Friction model [7],Growth model [8], Bacteria culture model [9]. It has been found that usage of FDE is a natural way in terms of modeling dynamical system under probabilistic uncertainty. First order linear FDE are considered to be one of the simplest FDE which may implement in many applications.

The advent of fuzzy derivative was first introduced by S.L.Change and L.A.Zadeh in [10].D.Dubois and Prade in [11] discussed differentiation with every aspects of fuzzy. The differential of fuzzy functions were immensely contributed by M.L.Puri and D.A.Ralesec in [12] andR.Goetschel and W.Voxman in [13]. The fuzzy differential equation and initial value problems were vastly studied by O.Kaleva in [14,15] and by S.Seikkala in [16]. Derivatives of fuzzy function was compared by Buckley and Feuring [17]which have been presented in the various manuscript by comparing the different solutions, one may obtain to the FDE's using these derivatives.

In many papers initial condition of a FDE was taken as different type of fuzzy numbers. Buckley et al [18] used triangular fuzzy number , Duraisamy&Usha [19] used Trapezoidal fuzzy number, Bede et al [20] used LR type fuzzy number.

Laplace transform is a very useful tool to solve differential equation. Laplace transforms give the solution of a differential equations satisfying the initial condition directly without use the general solution of the differential equation. Fuzzy Laplace Transform (FLT) was first introduced by Allahviranloo&Ahmadi [21].Here first order fuzzy differential equation with fuzzy initial condition is solved by FLT. Tolouti&Ahmadi [22] applied the FLT in 2^{nd} order FDE. FLT also used to solve many areas of differential equation. Salahshour et al [23] used FLT in Fuzzy fractional differential equation. Salahshour&Haghi used FLT in Fuzzy Heat Equation [24]. Ahmad et al [25] used FLT in Fuzzy Duffing's Equation.

The structure of this paper is as follows: In first two sections, we introduce some concepts and introductory material to deal with the FDE. Solution procedure of 1^{st} order linear non homogeneous fuzzy ordinary differential equation (FODE) is discussed in section 3. In section 4 there are an application. At the end in section 5 of the paper we present some conclusion and topics for future research.

2. Preliminary concept:

Definition 2.1: Fuzzy Set: A fuzzy set $\tilde{A}$ in a universe of discourse X is defined as the following set of pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X)\}$. Here $\mu_{\tilde{A}}$: X→[0,1] is a mapping called the membership value of x∈X in a fuzzy set $\tilde{A}$.

Definition 2.2: Height: The height $h(\tilde{A})$, of a fuzzy set$\tilde{A} = (x, \mu_{\tilde{A}}(x) : x \in X)$, is the largest membership grade obtained by any element in that set i.e. $h(\tilde{A})$=sup $\mu_{\tilde{A}}(x)$.

Definition 2.3: Convex Fuzzy sets:$\tilde{A}$ is fuzzy convex, i.e. ∀ x,y∈ R and $0 \leq \lambda \leq 1$, $\tilde{A}(\lambda x + (1-\lambda)y) \geq \min\{\tilde{A}(x), \tilde{A}(y)\}$.

Definition 2.4:α**-Level or $\boldsymbol{\alpha}$-cut of a fuzzy set:**The α-level set (or interval of confidence at level α or α-cut) of the fuzzy set $\tilde{A}$ of X is a crisp set A_α that contains all the elements of X that have membership values in A greater than or equal to α i.e. $\tilde{A} = \{x, \mu_{\tilde{A}}(x) \geq \alpha, x \in X, \alpha \in [0,1]\}$.

Definition 2.5: Fuzzy Number: $\tilde{A} \in \mathcal{F}(R)$ is called a fuzzy number where R denotes the set of whole real numbers if

i. $\tilde{A}$is normal i.e. $x_0 \in R$ exists such that $\mu_{\tilde{A}}(x_0) = 1$.
ii. $\forall \boldsymbol{\alpha} \in (0,1] A_\alpha$is a closed interval.

If $\tilde{A}$ is a fuzzy number then $\tilde{A}$ is a convex fuzzy set and if $\mu_{\tilde{A}}(x_0) = 1$ then $\mu_{\tilde{A}}(x)$ is non decreasing for $x \leq x_0$ and non increasing for $x \geq x_0$.
The membership function of a fuzzy number $\tilde{A}(a_1, a_2, a_3, a_4)$ is defined by

$$\mu_{\tilde{A}}(x) = \begin{cases} 1, & x \in [a_2, a_3] \neq \phi \\ L(x), & a_1 \leq x \leq a_2 \\ R(x), & a_3 \leq x \leq a_4 \end{cases}$$

Where L(x) denotes an increasing function and $0 < L(x) \leq 1$ and R(x) denotes a decreasing function and $0 \leq R(x) < 1$.

Definition 2.6:Generalized Fuzzy number (GFN): Generalized Fuzzy number $\tilde{A}$ as $\tilde{A} = (a_1, a_2, a_3, a_4; \omega)$,where $0 < \omega \leq 1$, and a_1, a_2, a_3, a_4 $(a_1 < a_2 < a_3 < a_4)$ are real numbers. The generalized fuzzy number $\tilde{A}$ is a fuzzy subset of real line R, whose membership function $\mu_{\tilde{A}}(x)$ satisfies the following conditions:

1) $\mu_{\tilde{A}}(x)$: R →[0, 1]

2) $\mu_{\tilde{A}}(x) = 0$ for$x \leq a_1$

3) $\mu_{\tilde{A}}(x)$ is strictly increasing function for $a_1 \leq x \leq a_2$

4) $\mu_{\tilde{A}}(x) = w$ for $a_2 \leq x \leq a_3$

5) $\mu_{\tilde{A}}(x)$ is strictly decreasing function for $a_3 \leq x \leq a_4$

6) $\mu_{\tilde{A}}(x) = 0$ for $a_4 \leq x$

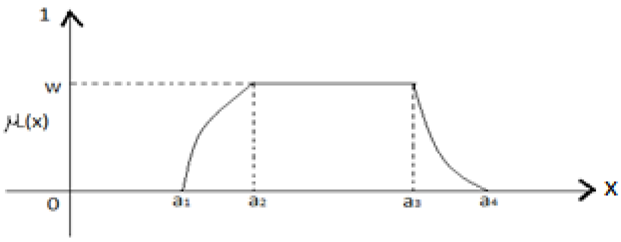

Fig-2.1:-Generalized Fuzzy Number

Definition 2.7:Generalized TFN: If a_2=a_3 then $\tilde{A}$ is called a GTFN as $\tilde{A} = (a_1, a_2, a_4;\omega)$ or $(a_1, a_3, a_4;\omega)$ with membership function

$$\mu_{\tilde{A}}(x) = \begin{cases} \omega \frac{x-a_1}{a_2-a_1} & if\ a_1 \leq x \leq a_2 \\ \omega \frac{a_4-x}{a_4-a_2} & if\ a_2 \leq x \leq a_4 \\ 0 & otherwise \end{cases}$$

Definition 2.8: TFN: If a_2=a_3, $\omega = 1$ then $\tilde{A}$ is called a TFN as $\tilde{A} = (a_1, a_2, a_3)$ or $\tilde{A} = (a_1, a_3, a_4)$

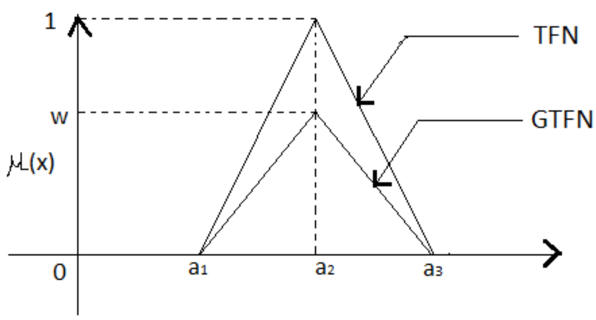

Fig-2.2:- GTFN and TFN

Definition 2.9: Multiplication of two GTFN: If$\tilde{A} = (a_1, a_2, a_3;\omega)$ and $\tilde{B} = (b_1, b_2, b_3;\beta)$ are two GTFN then $\tilde{A}.\tilde{B} \approx (a_1b_1, a_2b_2, a_3, b_3; \eta)$ where $\eta = \min\{\omega, \beta\}$.

Definition 2.10: Inverse GTFN: If $\tilde{A} = (a_1, a_2, a_3; \omega)$ is a GTFN then its inverse denoted by $\tilde{A}^{-1} \approx (\frac{1}{a_3}, \frac{1}{a_2}, \frac{1}{a_1}; \omega)$

Definition 2.11: Fuzzy ordinary differential equation (FODE): Consider a simple 1st Order Linear non-homogeneous Ordinary Differential Equation (ODE) as follows:

$\frac{dx}{dt} = kx + x_0$ with initial condition $x(t_0) = \gamma$

The above ODE is called FODE if any one of the following three cases holds:

(i) Only γ is a generalized fuzzy number (Type-I).

(ii) Only k is a generalized fuzzy number (Type-II).

(iii) Both k and γ are generalized fuzzy numbers (Type-III).

Definition 2.12: Strong and Weak solution of FODE: Consider the 1st order linear non homogeneous fuzzy ordinary differential equation $\frac{dx}{dt} = kx + x_0$with $(t_0) = x_0$.

Here k or (and) x_0 be generalized fuzzy number(s).

Let the solution of the above FODE be $\tilde{x}(t)$ and its α -cut be $x(t, \alpha) = [x_1(t, \alpha), x_2(t, \alpha)]$.

If $x_1(t, \alpha) \le x_2(t, \alpha) \forall\, \alpha \in [0, \omega]$where $0 < \omega \le 1$ then $\tilde{x}(t)$ is called strong solution otherwise $\tilde{x}(t)$ is called weak solution and in that case the α-cut of the solution is given by

$x(t, \alpha) = [\min\{x_1(t, \alpha), x_2(t, \alpha)\}, \max\{x_1(t, \alpha), x_2(t, \alpha)\}]$.

Definition 2.13: [26] Let $f: (a, b) \to E$ and $x_0 \in (a, b)$. We say that f is strongly generalized differential at x_0 (Bede-Gal differential) if there exists an element $f'(x_0) \in E$, such that

(i) for all $h > 0$ sufficiently small, $\exists\, f(x_0 + h) -^h f(x_0)$, $\exists\, f(x_0) -^h f(x_0 - h)$ and the limits(in the metric D)

$$\lim_{h \searrow 0} \frac{f(x_0 + h) -^h f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) -^h f(x_0 - h)}{h} = f'(x_0)$$

Or

(ii) for all $h > 0$ sufficiently small, $\exists\, f(x_0) -^h f(x_0 + h), \exists\, f(x_0 - h) -^h f(x_0)$ and the limits(in the metric D)

$$\lim_{h\searrow 0} \frac{f(x_0) -^h f(x_0 + h)}{-h} = \lim_{h\searrow 0} \frac{f(x_0 - h) -^h f(x_0)}{-h} = f'(x_0)$$

Or

(iii) for all $h > 0$ sufficiently small, $\exists\, f(x_0 + h) -^h f(x_0), \exists\, f(x_0 - h) -^h f(x_0)$ and the limits(in the metric D)

$$\lim_{h\searrow 0} \frac{f(x_0 + h) -^h f(x_0)}{h} = \lim_{h\searrow 0} \frac{f(x_0 - h) -^h f(x_0)}{-h} = f'(x_0)$$

Or

(iv) for all $h > 0$ sufficiently small, $\exists\, f(x_0) -^h f(x_0 + h), \exists\, f(x_0) -^h f(x_0 - h)$ and the limits(in the metric D)

$$\lim_{h\searrow 0} \frac{f(x_0) -^h f(x_0 + h)}{-h} = \lim_{h\searrow 0} \frac{f(x_0) -^h f(x_0 - h)}{h} = f'(x_0)$$

(h and $-h$ at denominators mean $\frac{1}{h}$ and $\frac{-1}{h}$, respectively).

Definition 2.14: [27] Let $f: R \to E$ be a function and denote $f(t) = (\underline{f}(t,r), \overline{f}(t,r))$, for each $r \in [0,1]$. Then (1) If f is (i)-differentiable, then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable function and $f'(t) = (\underline{f}'(t,r), \overline{f}'(t,r))$.(2)) If f is (ii)-differentiable, then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable function and $f'(t) = (\overline{f}'(t,r), \underline{f}'(t,r))$.

Definition 2.15: Let $f: [0,T] \to \Re_{\mathcal{F}}$. The integral of f in $[0,T]$, (denoted by $\int_{[0,T]} f(t)dt$ or, $\int_0^T f(t)dt$) is defined levelwise as the set if integrals of the (real) measurable selections for $[f]^r$, for each $r \in [0,1]$. We say that f is integrable over $[0,T]$ if $\int_{[0,T]} f(t)dt \in \Re_{\mathcal{F}}$ and we have

$\left[\int_0^T f(t)dt\right]^r = \left[\int_0^T \underline{f}^r(t)dt , \int_0^T \overline{f}^r(t)dt\right]$ for each $r \in [0,1]$.

3. Solution Procedure of 1st Order Linear Non Homogeneous FODE

The solution procedure of 1st order linear non homogeneous FODE of Type-I, Type-II and Type-III are described. Here fuzzy numbers are taken as GTFNs.

3.1. Solution Procedure of 1st Order Linear Non Homogeneous FODE of Type-I

Consider the initial value problem $\frac{dx}{dt} = Kx + x_0$ ....(3.1.1)

with fuzzy initial condition(IC) $\tilde{x}(0) = \widetilde{\gamma_0} = (\gamma_1, \gamma_2, \gamma_3; \omega)$

Let $\tilde{x}(t)$ be a solution of FODE (3.1.1) and $x(t,\alpha) = [x_1(t,\alpha), x_2(t,\alpha)]$ be the α-cut of $\tilde{x}(t)$.

Hence $(\widetilde{\gamma_0})_\alpha = \left[\gamma_1 + \frac{\alpha l_{\gamma_0}}{\omega}, \gamma_3 - \frac{\alpha r_{\gamma_0}}{\omega}\right] \forall\ \alpha \epsilon\ [0,\omega], 0 < \omega \leq 1$

Where $l_{\gamma_0} = \gamma_2 - \gamma_1andr_{\gamma_0} = \gamma_3 - \gamma_2$

Here we solve the given problem for $k > 0$ and $k < 0$ respecively.

Case 3.1.1: When $k > 0$

The FODE (3.1.1) becomes

$\frac{dx_1(t,\alpha)}{dt} = kx_1(t,\alpha) + x_0$ ....(3.1.2)

$\frac{dx_2(t,\alpha)}{dt} = kx_2(t,\alpha) + x_0$ ....(3.1.3)

Taking Laplace Transform both sides of (3.1.2) we get

$l\left\{\frac{dx_1(t,\alpha)}{dt}\right\} = l\{kx_1(t,\alpha)\} + l\{x_0\}$

Or, $sl\{x_1(t,\alpha)\} - x_1(0,\alpha) = kl\{x_1(t,\alpha)\} + \frac{x_0}{s}$

Or, $l\{x_1(t,\alpha)\} = \frac{\left(\gamma_1+\frac{\alpha l_{\gamma_0}}{\omega}\right)}{s-k} + \frac{x_0}{k}\left(\frac{1}{s-k} - \frac{1}{s}\right)$ [$\because$ $x_1(0,\alpha) = \gamma_1 + \frac{\alpha l_{\gamma_0}}{\omega}$]

Taking inverse Laplace transform we get

$$x_1(t,\alpha) = \left(\gamma_1 + \frac{\alpha l_{\gamma_0}}{\omega}\right) l^{-1}\left\{\frac{1}{s-k}\right\} + \frac{x_0}{k} l^{-1}\left\{\frac{1}{s-k}\right\} - \frac{x_0}{k} l^{-1}\left\{\frac{1}{s}\right\}$$

Or, $x_1(t,\alpha) = -\frac{x_0}{k} + \left\{\frac{x_0}{k} + \left(\gamma_1 + \frac{\alpha l_{\gamma_0}}{\omega}\right)\right\} e^{kt}$ ....(3.1.4)

Similarly using Laplace transform of (3.1.3) we get

$x_2(t,\alpha) = -\frac{x_0}{k} + \left\{\frac{x_0}{k} + \left(\gamma_3 - \frac{\alpha r_{\gamma_0}}{\omega}\right)\right\} e^{kt}$ ....(3.1.5)

Now $\frac{\partial}{\partial\alpha}[x_1(t,\alpha)] = \frac{l_{\gamma_0}}{\omega} e^{kt} > 0$ and $\frac{\partial}{\partial\alpha}[x_2(t,\alpha)] = -\frac{r_{\gamma_0}}{\omega} e^{kt} < 0$

and $x_1(t,\omega) = -\frac{x_0}{k} + \left\{\frac{x_0}{k} + \gamma_2\right\} e^{kt} = x_2(t,\omega)$

So the solution of (3.1.1) is a strong solution

The α-cut of the solution is

$$(\tilde{x}(t))_\alpha = -\frac{x_0}{k} + \left[\frac{x_0}{k} + \gamma_1 + \frac{\alpha}{\omega}(\gamma_2 - \gamma_1), \frac{x_0}{k} + \gamma_3 - \frac{\alpha}{\omega}(\gamma_3 - \gamma_2)\right] e^{kt}$$

$$= -\frac{x_0}{k} + \left[\frac{x_0}{k} + \left(\gamma_1 + \frac{\alpha l_{\gamma_0}}{\omega}\right), \frac{x_0}{k} + (\gamma_3 - \frac{\alpha r_{\gamma_0}}{\omega})\right] e^{kt}$$

So, $\tilde{x}(t) = (e^{kt} - 1)\frac{x_0}{k} + (\gamma_1, \gamma_2, \gamma_3; \omega)e^{kt}$

Example-3.1.1: Consider the FODE $\frac{dx}{dt} = \frac{1}{10}x + 5$ with IC $\tilde{x}(t=0) = (8,12,16; 0.8)$.

The strong solution is $\tilde{x}(t) = 50(e^{\frac{1}{10}t} - 1) + (8,12,16; 0.8)e^{\frac{1}{10}t}$

Case 3.1.2: when $k < 0$, let $k = -m$ where m is a positive real number.

Then the FODE (3.1.1) becomes

$\frac{dx_1(t,\alpha)}{dt} = -mx_2(t,\alpha) + x_0$ ....(3.1.6)

$\frac{dx_2(t,\alpha)}{dt} = -mx_1(t,\alpha) + x_0$ ....(3.1.7)

Taking Laplace transform both sides of (3.6.1) we get

$l\left\{\frac{dx_1(t,\alpha)}{dt}\right\} = l\{-mx_2(t,\alpha)\} + l\{x_0\}$

Or, $sl\{x_1(t,\alpha)\} - x_1(0,\alpha) = -ml\{x_2(t,\alpha)\} + l\{x_0\}$

Or, $sl\{x_1(t,\alpha)\} + ml\{x_2(t,\alpha)\} = \left(\gamma_1 + \frac{\alpha l_{\gamma_0}}{\omega}\right) + \frac{x_0}{s}$ ....(3.1.8)

Taking Laplace Transform both sides of (3.1.7) we get

$l\left\{\frac{dx_2(t,\alpha)}{dt}\right\} = l\{-mx_1(t,\alpha) + x_0\}$

Or, $sl\{x_2(t,\alpha)\} - x_2(0,\alpha) = -ml\{x_1(t,\alpha)\} + l\{x_0\}$

Or, $ml\{x_1(t,\alpha)\} + sl\{x_2(t,\alpha)\} = \left(\gamma_3 - \frac{\alpha r_{\gamma_0}}{\omega}\right) + \frac{x_0}{s}$ ....(3.1.9)

Solving (4) and (5) we get

$l\{x_1(t,\alpha)\} = \left(\gamma_1 + \frac{\alpha l_{\gamma_0}}{\omega}\right)\frac{s}{s^2-m^2} - \left(\gamma_3 - \frac{\alpha r_{\gamma_0}}{\omega}\right)\frac{m}{s^2-m^2} + \frac{x_0}{m}\left\{\frac{1}{s} - \frac{1}{s+m}\right\}$..............(3.1.10)

and

$l\{x_2(t,\alpha)\} = \left(\gamma_3 - \frac{\alpha r_{\gamma_0}}{\omega}\right)\frac{s}{s^2-m^2} - \left(\gamma_1 + \frac{\alpha l_{\gamma_0}}{\omega}\right)\frac{m}{s^2-m^2} + \frac{x_0}{m}\left(\frac{1}{s} - \frac{1}{s+m}\right)$ ....(3.1.11)

Taking inverse Laplace Transform of (3.1.10) we get

$x_1(t,\alpha)$

$$= \left(\gamma_1 + \frac{\alpha l_{\gamma_0}}{\omega}\right) l^{-1}\left\{\frac{s}{s^2 - m^2}\right\} - \left(\gamma_3 - \frac{\alpha r_{\gamma_0}}{\omega}\right) l^{-1}\left\{\frac{m}{s^2 - m^2}\right\} + \frac{x_0}{m} l^{-1}\left\{\frac{1}{s}\right\} - \frac{x_0}{m} l^{-1}\left\{\frac{1}{s+m}\right\}$$

$$= \left(\gamma_1 + \frac{\alpha l_{\gamma_0}}{\omega}\right)\cosh mt - \left(\gamma_3 - \frac{\alpha r_{\gamma_0}}{\omega}\right)\sinh mt + \frac{x_0}{m} - \frac{x_0}{m}e^{-mt}$$

$$= \frac{x_0}{m} + \frac{1}{2}\left\{-\frac{x_0}{m} + \gamma_1 + \gamma_3 + \frac{\alpha}{\omega}\left(l_{\gamma_0} - r_{\gamma_0}\right)\right\} e^{-mt} + \frac{1}{2}\left(\frac{\alpha}{\omega} - 1\right)\left(l_{\gamma_0} + r_{\gamma_0}\right)e^{mt}$$

…..……(3.1.12)

Similarly taking inverse Laplace transform of (3.1.11) we get

$$x_2(t,\alpha) = \frac{x_0}{m} + \frac{1}{2}\left\{-\frac{x_0}{m} + \gamma_1 + \gamma_3 + \frac{\alpha}{\omega}\left(l_{\gamma_0} - r_{\gamma_0}\right)\right\} e^{-mt} - \frac{1}{2}\left(\frac{\alpha}{\omega} - 1\right)\left(l_{\gamma_0} + r_{\gamma_0}\right)e^{mt}$$

………..(3.1.13)

Here three cases arise.

Case1: When left spread= $l_{\gamma_0} = \gamma_2 - \gamma_1$ =right spread= $r_{\gamma_0} = \gamma_3 - \gamma_2$

i.e., $\widetilde{\gamma_0} = (\gamma_1, \gamma_2, \gamma_3; \omega)$ is a symmetric GTFN.

$\therefore \frac{\partial}{\partial \alpha}[x_1(t,\alpha)] = \frac{1}{2\omega}\left(l_{\gamma_0} + r_{\gamma_0}\right)e^{mt} > 0$, $\frac{\partial}{\partial \alpha}[x_2(t,\alpha)] = -\frac{1}{2\omega}\left(l_{\gamma_0} + r_{\gamma_0}\right)e^{mt} < 0$

and $x_1(t,\omega) = \left(1 - \frac{1}{2}e^{-mt}\right)\frac{x_0}{m} + \gamma_2 e^{-mt} = x_2(t,\omega)$

Hence, $[\frac{x_0}{m} + \frac{1}{2}\left\{-\frac{x_0}{m} + \gamma_1 + \gamma_3\right\} e^{-mt} + \frac{1}{2}\left(\frac{\alpha}{\omega} - 1\right)\left(l_{\gamma_0} + r_{\gamma_0}\right)e^{mt}$,

$\frac{x_0}{m} + \frac{1}{2}\left\{-\frac{x_0}{m} + \gamma_1 + \gamma_3\right\} e^{-mt} - \frac{1}{2}\left(\frac{\alpha}{\omega} - 1\right)\left(l_{\gamma_0} + r_{\gamma_0}\right)e^{mt}]$ is the α-cut of the strong solution of the FODE (3.1.1).

So, $\tilde{x}(t) = (1 - \frac{1}{2}e^{-mt})\frac{x_0}{m} + \frac{\gamma_1+\gamma_3}{2}e^{-mt} + \tilde{0}\,(\gamma_2 - \gamma_1)e^{mt}$ where $\tilde{0} = (-1,0,1;\omega)$ be a GTFN is the solution of (3.1.1).

Example-3.1.2: Consider the FODE $\frac{dx}{dt} = -\frac{1}{10}x + 5$ with IC $\tilde{x}(t=0) = (8,12,16; 0.8)$

The strong solution is $\tilde{x}(t) = 50(1 - \frac{1}{2}e^{-\frac{t}{10}}) + 12e^{-\frac{t}{10}} + 4\,\tilde{0}e^{\frac{t}{10}}$ where $\tilde{0} = (-1,0,1; 0.8)$

Case2: When $l_{\gamma_0} < r_{\gamma_0}$

Then $\frac{\partial}{\partial\alpha}[x_2(t,\alpha)] = \frac{1}{2\omega}\left(l_{\gamma_0} - r_{\gamma_0}\right)e^{-mt} - \frac{1}{2\omega}\left(l_{\gamma_0} + r_{\gamma_0}\right)e^{mt} < 0$

In this case the classical solution exists if

$$\frac{\partial}{\partial\alpha}[x_1(t,\alpha)] = \frac{1}{2\omega}\left(l_{\gamma_0} - r_{\gamma_0}\right)e^{-mt} + \frac{1}{2\omega}\left(l_{\gamma_0} + r_{\gamma_0}\right)e^{mt} > 0$$

i.e.,$\frac{1}{2\omega}\left(l_{\gamma_0} - r_{\gamma_0}\right)e^{-mt} + \frac{1}{2\omega}\left(l_{\gamma_0} + r_{\gamma_0}\right)e^{mt} > 0$

i.e.,$e^{mt} > \frac{r_{\gamma_0} - l_{\gamma_0}}{l_{\gamma_0} + r_{\gamma_0}}$

i.e.,$t > \frac{1}{2m}\log\left[\frac{r_{\gamma_0} - l_{\gamma_0}}{l_{\gamma_0} + r_{\gamma_0}}\right]$

Hence, $[\frac{x_0}{m} + \frac{1}{2}\left\{-\frac{x_0}{m} + \gamma_1 + \gamma_3 + \frac{\alpha}{\omega}\left(l_{\gamma_0} - r_{\gamma_0}\right)\right\}e^{-mt} + \frac{1}{2}\left(\frac{\alpha}{\omega} - 1\right)\left(l_{\gamma_0} + r_{\gamma_0}\right)e^{mt},$

$\frac{x_0}{m} + \frac{1}{2}\left\{-\frac{x_0}{m} + \gamma_1 + \gamma_3 + \frac{\alpha}{\omega}\left(l_{\gamma_0} - r_{\gamma_0}\right)\right\}e^{-mt} - \frac{1}{2}\left(\frac{\alpha}{\omega} - 1\right)\left(l_{\gamma_0} + r_{\gamma_0}\right)e^{mt}]$ is the α-cut of the strong solution of the FODE (3.1.1) if $t > \frac{1}{2m}\log\left[\frac{r_{\gamma_0} - l_{\gamma_0}}{l_{\gamma_0} + r_{\gamma_0}}\right]$.

Case3: When $l_{\gamma_0} > r_{\gamma_0}$

Then $\frac{\partial}{\partial\alpha}[x_1(t,\alpha)] = \frac{1}{2\omega}\left(l_{\gamma_0} - r_{\gamma_0}\right)e^{-mt} + \frac{1}{2\omega}\left(l_{\gamma_0} + r_{\gamma_0}\right)e^{mt} > 0$

In this case the classical solution exists if

$$\frac{\partial}{\partial\alpha}[x_2(t,\alpha)] = \frac{1}{2\omega}\left(l_{\gamma_0} - r_{\gamma_0}\right)e^{-mt} - \frac{1}{2\omega}\left(l_{\gamma_0} + r_{\gamma_0}\right)e^{mt} < 0$$

i.e.$t > \frac{1}{2m}\log\left[\frac{l_{\gamma_0} - r_{\gamma_0}}{l_{\gamma_0} + r_{\gamma_0}}\right]$

Hence , $[\frac{x_0}{m} + \frac{1}{2}\left\{-\frac{x_0}{m} + \gamma_1 + \gamma_3 + \frac{\alpha}{\omega}\left(l_{\gamma_0} - r_{\gamma_0}\right)\right\}e^{-mt} + \frac{1}{2}\left(\frac{\alpha}{\omega} - 1\right)\left(l_{\gamma_0} + r_{\gamma_0}\right)e^{mt},$

$\frac{x_0}{m}+\frac{1}{2}\left\{-\frac{x_0}{m}+\gamma_1+\gamma_3+\frac{\alpha}{\omega}\left(l_{\gamma_0}-r_{\gamma_0}\right)\right\}e^{-mt}-\frac{1}{2}\left(\frac{\alpha}{\omega}-1\right)\left(l_{\gamma_0}+r_{\gamma_0}\right)e^{mt}]$ is the α-cut of the strong solution of the FODE (3.1.1) if $t>\frac{1}{2m}\log\left[\frac{l_{\gamma_0}-r_{\gamma_0}}{l_{\gamma_0}+r_{\gamma_0}}\right]$.

In both Case 2 and Case 3 the strong solution is,

$\tilde{x}(t)=(1-\frac{1}{2}e^{-mt})\frac{x_0}{m}+\frac{1}{2}\tilde{\Gamma}e^{-mt}+\frac{\gamma_3-\gamma_1}{2}\tilde{0}e^{mt}$ where $\tilde{\Gamma}=(\gamma_1+\gamma_3,2\gamma_2,\gamma_1+\gamma_3;\omega)$, $\tilde{0}=(-1,0,1;\omega)$ are two symmetric GTFN.

Example-3.1.3:$(l_{\gamma_0}<r_{\gamma_0})$Consider the FODE $\frac{dx}{dt}=-\frac{1}{10}x+5$ and the IC is $\tilde{x}(t=0)=(10,15,25;0.7)$.

The strong solution is $\tilde{x}(t)=50\left(1-\frac{1}{2}e^{-\frac{1}{10}t}\right)+\frac{1}{2}\ (35,30,35;0.7)e^{-\frac{1}{10}t}+5\ (-1,0,1;0.7)e^{\frac{1}{10}t}$

Example-3.1.4: $(l_{\gamma_0}>r_{\gamma_0})$Consider the FODE $\frac{dx}{dt}=-\frac{1}{10}x+5$ and the IC is $\tilde{x}(t=0)=(5,15,20;0.7)$

The strong solution is $\tilde{x}(t)=50\left(1-\frac{1}{2}e^{-\frac{1}{10}t}\right)+\frac{1}{2}(25,3025;0.7)e^{-\frac{1}{10}t}+7.5\ (-1,0,1;0.7)e^{\frac{1}{10}t}$

3.2. Solution Procedure of 1st Order Linear Non Homogeneous FODE of Type-II

Consider the initial value problem $\frac{dx}{dt}=\tilde{k}x+x_0$ ………….(3.2.1)

with IC $x(0)=\gamma$ where $\tilde{k}=(\beta_1,\beta_2,\beta_3;\lambda)$

Let $\tilde{x}(t)$ be the solution of FODE (3.2.1)

Let $x(t,\alpha)=[x_1(t,\alpha),x_2(t,\alpha)]$ be the α-cut of the solution and the α-cut of $\tilde{k}$ be

$$\left(\tilde{k}\right)_\alpha=\left[\beta_1+\frac{\alpha}{\lambda}(\beta_2-\beta_1),\beta_3-\frac{\alpha}{\lambda}(\beta_3-\beta_2)\right]=\left[\beta_1+\frac{\alpha l_k}{\lambda},\beta_3-\frac{\alpha r_k}{\lambda}\right]\forall\ \alpha\ \epsilon\ [0,\lambda],0<\lambda\leq 1$$

Where $l_k = \beta_2 - \beta_1$ and $r_k = \beta_3 - \beta_2$

Here we solve the given problem for $\tilde{k} > 0$ and $\tilde{k} < 0$ respecively.

Case 3.2.1: when $\tilde{k} > 0$

The equation (3.2.1) becomes

$$\frac{dx_i(t,\alpha)}{dt} = k_i(\alpha)x_i(t,\alpha) + x_0 \text{ for } i = 1,2 \qquad \text{……….(3.2.2)}$$

The FODE (3.2.2) becomes

$$\frac{dx_1(t,\alpha)}{dt} = \left(\beta_1 + \frac{\alpha l_k}{\lambda}\right) x_1(t,\alpha) + x_0 \qquad \text{..……….(3.2.3)}$$

and $$\frac{dx_2(t,\alpha)}{dt} = \left(\beta_3 - \frac{\alpha r_k}{\lambda}\right) x_2(t,\alpha) + x_0 \qquad \text{……….(3.2.4)}$$

Taking Laplace transform both sides of (3.2.3) we get

$$l\left\{\frac{dx_1(t,\alpha)}{dt}\right\} = l\{\left(\beta_1 + \frac{\alpha l_k}{\lambda}\right) x_1(t,\alpha) + x_0\}$$

Or, $sl\{x_1(t,\alpha)\} - x_1(0,\alpha) = \left(\beta_1 + \frac{\alpha l_k}{\lambda}\right) l\{x_1(t,\alpha)\} + l\{x_0\}$

Or, $\left(s - (\beta_1 + \frac{\alpha l_k}{\lambda})\right) l\{x_1(t,\alpha)\} = \gamma + \frac{x_0}{s}$

Or, $l\{x_1(t,\alpha)\} = \frac{\gamma}{s-(\beta_1+\frac{\alpha l_k}{\lambda})} + \frac{x_0}{s\left(s-(\beta_1+\frac{\alpha l_k}{\lambda})\right)} = \frac{\gamma}{s-(\beta_1+\frac{\alpha l_k}{\lambda})} + \frac{x_0}{(\beta_1+\frac{\alpha l_k}{\lambda})}\left(\frac{1}{s-(\beta_1+\frac{\alpha l_k}{\lambda})} - \frac{1}{s}\right)$

Taking inverse Laplace transform we get

$$x_1(t,\alpha) = \gamma l^{-1}\left\{\frac{1}{s-(\beta_1+\frac{\alpha l_k}{\lambda})}\right\} + x_0 l^{-1}\left\{\frac{1}{s\left(s-(\beta_1+\frac{\alpha l_k}{\lambda})\right)}\right\}$$

Or, $x_1(t,\alpha) = \gamma l^{-1}\left\{\frac{1}{s-(\beta_1+\frac{\alpha l_k}{\lambda})}\right\} + \frac{x_0}{(\beta_1+\frac{\alpha l_k}{\lambda})} l^{-1}\left\{\frac{1}{s-(\beta_1+\frac{\alpha l_k}{\lambda})}\right\} - \frac{x_0}{(\beta_1+\frac{\alpha l_k}{\lambda})} l^{-1}\left\{\frac{1}{s}\right\}$

Or, $x_1(t,\alpha) = \gamma e^{(\beta_1+\frac{\alpha l_k}{\lambda})t} + \frac{x_0}{(\beta_1+\frac{\alpha l_k}{\lambda})} e^{(\beta_1+\frac{\alpha l_k}{\lambda})t} - \frac{x_0}{(\beta_1+\frac{\alpha l_k}{\lambda})}$

Or, $x_1(t,\alpha) = -\frac{x_0}{(\beta_1+\frac{\alpha l_k}{\lambda})} + \left\{\gamma + \frac{x_0}{(\beta_1+\frac{\alpha l_k}{\lambda})}\right\} e^{(\beta_1+\frac{\alpha l_k}{\lambda})(t-t_0)}$ ....(3.2.5)

Similarly using Laplace transform both sides of (3.2.3) we get

$x_2(t,\alpha) = -\frac{x_0}{(\beta_3-\frac{\alpha r_k}{\lambda})} + \left\{\gamma + \frac{x_0}{(\beta_3-\frac{\alpha r_k}{\lambda})}\right\} e^{(\beta_3-\frac{\alpha r_k}{\lambda})(t-t_0)}$ ....(3.2.6)

Example 3.2.1:- Consider the FODE $\frac{dx}{dt} = (.06,.1,.14;.7)x + 2$ with IC x(t=0) =15.

Therefore the α-cut of the solution is ,$x_1(t,\alpha) = -\frac{2}{(0.06+0.057\alpha)} + \left\{15 + \frac{2}{(0.06+0.057\alpha)}\right\} e^{(0.06+0.057\alpha)t}$

and $x_2(t,\alpha) = -\frac{2}{(0.14-0.057\alpha)} + \left\{15 + \frac{2}{(0.14-0.057\alpha)}\right\} e^{(0.14-0.057\alpha)t}$

Table-5: Value of $x_1(t,\alpha)$ and $x_2(t,\alpha)$ for different α and t=5

α	$x_1(t,\alpha)$	$x_2(t,\alpha)$
0	31.9098	44.6885
0.1	32.6714	43.6118
0.2	33.4533	42.5635
0.3	34.2563	41.5427
0.4	35.0807	40.5488
0.5	35.9273	39.5811
0.6	36.7966	38.6387
0.7	37.6894	37.7211

From the above table we see that for this particular value of t, $x_1(t,\alpha)$ is an increasing function, $x_2(t,\alpha)$ is a decreasing function and $x_1(t,0.7) < x_2(t,0.7)$. Hence this solution is a strong solution.

Case 3.2.2: when $\tilde{K} < 0$

When $\tilde{k} < 0$, let $\tilde{k} = -\tilde{m}$, where $\tilde{m} = (\beta_1, \beta_2, \beta_3; \lambda)$ is a positive GTFN.

So $(\tilde{m})_\alpha = [m_1(\alpha), m_2(\alpha)] = \left[\beta_1 + \frac{\alpha l_m}{\lambda}, \beta_3 - \frac{\alpha r_m}{\lambda}\right] \forall\, \alpha\, \epsilon\, [0,\lambda], 0 < \lambda \leq 1$

where $l_m = \beta_2 - \beta_1$ and $r_m = \beta_3 - \beta_2$

$$\frac{dx_1(t,\alpha)}{dt} = -m_2(\alpha)x_2(t,\alpha) + x_0 = -\left(\beta_3 - \frac{\alpha r_m}{\lambda}\right)x_2(t,\alpha) + x_0 \quad . \quad \ldots\ldots\ldots(3.2.7)$$

and

$$\frac{dx_2(t,\alpha)}{dt} = -m_1(\alpha)x_1(t,\alpha) + x_0 = -\left(\beta_1 + \frac{\alpha l_m}{\lambda}\right)x_1(t,\alpha) + x_0 \quad \ldots\ldots\ldots(3.2.8)$$

Taking Laplace transform both sides of (3.2.7) we get

$$l\left\{\frac{dx_1(t,\alpha)}{dt}\right\} = l\{-m_2(\alpha)x_2(t,\alpha)\} + l\{x_0\}$$

Or, $sl\{x_1(t,\alpha)\} - x_1(0,\alpha) = -m_2(\alpha)l\{x_2(t,\alpha)\} + \frac{x_0}{s}$

Or, $sl\{x_1(t,\alpha)\} + m_2(\alpha)l\{x_2(t,\alpha)\} = \gamma + \frac{x_0}{s}$ …………(3.2.9)

Taking Laplace transform both sides of (3.2.8) we get

$$l\left\{\frac{dx_2(t,\alpha)}{dt}\right\} = l\{-m_1(\alpha)x_1(t,\alpha)\} + l\{x_0\}$$

Or, $sl\{x_2(t,\alpha)\} - x_2(0,\alpha) = -m_1(\alpha)l\{x_1(t,\alpha)\} + \frac{x_0}{s}$

Or, $m_1(\alpha)l\{x_1(t,\alpha)\} + sl\{x_2(t,\alpha)\} = \gamma + \frac{x_0}{s}$ ………..(3.2.10)

Solving (3.2.9) and (3.2.10) we get

$$l\{x_2(t,\alpha)\} = \frac{\left(\gamma + \frac{x_0}{s}\right)\{\mathrm{s} - m_1(\alpha)\}}{\mathrm{s}^2 - m_1(\alpha)m_2(\alpha)}$$

$$= \gamma \frac{s}{\mathrm{s}^2 - m_1(\alpha)m_2(\alpha)} - \sqrt{\frac{m_1(\alpha)}{m_2(\alpha)}} \frac{\sqrt{m_1(\alpha)m_2(\alpha)}}{\mathrm{s}^2 - m_1(\alpha)m_2(\alpha)} + x_0\left\{\frac{1}{sm_2(\alpha)} + \frac{s\left(-\frac{1}{m_2(\alpha)}\right)+1}{\mathrm{s}^2 - m_1(\alpha)m_2(\alpha)}\right\} \ldots\ldots\ldots\ldots(3.2.11)$$

and

$$l\{x_1(t,\alpha)\} = \frac{\left(\gamma + \frac{x_0}{s}\right)\{s - m_2(\alpha)\}}{s^2 - m_1(\alpha)m_2(\alpha)}$$

$$= \gamma \frac{s}{s^2 - m_1(\alpha)m_2(\alpha)} - \sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}} \frac{\sqrt{m_1(\alpha)m_2(\alpha)}}{s^2 - m_1(\alpha)m_2(\alpha)} + x_0 \left\{ \frac{1}{s m_1(\alpha)} + \frac{s\left(-\frac{1}{m_1(\alpha)}\right)+1}{s^2 - m_1(\alpha)m_2(\alpha)} \right\}$$

..……….(3.2.12)

Taking inverse Laplace transform of (3.2.11) we get

$$x_2(t,\alpha) = \gamma\, l^{-1}\left\{\frac{s}{s^2 - m_1(\alpha)m_2(\alpha)}\right\} + \sqrt{\frac{m_1(\alpha)}{m_2(\alpha)}}\, l^{-1}\left\{\frac{\sqrt{m_1(\alpha)m_2(\alpha)}}{s^2 - m_1(\alpha)m_2(\alpha)}\right\} + \frac{x_0}{m_2(\alpha)}\left\{\frac{1}{s}\right\} - \frac{x_0}{m_2(\alpha)}\, l^{-1}\left\{\frac{s}{s^2 - m_1(\alpha)m_2(\alpha)}\right\} + \frac{x_0}{\sqrt{m_1(\alpha)m_2(\alpha)}}\, l^{-1}\left\{\frac{\sqrt{m_1(\alpha)m_2(\alpha)}}{s^2 - m_1(\alpha)m_2(\alpha)}\right\}$$

$$= \gamma \cosh\sqrt{m_1(\alpha)m_2(\alpha)}t + \sqrt{\frac{m_1(\alpha)}{m_2(\alpha)}} \sinh\sqrt{m_1(\alpha)m_2(\alpha)}t + \frac{x_0}{m_2(\alpha)} - \frac{x_0}{m_2(\alpha)}\cosh\sqrt{m_1(\alpha)m_2(\alpha)}t + \frac{x_0}{\sqrt{m_1(\alpha)m_2(\alpha)}}\sinh\sqrt{m_1(\alpha)m_2(\alpha)}t$$

$$= -\frac{1}{2}\sqrt{\frac{\beta_1 + \frac{\alpha l_m}{\lambda}}{\beta_3 - \frac{\alpha r_m}{\lambda}}}\left\{\gamma\left(1 - \sqrt{\frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\beta_1 + \frac{\alpha l_m}{\lambda}}}\right) - x_0\left(\frac{1}{\beta_1 + \frac{\alpha l_m}{\lambda}} - \frac{1}{\sqrt{(\beta_1 + \frac{\alpha l_m}{\lambda})(\beta_3 - \frac{\alpha r_m}{\lambda})}}\right)\right\} e^{\sqrt{(\beta_1 + \frac{\alpha l_m}{\lambda}(\beta_3 - \frac{\alpha r_m}{\lambda})}t}$$

$$+\frac{1}{2}\sqrt{\frac{\beta_1 + \frac{\alpha l_m}{\lambda}}{\beta_3 - \frac{\alpha r_m}{\lambda}}}\left\{\gamma\left(1 + \sqrt{\frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\beta_1 + \frac{\alpha l_m}{\lambda}}}\right) - x_0\left(\frac{1}{\beta_1 + \frac{\alpha l_m}{\lambda}} + \frac{1}{\sqrt{(\beta_1 + \frac{\alpha l_m}{\lambda})(\beta_3 - \frac{\alpha r_m}{\lambda})}}\right)\right\} e^{-\sqrt{\left(\beta_1 + \frac{\alpha l_m}{\lambda}\right)(\beta_3 - \frac{\alpha r_m}{\lambda})}t} + \frac{x_0}{\beta_3 - \frac{\alpha r_m}{\lambda}}$$

Taking inverse Laplace transform of (3.2.12) we get

$$x_1(t,\alpha) = \gamma\, l^{-1}\left\{\frac{s}{s^2 - m_1(\alpha)m_2(\alpha)}\right\} + \sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}}\, l^{-1}\left\{\frac{\sqrt{m_1(\alpha)m_2(\alpha)}}{s^2 - m_1(\alpha)m_2(\alpha)}\right\} + \frac{x_0}{m_1(\alpha)}\left\{\frac{1}{s}\right\} - \frac{x_0}{m_1(\alpha)}\, l^{-1}\left\{\frac{s}{s^2 - m_1(\alpha)m_2(\alpha)}\right\} + \frac{x_0}{\sqrt{m_1(\alpha)m_2(\alpha)}}\, l^{-1}\left\{\frac{\sqrt{m_1(\alpha)m_2(\alpha)}}{s^2 - m_1(\alpha)m_2(\alpha)}\right\}$$

$$= \gamma \cosh \sqrt{m_1(\alpha)m_2(\alpha)}t + \sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}} \sinh \sqrt{m_1(\alpha)m_2(\alpha)}t + \frac{x_0}{m_1(\alpha)} - \frac{x_0}{m_1(\alpha)} \cosh \sqrt{m_1(\alpha)m_2(\alpha)}t + \frac{x_0}{\sqrt{m_1(\alpha)m_2(\alpha)}} \sinh \sqrt{m_1(\alpha)m_2(\alpha)}t$$

$$= \frac{1}{2}\left\{\gamma\left(1 - \sqrt{\frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\beta_1 + \frac{\alpha l_m}{\lambda}}}\right) - x_0\left(\frac{1}{\beta_1 + \frac{\alpha l_m}{\lambda}} - \frac{1}{\sqrt{(\beta_1 + \frac{\alpha l_m}{\lambda})(\beta_3 - \frac{\alpha r_m}{\lambda})}}\right)\right\} e^{\sqrt{(\beta_1+\frac{\alpha l_m}{\lambda})(\beta_3-\frac{\alpha r_m}{\lambda})}t}$$

$$+ \frac{1}{2}\left\{\gamma\left(1 + \sqrt{\frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\beta_1 + \frac{\alpha l_m}{\lambda}}}\right) - x_0\left(\frac{1}{\beta_1 + \frac{\alpha l_m}{\lambda}} + \frac{1}{\sqrt{(\beta_1 + \frac{\alpha l_m}{\lambda})(\beta_3 - \frac{\alpha r_m}{\lambda})}}\right)\right\} e^{-\sqrt{(\beta_1+\frac{\alpha l_m}{\lambda})(\beta_3-\frac{\alpha r_m}{\lambda})}t} + \frac{x_0}{\beta_1 + \frac{\alpha l_m}{\lambda}}$$

Example 3.2.2: Consider the FODE $\frac{dx}{dt} = -(.06,.1,.14;.7)x + 2$ with IC x(t=0) = 15

Here $\widetilde{m} = (.06,.1,.14;.7)$

Therefore the α-cut of the solution is

$$x_1(t,\alpha) = \frac{1}{2}\left\{15\left(1 - \sqrt{\frac{0.14 - 0.057\,\alpha}{0.06 + 0.057\,\alpha}}\right) - 2\left(\frac{1}{0.06 + 0.057\,\alpha} - \frac{1}{\sqrt{(0.06 + 0.057\,\alpha)(0.14 - 0.057\,\alpha)}}\right)\right\} e^{\sqrt{(0.06+0.057\,\alpha)(0.14-0.057\,\alpha)}t}$$

$$+ \frac{1}{2}\left\{15\left(1 + \sqrt{\frac{0.14 - 0.057\,\alpha}{0.06 + 0.057\,\alpha}}\right) - 2\left(\frac{1}{0.06 + 0.057\,\alpha} + \frac{1}{\sqrt{(0.06 + 0.057\,\alpha)(0.14 - 0.057\,\alpha)}}\right)\right\} e^{-\sqrt{(0.06+0.057\,\alpha)(0.14-0.057\,\alpha)}t} + \frac{2}{0.06 + 0.057\,\alpha}$$

$$x_2(t,\alpha) = -\frac{1}{2}\sqrt{\frac{0.06+0.057\,\alpha}{0.14-0.057\,\alpha}}\left\{15\left(1-\sqrt{\frac{0.14-0.057\,\alpha}{0.06+0.057\,\alpha}}\right)\right.$$

$$-2\left(\frac{1}{0.06+0.057\,\alpha}\right.$$

$$\left.\left.-\frac{1}{\sqrt{(0.06+0.057\,\alpha)(0.14-0.057\,\alpha)}}\right)\right\}e^{\sqrt{(0.06+0.057\,\alpha)(0.14-0.057\,\alpha)}t}$$

$$+\frac{1}{2}\sqrt{\frac{0.06+0.057\,\alpha}{0.14-0.057\,\alpha}}\left\{15\left(1+\sqrt{\frac{0.14-0.057\,\alpha}{0.06+0.057\,\alpha}}\right)\right.$$

$$-2\left(\frac{1}{0.06+0.057\,\alpha}\right.$$

$$\left.\left.+\frac{1}{\sqrt{(0.06+0.057\,\alpha)(0.14-0.057\,\alpha)}}\right)\right\}e^{-\sqrt{(0.06+0.057\,\alpha)(0.14-0.057\,\alpha)}t}$$

$$+\frac{2}{0.14-0.057\,\alpha}$$

Table-6: Value of $x_1(t,\alpha)$ and $x_2(t,\alpha)$ for different α and t=5

α	$x_1(t,\alpha)$	$x_2(t,\alpha)$
0	12.5234	20.7709
0.1	13.1902	20.2732
0.2	13.8466	19.7602
0.3	14.4922	19.2322
0.4	15.1264	18.6895
0.5	15.7489	18.1327
0.6	16.3593	17.5619
0.7	16.9570	16.9777

From the above table we see that for this particular value of t, $x_1(t,\alpha)$ is an increasing function, $x_2(t,\alpha)$ is a decreasing function and $x_1(t,0.7) < x_2(t,0.7)$. Hence this solution is a strong solution.

3.3.Solution Procedure of 1st Order Linear Non Homogeneous FODE of Type-III

Consider the initial value problem $\frac{dx}{dt} = \widetilde{K}x + x_0$ ....(3.3.1)

With fuzzy IC $\tilde{x}(0) = \widetilde{\gamma_0} = (\gamma_1, \gamma_2, \gamma_3; \omega)$, where $\tilde{k} = (\beta_1, \beta_2, \beta_3; \lambda)$

Let $\tilde{x}(t)$ be the solution of FODE (3.3.1) .

Let $x(t, \alpha) = [x_1(t, \alpha), x_2(t, \alpha)]$ be the α-cut of the solution.

Also $\left(\tilde{k}\right)_\alpha = \left[\beta_1 + \frac{\alpha l_k}{\lambda}, \beta_3 - \frac{\alpha r_k}{\lambda}\right] \forall\, \alpha \in [0, \lambda], 0 < \lambda \leq 1$

where $l_k = \beta_2 - \beta_1$ and $r_k = \beta_3 - \beta_2$

and $(\widetilde{\gamma_0})_\alpha = \left[\gamma_1 + \frac{\alpha l_{\gamma_0}}{\omega}, \gamma_3 - \frac{\alpha r_{\gamma_0}}{\omega}\right] \forall\, \alpha \in [0, \omega], 0 < \omega \leq 1$

where $l_{\gamma_0} = \gamma_2 - \gamma_1$ and $r_{\gamma_0} = \gamma_3 - \gamma_2$

Let $\eta = \min(\lambda, \omega)$

Here we solve the given problem for $\tilde{k} > 0$ and $\tilde{k} < 0$ respecively.

Case 3.3.1: when $\tilde{k} > 0$

From equation (3.3.1) we get

$$\frac{dx_1(t,\alpha)}{\mathrm{dt}} = k_1(\alpha)x_1(t, \alpha) + x_0 \quad \text{.........(3.3.2)}$$

and

$$\frac{dx_2(t,\alpha)}{\mathrm{dt}} = k_2(\alpha)x_2(t, \alpha) + x_0 \quad \text{.........(3.3.3)}$$

Taking Laplace transform both sides of (3.3.2) we get

$$l\left\{\frac{dx_1(t,\alpha)}{dt}\right\} = l\{k_1(\alpha)x_1(t, \alpha)\} + l\{x_0\}$$

Or, $sl\{x_1(t, \alpha)\} - x_1(0, \alpha) = k_1(\alpha)l\{x_1(t, \alpha)\} + \frac{x_0}{s}$

Or, $\left(s - \left(\beta_1 + \frac{\alpha l_k}{\eta}\right)\right) l\{x_1(t, \alpha)\} = \gamma_1 + \frac{\alpha l_{\gamma_0}}{\eta} + \frac{x_0}{s}$

Or, $l\{x_1(t,\alpha)\} = \frac{\left(\gamma_1+\frac{\alpha l_{\gamma_0}}{\eta}\right)}{s-\left(\beta_1+\frac{\alpha l_k}{\eta}\right)} + \frac{x_0}{s\left(s-\left(\beta_1+\frac{\alpha l_k}{\eta}\right)\right)}$ …………(3.3.4)

Taking inverse Laplace transform of (3.3.4) we get

$$x_1(t,\alpha) = \left(\gamma_1 + \frac{\alpha l_{\gamma_0}}{\eta}\right) l^{-1}\left\{\frac{1}{s-\left(\beta_1+\frac{\alpha l_k}{\eta}\right)}\right\} + \frac{x_0}{\left(\beta_1+\frac{\alpha l_k}{\eta}\right)} l^{-1}\left\{\frac{1}{s-\left(\beta_1+\frac{\alpha l_k}{\eta}\right)}\right\} - \frac{x_0}{\left(\beta_1+\frac{\alpha l_k}{\eta}\right)} l^{-1}\left\{\frac{1}{s}\right\}$$

Or, $x_1(t,\alpha) = \left(\gamma_1 + \frac{\alpha l_{\gamma_0}}{\eta}\right) e^{\left(\beta_1+\frac{\alpha l_k}{\eta}\right)t} + \frac{x_0}{\left(\beta_1+\frac{\alpha l_k}{\eta}\right)} e^{\left(\beta_1+\frac{\alpha l_k}{\lambda}\right)t} - \frac{x_0}{\left(\beta_1+\frac{\alpha l_k}{\eta}\right)}$

Or, $x_1(t,\alpha) = -\frac{x_0}{\left(\beta_1+\frac{\alpha l_k}{\eta}\right)} + \left\{\left(\gamma_1 + \frac{\alpha l_{\gamma_0}}{\eta}\right) + \frac{x_0}{\left(\beta_1+\frac{\alpha l_k}{\eta}\right)}\right\} e^{\left(\beta_1+\frac{\alpha l_k}{\eta}\right)t}$ ………..(3.3.5)

Similarly from (3.3.3) we get

$x_2(t,\alpha) = -\frac{x_0}{\left(\beta_3-\frac{\alpha r_k}{\eta}\right)} + \left\{\left(\gamma_3 - \frac{\alpha r_{\gamma_0}}{\eta}\right) + \frac{x_0}{\left(\beta_3-\frac{\alpha r_k}{\eta}\right)}\right\} e^{\left(\beta_3-\frac{\alpha r_k}{\eta}\right)t}$ ………..(3.3.6)

Example 3.3.1: Consider the FODE $\frac{dx}{dt} = (.06,.1,.12;.7)x + 2$ with IC

$\tilde{x}$(t=0)=(10,12,14;0.8)

Therefore the α-cut of the solution is

$x_1(t,\alpha) = -\frac{2}{(0.06+0.057\alpha)} + \left((10+2.86\alpha) + \frac{2}{(0.06+0.057\alpha)}\right) e^{(0.06+0.057\alpha)t}$ and

$$x_2(t,\alpha) = -\frac{2}{(0.12-0.028\alpha)} + \left((14-2.86\alpha) + \frac{2}{(0.12-0.028\alpha)}\right) e^{(0.12-0.028\alpha)t}$$

Table-7: Value of $x_1(t,\alpha)$ and $x_2(t,\alpha)$ for different α and t=15

α	$x_1(t,\alpha)$	$x_2(t,\alpha)$
0	73.2495	168.8559
0.1	78.6735	161.4778
0.2	84.5860	154.4490
0.3	91.0338	147.7529
0.4	98.0679	141.3735
0.5	105.7445	135.2958
0.6	114.1252	129.5053
0.7	123.2776	123.9885

From the above table we see that for this particular value of t, $x_1(t,\alpha)$ is an increasing function, $x_2(t,\alpha)$ is a decreasing function and $x_1(t,0.7) < x_2(t,0.7)$. Hence this solution is a strong solution.

Case 3.3.2: when $\tilde{k} < 0$

Let $\tilde{k} = -\widetilde{m}$

Then equation (3.3.1) becomes

$$\frac{dx_1(t,\alpha)}{\mathrm{dt}} = -m_2(\alpha)x_2(t,\alpha) + x_0 \qquad \text{............(3.3.7)}$$

and

$$\frac{dx_2(t,\alpha)}{\mathrm{dt}} = -m_1(\alpha)x_1(t,\alpha) + x_0 \qquad \text{............(3.3.8)}$$

Taking Laplace transform both sides of (3.3.7) we get

$$l\left\{\frac{dx_1(t,\alpha)}{dt}\right\} = l\{-m_2(\alpha)x_2(t,\alpha)\} + l\{x_0\}$$

Or, $sl\{x_1(t,\alpha)\} - x_1(0,\alpha) = -m_2(\alpha)l\{x_2(t,\alpha)\} + \frac{x_0}{s}$

Or, $sl\{x_1(t,\alpha)\} + m_2(\alpha)l\{x_2(t,\alpha)\} = \gamma_1(\alpha) + \frac{x_0}{s}$ ....(3.3.9)

Taking Laplace transform both sides of (3.3.8) we get

$$l\left\{\frac{dx_2(t,\alpha)}{dt}\right\} = l\{-m_1(\alpha)x_1(t,\alpha)\} + l\{x_0\}$$

Or, $sl\{x_2(t,\alpha)\} - x_2(0,\alpha) = -m_1(\alpha)l\{x_1(t,\alpha)\} + \frac{x_0}{s}$

Or, $m_1(\alpha)l\{x_1(t,\alpha)\} + sl\{x_2(t,\alpha)\} = \gamma_2(\alpha) + \frac{x_0}{s}$ ....(3.3.10)

Solving (3.3.9) and (3.3.10) we get

$$l\{x_2(t,\alpha)\} = \frac{\mathrm{s}\left(\gamma_2(\alpha)+\frac{x_0}{s}\right) - m_1(\alpha)\left(\gamma_1(\alpha)+\frac{x_0}{s}\right)}{\mathrm{s}^2 - m_1(\alpha)m_2(\alpha)} \qquad \text{............(3.3.11)}$$

$$l\{x_1(t,\alpha)\} = \frac{s\left(\gamma_1(\alpha)+\frac{x_0}{s}\right)-m_2(\alpha)\left(\gamma_2(\alpha)+\frac{x_0}{s}\right)}{s^2-m_1(\alpha)m_2(\alpha)} \qquad \ldots\ldots\ldots\ldots..(3.3.12)$$

Taking inverse Laplace transform of (3.3.11) we get

$$x_2(t,\alpha) = \gamma_2(\alpha)l^{-1}\left\{\frac{s}{s^2-m_1(\alpha)m_2(\alpha)}\right\} + \frac{x_0}{\sqrt{m_1(\alpha)m_2(\alpha)}}l^{-1}\left\{\frac{\sqrt{m_1(\alpha)m_2(\alpha)}}{s^2-m_1(\alpha)m_2(\alpha)}\right\}$$
$$-\gamma_1(\alpha)\sqrt{\frac{m_1(\alpha)}{m_2(\alpha)}}l^{-1}\left\{\frac{\sqrt{m_1(\alpha)m_2(\alpha)}}{s^2-m_1(\alpha)m_2(\alpha)}\right\} - \frac{x_0}{2m_2(\alpha)}l^{-1}\left\{\frac{1}{s-\sqrt{m_1(\alpha)m_2(\alpha)}}\right\}$$
$$-\frac{x_0}{2m_2(\alpha)}l^{-1}\left\{\frac{1}{s+\sqrt{m_1(\alpha)m_2(\alpha)}}\right\} + \frac{x_0}{m_2(\alpha)}l^{-1}\left\{\frac{1}{s}\right\}$$

$$= \gamma_2(\alpha)\cosh\sqrt{m_1(\alpha)m_2(\alpha)}\,t + \frac{x_0}{\sqrt{m_1(\alpha)m_2(\alpha)}}\sinh\sqrt{m_1(\alpha)m_2(\alpha)}\,t$$
$$-\gamma_1(\alpha)\sqrt{\frac{m_1(\alpha)}{m_2(\alpha)}}\sinh\sqrt{m_1(\alpha)m_2(\alpha)}\,t - \frac{x_0}{2m_2(\alpha)}e^{\sqrt{m_1(\alpha)m_2(\alpha)}t}$$
$$-\frac{x_0}{2m_2(\alpha)}e^{-\sqrt{m_1(\alpha)m_2(\alpha)}t} + \frac{x_0}{m_2(\alpha)}$$

$$= -\frac{1}{2}\sqrt{\frac{(\beta_1+\frac{\alpha l_m}{\eta})}{(\beta_3-\frac{\alpha r_m}{\eta})}}\left\{\left(\gamma_1+\frac{\alpha l_{\gamma_0}}{\eta}-\sqrt{\frac{(\beta_3-\frac{\alpha r_m}{\eta})}{(\beta_1+\frac{\alpha l_m}{\eta})}}(\gamma_3-\frac{\alpha r_{\gamma_0}}{\eta})\right)\right.$$
$$\left.-\left(\frac{1}{(\beta_1+\frac{\alpha l_m}{\eta})}-\frac{1}{\sqrt{(\beta_1+\frac{\alpha l_m}{\eta})(\beta_3-\frac{\alpha r_m}{\eta})}}\right)x_0\right\}e^{\sqrt{(\beta_1+\frac{\alpha l_m}{\eta})(\beta_3-\frac{\alpha r_m}{\eta})}t}$$
$$+\frac{1}{2}\sqrt{\frac{(\beta_1+\frac{\alpha l_m}{\eta})}{(\beta_3-\frac{\alpha r_m}{\eta})}}\left\{\left(\gamma_1+\frac{\alpha l_{\gamma_0}}{\eta}+\sqrt{\frac{(\beta_3-\frac{\alpha r_m}{\eta})}{(\beta_1+\frac{\alpha l_m}{\eta})}}(\gamma_3-\frac{\alpha r_{\gamma_0}}{\eta})\right)\right.$$
$$\left.-\left(\frac{1}{(\beta_1+\frac{\alpha l_m}{\eta})}+\frac{1}{\sqrt{(\beta_1+\frac{\alpha l_m}{\eta})(\beta_3-\frac{\alpha r_m}{\eta})}}\right)x_0\right\}e^{-\sqrt{(\beta_1+\frac{\alpha l_m}{\eta})(\beta_3-\frac{\alpha r_m}{\eta})}t}+\frac{x_0}{(\beta_3-\frac{\alpha r_m}{\eta})}$$

Similarly taking inverse Laplace transform of (3.3.12) we get

$x_1(t,\alpha)$

$$= \frac{1}{2}\left\{\left(\gamma_1 + \frac{\alpha l_{\gamma_0}}{\eta} - \sqrt{\frac{\left(\beta_3 - \frac{\alpha r_m}{\eta}\right)}{\left(\beta_1 + \frac{\alpha l_m}{\eta}\right)}}(\gamma_3 - \frac{\alpha r_{\gamma_0}}{\eta}) \right.\right.$$
0)

$$\left.\left. - \left(\frac{1}{\left(\beta_1 + \frac{\alpha l_m}{\eta}\right)} - \frac{1}{\sqrt{\left(\beta_1 + \frac{\alpha l_m}{\eta}\right)\left(\beta_3 - \frac{\alpha r_m}{\eta}\right)}}\right) x_0 \right\} e^{\sqrt{\left(\beta_1 + \frac{\alpha l_m}{\eta}\right)\left(\beta_3 - \frac{\alpha r_m}{\eta}\right)}t}$$

$$+ \frac{1}{2}\left\{\left(\gamma_1 + \frac{\alpha l_{\gamma_0}}{\eta} + \sqrt{\frac{\left(\beta_3 - \frac{\alpha r_m}{\eta}\right)}{\left(\beta_1 + \frac{\alpha l_m}{\eta}\right)}}((\gamma_3 - \frac{\alpha r_{\gamma_0}}{\eta})\right)\right.$$

$$\left. - \left(\frac{1}{(\beta_1 + \frac{\alpha l_m}{\eta})} + \frac{1}{\sqrt{(\beta_1 + \frac{\alpha l_m}{\eta})(\beta_3 - \frac{\alpha r_m}{\eta})}}\right) x_0 \right\} e^{-\sqrt{(\beta_1 + \frac{\alpha l_m}{\eta})(\beta_3 - \frac{\alpha r_m}{\eta})}t}$$

$$+ \frac{x_0}{(\beta_1 + \frac{\alpha l_m}{\eta})}$$

Example 3.3.2:- Consider the FODE $\frac{dx}{dt} = -(0.05,0.07,0.10;0.7)x + 2$ with IC x(t=0)=(9,12,14;0.9)

Here α -cut of the solution is

$$x_1(t,\alpha) = \frac{1}{2}\left[\left\{\left((9 + 4.29\alpha) - \sqrt{\frac{0.1 - 0.042\alpha}{0.05 + 0.029\alpha}}(14 - 2.86\alpha)\right)\right.\right.$$

$$\left. - 2\left(\frac{1}{0.05 + 0.029\alpha} - \frac{1}{\sqrt{(0.05 + 0.029\alpha)(0.1 - 0.042\alpha)}}\right)\right\} e^{\sqrt{(0.05+0.029\alpha)(0.1-0.042\alpha)}t}$$

$$+ \left\{\left((9 + 4.29\alpha) + \sqrt{\frac{0.1 - 0.042\alpha}{0.05 + 0.029\alpha}}(14 - 2.86\alpha)\right)\right.$$

$$\left.\left. - 2\left(\frac{1}{0.05 + 0.029\alpha} + \frac{1}{\sqrt{(0.05 + 0.029\alpha)(0.1 - 0.042\alpha)}}\right)\right\} e^{-\sqrt{(0.05+0.029\alpha)(0.1-0.042\alpha)}t}\right]$$

$$+ \frac{2}{0.05 + 0.029\alpha}$$

$$x_2(t,\alpha)=\frac{1}{2}\sqrt{\frac{0.05+0.029\alpha}{0.1-0.042\alpha}}\Bigg[-\Bigg\{\Bigg((9+4.29\alpha)-\sqrt{\frac{0.1-0.042\alpha}{0.05+0.029\alpha}}(14-2.86\alpha)\Bigg)$$

$$-2\left(\frac{1}{0.05+0.029\alpha}\right.$$

$$\left.-\frac{1}{\sqrt{(0.05+0.029\alpha)(0.1-0.042\alpha)}}\right)\Bigg\}e^{\sqrt{(0.05+0.029\alpha)(0.1-0.042\alpha)}t}$$

$$+\Bigg\{\Bigg((9+4.29\alpha)+\sqrt{\frac{0.1-0.042\alpha}{0.05+0.029\alpha}}(14-2.86\alpha)\Bigg)$$

$$-2\left(\frac{1}{0.05+0.029\alpha}\right.$$

$$\left.+\frac{1}{\sqrt{(0.05+0.029\alpha)(0.1-0.042\alpha)}}\right)\Bigg\}e^{-\sqrt{(0.05+0.029\alpha)(0.1-0.042\alpha)}t}\Bigg]$$

$$+\frac{2}{0.1-0.042\alpha}$$

Table-8: Value of $x_1(t,\alpha)$ and $x_2(t,\alpha)$ for different α and t=14

α	$x_1(t,\alpha)$	$x_2(t,\alpha)$
0	2.3691	36.2340
0.1	5.4072	34.5254
0.2	8.3910	32.7160
0.3	11.3123	30.8105
0.4	14.1636	28.8139
0.5	16.9371	26.7315
0.6	19.6256	24.5687
0.7	22.2221	22.3314

From the above table we see that for this particular value of t=14, $x_1(t,\alpha)$ is an increasing function, $x_2(t,\alpha)$ is a decreasing function and $x_1(t,0.7)<x_2(t,0.7)$. Hence this solution is a strong solution.

4.Application: A tank initially contains V_0 liters of brine (salt solution) with a salt concentration of C_0 grams per liter. At some instant brine with a salt concentration of .4 grams per liter begins to flow into the tank at a rate of 3 liters per minute, while the well-stirred mixture flows out at the same rate. Solve the problem when

(i) $\widetilde{C_0} = (4,5,7; 0.7)$ gr/lit and $V_0 = 200$

(ii) $\widetilde{V_0} = (180,200,210; 0.8)$, C_0 = 5 gr/lit

(iii) $\widetilde{V_0} = (190,200,220; 0.8)$, $\widetilde{C_0} = (3,5,6; 0.7)$ gr/lit

Solution: Let V (t) be the volume (lit) of brine in the tank at time t minutes. Let S(t) be the mass (gr) of salt in the tank at time t minutes. Because the mixture is assumed to be well-stirred, the salt concentration of the brine in the tank at time t is C(t) = S(t)/V (t). In particular, this will be the concentration of the brine that flows out of the tank.

(i): when $\widetilde{C_0} = (4,5,7; 0.7)$ gr/lit and $V_0 = 200$

Therefore $\frac{dS}{dt} = 0.4 \cdot 3 - \frac{S}{V_0} \cdot 31.2 - .015$ S where $V_0 = 200$

With initial condition $\tilde{S}(0) = V_0\widetilde{C_0} = 200(4,5,7; 0.7)$

i.e., $\frac{dS}{dt} = 1.2 - .015$ S with $\tilde{S}(0) = (800,1000,1400; 0.7)$ …………(4.1)

The α -cut of the solution is

$S_1(t,\alpha) = 80 + (1060 - 142.85\alpha)e^{-0.015t} + (428.57\alpha - 300)e^{0.015t}$

and

$S_2(t,\alpha) = 80 + (1060 - 142.85\alpha)e^{-0.015t} - (428.57\alpha - 300)e^{0.015t}$

Table 9: Value of $S_1(t,\alpha)$ and $S_2(t,\alpha)$ for different $\alpha$and t=30 min

α	$S_1(t,\alpha)$	$S_2(t,\alpha)$
0	285.3922	1226.3795
0.1	343.4968	1150.0578
0.2	401.6015	1073.7361
0.3	459.7061	997.4145
0.4	517.8107	921.0928
0.5	575.9154	844.7711
0.6	634.0200	768.4495
0.7	692.1246	692.1278

From the above table we see that for this particular value of t, $S_1(t,\alpha)$ is an increasing function, $S_2(t,\alpha)$ is a decreasing function and $S_1(t,0.7) < S_2(t,0.7)$. Hence this solution is a strong solution.

(ii):when $\widetilde{V_0} = (180,200,210;0.8)$, C_0 = 5 gr/lit

Therefore

$$\frac{dS}{dt} = 0.4 \cdot 3 - \frac{S}{\widetilde{V_0}} \cdot 3 = 1.2 - \frac{S}{(180,200,210;0.8)} \cdot 3 = 1.2 - \frac{3}{(180,200,210;0.8)} \cdot S$$

$$\approx 1.2 - 3\left(\frac{1}{210},\frac{1}{200},\frac{1}{180};0.8\right)S = 1.2 - (0.014,0.015,0.017;0.8)S$$

With initial condition $\tilde{S}(0) = \widetilde{V_0}C_0 = 5(180,200,210;0.8) = (900,1000,1050;0.8)$

i.e., $\frac{dS}{dt} = 1.2 - (0.014,0.015,0.017;0.8)S$ with $\tilde{S}(0) = (900,1000,1050;0.8)$

…………..(4.2)

The α -cut of the solution is

$$S_1(t,\alpha) = \frac{1}{2}\Bigg[\Bigg\{\Bigg((900+125\alpha) - \sqrt{\frac{0.017-0.0025\alpha}{0.014+0.00125\alpha}}(1050-62.5\alpha)\Bigg) - 1.2\Bigg(\frac{1}{0.014+0.00125\alpha} - \frac{1}{\sqrt{(0.014+0.00125\alpha)(0.017-0.0025\alpha)}}\Bigg)\Bigg\}e^{\sqrt{(0.014+0.00125\alpha)(0.017-0.0025\alpha)}t}$$
$$+\Bigg\{\Bigg((900+125\alpha) + \sqrt{\frac{0.017-0.0025\alpha}{0.014+0.00125\alpha}}(1050-62.5\alpha)\Bigg) - 1.2\Bigg(\frac{1}{0.014+0.00125\alpha} + \frac{1}{\sqrt{(0.014+0.00125\alpha)(0.017-0.0025\alpha)}}\Bigg)\Bigg\}e^{-\sqrt{(0.014+0.00125\alpha)(0.017-0.0025\alpha)}t}\Bigg]$$
$$+\frac{1.2}{0.014+0.00125\alpha}$$

and

$$S_2(t,\alpha)=\frac{1}{2}\sqrt{\frac{0.014+0.00125\alpha}{0.017-0.0025\alpha}}\Bigg[-\Bigg\{\Bigg((900+125\alpha)-\sqrt{\frac{0.017-0.0025\alpha}{0.014+0.00125\alpha}}(1050-62.5\alpha)\Bigg)$$

$$-1.2\Bigg(\frac{1}{0.014+0.00125\alpha}$$

$$-\frac{1}{\sqrt{(0.014+0.00125\alpha)(0.017-0.0025\alpha)}}\Bigg)\Bigg\}e^{\sqrt{(0.014+0.00125\alpha)(0.017-0.0025\alpha)}t}$$

$$+\Bigg\{\Bigg((900+125\alpha)+\sqrt{\frac{0.017-0.0025\alpha}{0.014+0.00125\alpha}}(1050-62.5\alpha)\Bigg)$$

$$-1.2\Bigg(\frac{1}{0.014+0.00125\alpha}$$

$$+\frac{1}{\sqrt{(0.014+0.00125\alpha)(0.017-0.0025\alpha)}}\Bigg)\Bigg\}e^{-\sqrt{(0.014+0.00125\alpha)(0.017-0.0025\alpha)}t}\Bigg]$$

$$+\frac{1.2}{0.017-0.0025\alpha}$$

Table 10: Value of $S_1(t,\alpha)$ and $S_2(t,\alpha)$ for different α and t=30

α	$S_1(t,\alpha)$	$S_2(t,\alpha)$
0	471.2537	802.4400
0.1	496.1751	785.8472
0.2	520.9560	769.1426
0.3	545.5951	752.3272
0.4	570.0911	735.4017
0.5	594.4427	718.3672
0.6	618.6485	701.2245
0.7	642.7074	683.9744
0.8	666.6179	666.6179

From the above table we see that for this particular value of t, $S_1(t,\alpha)$ is an increasing function, $S_2(t,\alpha)$ is a decreasing function and $S_1(t,0.8)=S_2(t,0.8)$. Hence this solution is a strong solution.

Case 3: when $\widetilde{V_0}=(190,200,220;0.8)$, $\widetilde{C_0}=(3,5,6;0.7)$ gr/lit

$$\frac{dS}{dt} = 0.4 \cdot 3 - \frac{S}{\widetilde{V_0}} \cdot 3 = 1.2 - \frac{S}{(190,200,220;0.8)} \cdot 3 = 1.2 - \frac{3}{(190,200,220;0.8)} \cdot S$$

$$\approx 1.2 - 3\left(\frac{1}{220}, \frac{1}{200}, \frac{1}{190}; 0.8\right) S = 1.2 - (0.0136, 0.0150, 0.0157; 0.8) S \quad \ldots\ldots\ldots(4.3)$$

With initial condition

$$\tilde{S}(0) = \widetilde{V_0}\widetilde{C_0} = (190,200,220;0.8) \cdot (3,5,6;0.7) \approx (570,1000,1320;0.7)$$

The α -cut of the solution is

$$S_1(t,\alpha) = \frac{1}{2}\left[\left\{\left((570 + 614.28\alpha) - \sqrt{\frac{0.0157 - 0.001\alpha}{0.0136 + 0.002\alpha}}(1320 - 457.14\alpha)\right)\right.\right.$$

$$- 1.2\left(\frac{1}{0.0136 + 0.002\alpha}\right.$$

$$\left.\left. - \frac{1}{\sqrt{(0.0136 + 0.002\alpha)(0.0157 - 0.001\alpha)}}\right)\right\} e^{\sqrt{(0.0136+0.002\alpha)(0.0157-0.001\alpha)}t}$$

$$+ \left\{\left((570 + 614.28\alpha) + \sqrt{\frac{0.0157 - 0.001\alpha}{0.0136 + 0.002\alpha}}(1320 - 457.14\alpha)\right)\right.$$

$$- 1.2\left(\frac{1}{0.0136 + 0.002\alpha}\right.$$

$$\left.\left.\left. + \frac{1}{\sqrt{(0.0136 + 0.002\alpha)(0.0157 - 0.001\alpha)}}\right)\right\} e^{-\sqrt{(0.0136+0.002\alpha)(0.0157-0.001\alpha)}t}\right]$$

$$+ \frac{1.2}{0.0136 + 0.002\alpha}$$

and

$S_2(t,\alpha)$

$$= \frac{1}{2}\sqrt{\frac{0.0136+0.002\alpha}{0.0157-0.001\alpha}}\Bigg[-\Bigg\{\Bigg((570+614.28\alpha)-\sqrt{\frac{0.0157-0.001\alpha}{0.0136+0.002\alpha}}(1320-457.14\alpha)\Bigg)$$

$$-1.2\Bigg(\frac{1}{0.0136+0.002\alpha}$$

$$-\frac{1}{\sqrt{(0.0136+0.002\alpha)(0.0157-0.001\alpha)}}\Bigg)\Bigg\}e^{\sqrt{(0.0136+0.002\alpha)(0.0157-0.001\alpha)}t}$$

$$+\Bigg\{\Bigg((570+614.28\alpha)+\sqrt{\frac{0.0157-0.001\alpha}{0.0136+0.002\alpha}}(1320-457.14\alpha)\Bigg)$$

$$-1.2\Bigg(\frac{1}{0.0136+0.002\alpha}$$

$$+\frac{1}{\sqrt{(0.0136+0.002\alpha)(0.0157-0.001\alpha)}}\Bigg)\Bigg\}e^{-\sqrt{(0.0136+0.002\alpha)(0.0157-0.001\alpha)}t}\Bigg]$$

$$+\frac{1.2}{0.0157-0.001\alpha}$$

Table 11: Value of $S_1(t,\alpha)$ and $S_2(t,\alpha)$ for different $\alpha$and t=30 min

α	$S_1(t,\alpha)$	$S_2(t,\alpha)$
0	12.3754	1238.4965
0.1	106.3941	1159.4057
0.2	200.2341	1079.4423
0.3	293.8915	998.6086
0.4	387.3620	916.9072
0.5	480.6418	834.3406
0.6	573.7266	750.9113
0.7	666.6126	666.6220

From the above table we see that for this particular value of t, $S_1(t,\alpha)$ is an increasing function, $S_2(t,\alpha)$ is a decreasing function and $S_1(t,0.7) < S_2(t,0.7)$. Hence this solution is a strong solution.

5. Conclusion: In this paper, we have used Laplace transform to obtain the solution of first order linear non homogeneous ordinary differential equation in fuzzy environment. Here all fuzzy numbers are taken as GTFNs. The method is discussed with several examples. Further research is in progress to apply and extend the Laplace transform to solve n^{th} order FDEs as well as a system of FDEs. This process can be applied for any economical or bio-mathematical model and problems in engineering and physical sciences.

Conflict of Interests

The author declares that there is no conflict of interests.

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