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THE COMBINED LAPLACE TRANSFORM-DIFFERENTIAL TRANSFORM METHOD FOR SOLVING LINEAR NON-HOMOGENEOUS PDES

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Abstract. In this work, a combined form of the Laplace transform method (LTM) with the differential transform method (DTM) will be used to solve non-homogeneous linear partial differential equations (PDEs). The combined method is capable of handling non-homogeneous linear partial differential equations with variable coefficient. The aim of using the Laplace transform is to overcome the deficiency that is caused by unsatisfied boundary conditions in using differential transform method. Illustrative examples will be examined to support the proposed analysis.

Keywords: Differential transform method (DTM); Laplace transform method (LTM); Non-homogenous PDEs.

2000 AMS Subject Classification: 41A58; 44A10; 35K05; 35L05

1. Introduction

Many interesting phenomena in scientific and engineering applications are governed by partial differential equations, employed to model innumerable nonlinear phenomenon, for instance in solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics. The wave equation, heat equation and Laplaces equations are known as three fundamental equations in mathematical physics, and

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occur in many branches of physics, in applied mathematics as well as in engineering. It is also known that there are two types of these equations, the homogeneous equations that have constant coefficients with many classical solutions, and the non-homogeneous equations with constant coefficients. Most of these types of equations do not have an analytical solution, these equations should be solved by using numerical or approximate methods. In the last decade, there has been some advanced developments including, Adomian decomposition method [2, 12], Differential transform method [3, 7], Variational iteration method [4, 6, 8], and Homotopy perturbation method [5, 13] for solving various types of partial differential equations (PDEs).

The recent development, so-called the Laplace Homotopy Perturbation Method (LHPM) [9], which coupled the homotopy perturbation method (HPM) and the Laplace transform being used to solve one-dimensional non-homogeneous partial differential equations with variable coefficients. DTM can be used to solve linear or nonlinear non-homogeneous PDEs with accurate approximation, which is acceptable for a small range, because boundary conditions are satisfied via the method, and the remaining unsatisfied conditions play no roles in the final results [9]. The aim of using the Laplace transform is to overcome the deficiency that is mainly caused by the unsatisfied conditions. A combined form of the Laplace transform method with the ADM was developed in [11] for analytic treatment of the of the nonlinear Volterra integro-differential equations.

The basic motivation of this work, is to propose a new modification of DTM, which is based on a combination of the Laplace transform and the differential transform methods, called (LDTM), that will be used together to establish exact solutions, or approximations of homogeneous and non-homogeneous partial differential equations. In [9] the numerical experiments show that it works well when coupling the Laplace transform with HPM. Therefore, as mentioned above, the objective of this paper is to show that the LDTM offers a reasonable, reliable solution to PDEs. To demonstrate this we intend to solve several examples in the succeeding sections considering the homogenous, non-homogenous, linear partial differential equations. We observe that the LDTM shows its validity and potential for the solution of non-homogenous, linear partial differential equations in science and engineering applications.

2. Basic idea of the LDTM

To illustrate the basic idea of this method, we consider the general form of one-dimensional non-homogeneous partial differential equations with a variable coefficients of the form [9]

$$u_t + a_0(x)u + a_1(x)u_x + a_2(x)u_{xx} = f(x,t), \quad t > 0, \quad x > 0, \tag{1}$$

subject to

$$u(x,0) = g(x), \tag{2}$$

$$u(0,t) = h_1(t), \qquad u_x(0,t) = h_2(t). \tag{3}$$

and

$$u_{tt} + a_0(x)u + a_1(x)u_x + a_2(x)u_{xx} = f(x,t), \quad t > 0, \quad x > 0, \tag{4}$$

subject to

$$u(x,0) = g_1(x), \qquad u_t(x,0) = g_2(x) \tag{5}$$

$$u(0,t) = h_1(t), \qquad u_x(0,t) = h_2(t). \tag{6}$$

The methodology consists of applying a Laplace transforms to Equations (1)-(3), and by the initial condition (2), we get

$$s\bar{u}(x,s) - g(x) + a_0(x)\bar{u} + a_1(x)\frac{d\bar{u}}{dx} + a_2(x)\frac{d^2\bar{u}}{dx^2} = \bar{f}(x,s), \tag{7}$$

subject to

$$\bar{u}(0,s) = \bar{h}_1(s), \qquad \frac{d\bar{u}}{dx}(0,s) = \bar{h}_2(s). \tag{8}$$

which is second-order initial value problem.

According to Differential transform method, the solution of (7), (8) can be written as:

$$\bar{u}(x,s) = \sum_{k=0}^{\infty} U(k)x^k, \tag{9}$$

where $U(k)$ is the differential transform of $\bar{u}(x,s)$, and $U(k)$ is a function of the parameter s. Once $\bar{u}(x,s)$ is determined, we then apply the Inverse Laplace transform to (9) to get $u(x,t)$. The same procedure will be applied to (4)-(6).

3. Applications and Results

In this section, we demonstrate the effectiveness of the LDTM. The advantage of the proposed method is its capability of combining the two powerful methods for obtaining exact solutions for several illustrative examples.

3.1 Example

We consider the linear-wave equation [10]

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}, x \in R, t > 0 \tag{10}$$

subject to

$$u(x,0) = x, \qquad u_t(x,0) = x^2, \tag{11}$$

and

$$u(0,t) = 0, \qquad u_x(0,t) = 1. \tag{12}$$

By applying the aforesaid method subject to the initial conditions (11), we have

$$s^2\bar{u}(x,s) - sx - x^2 = \frac{1}{2}x^2\frac{d^2\bar{u}}{dx^2}, \tag{13}$$

subject to

$$\bar{u}(0,s) = 0, \qquad \frac{d\bar{u}}{dx}(0,s) = \frac{1}{s}. \tag{14}$$

Now, we apply the Differential transform to equations (13)-(14) to get

$$s^2U(k) - s\delta(k-1) - \delta(k-2) = \frac{1}{2}\sum_{i=0}^{k}(i+1)(i+2)U(i+2)\delta(k-i-2) \quad : k \geq 0, \tag{15}$$

and,

$$U(0) = 0, \qquad U(1) = \frac{1}{s}. \tag{16}$$

By the above recurrence equation (15) and the initials (16), we get the following outcomes

$$U(2) = \frac{1}{s^2-1}, \qquad U(k) = 0 : k \geq 3. \tag{17}$$

Therefore,

$$\begin{aligned} \bar{u}(x,s) &= U(1)x + U(2)x^2 \\ &= \frac{1}{s}x + \frac{1}{s^2-1}x^2. \end{aligned} \tag{18}$$

Apply Inverse Laplace transform to (18) yields

$$u(x,t) = x + x^2\sinh t, \tag{19}$$

which is the exact for the problem in (10-12).

3.2 Example

In this example we consider the one-dimensional inhomogeneous equation given by:

$$u_t + xu_x + u_{xx} = 2t + 2x^2 + 2, \quad t > 0, \quad x \in R, \tag{20}$$

subject to,

$$u(x,0) = x^2, \tag{21}$$

and

$$u(0,t) = t^2, \qquad u_x(0,t) = 0. \tag{22}$$

By using (21), the Laplace transform of (20) is

$$s\bar{u}(x,s) - x^2 + x\frac{d\bar{u}}{dx} + \frac{d^2\bar{u}}{dx^2} = \frac{2}{s^2} + \frac{2}{s} + \frac{2x^2}{s}, \tag{23}$$

subject to

$$\bar{u}(0,s) = \frac{2}{s^3}, \qquad \frac{d\bar{u}}{dx}(0,s) = 0. \tag{24}$$

Apply the Differential transform to (23)-(24) yields

(25)

$$sU(k) - \delta(k-2) + (k+1)(k+2)U(k+2) + \sum_{i=0}^{k}(i+1)U(i+1)\delta(k-i-1) = (\frac{2}{s^2} + \frac{2}{s})\delta(k) + \frac{2}{s}\delta(k-2) \quad : k \geq 0,$$

and

$$U(0) = \frac{2}{s^3}, \qquad U(1) = 0. \tag{26}$$

By the above recurrence equation (25) and the initials (26) we get the following outcomes

$$U(2) = \frac{1}{s}, \qquad U(k) = 0 : k \geq 3. \tag{27}$$

Therefore,

$$\begin{aligned} \bar{u}(x,s) &= U(0) + U(2)x^2 \\ &= \frac{2}{s^3} + \frac{1}{s}x^2. \end{aligned} \tag{28}$$

Apply Inverse Laplace transform to (28) yields

$$u(x,t) = t^2 + x^2, \tag{29}$$

which is the exact for the problem in (20)-(22).

3.3 Example

We consider the one-dimensional wave-like equation with variable coefficients [1]

$$u_{tt} + \frac{1}{2}x^2 u_x + u_{xx} = kxt, \quad t > 0, \quad x > 0, \tag{30}$$

subject to,

$$u(x,0) = c_1 x, \qquad u_t(x,0) = c_2 x^2 \tag{31}$$

and

$$u(0,t) = 0, \qquad u_x(0,t) = c_1 + \frac{k}{6}t^3. \tag{32}$$

where $x \in [0,1], t \in [0,\pi]$, and k, c_1, c_2 are constants. By using (31), the Laplace transform of (30) is

$$s^2\bar{u}(x,s) - c_1 s x - c_2 x^2 + \frac{1}{2}x^2\frac{d^2\bar{u}}{dx^2} = \frac{kx}{s^2}, \tag{33}$$

subject to

$$\bar{u}(0,s) = 0, \qquad \frac{d\bar{u}}{dx}(0,s) = \frac{c_1}{s} + \frac{k}{s^4}. \tag{34}$$

Apply the Differential transform to (33-34) yields

$$s^2U(k) - c_1 s\delta(k-1) - c_2\delta(k-2) - \frac{k}{s^2}\delta(k-1) - \frac{1}{2}\sum_{i=0}^{k}(i+1)(i+2)U(i+2)\delta(k-i-2) = 0 \quad : k \geq 2, \tag{35}$$

and

$$U(0) = 0, \qquad U(1) = \frac{c_1}{s} + \frac{k}{s^4}. \tag{36}$$

By the above recurrence equation (35) and the initials (36) we get

$$U(2) = \frac{c_2}{s^2+1}, \qquad U(k) = 0 : k \geq 3. \tag{37}$$

Therefore,

$$\begin{aligned} \bar{u}(x,s) &= U(1)x + U(2)x^2 \\ &= \left(\frac{c_1}{s} + \frac{k}{s^4}\right)x + \frac{c_2}{s^2+1}x^2. \end{aligned} \tag{38}$$

Apply inverse Laplace transform to (38) yields

$$u(x,t) = c_1 x + \frac{k}{6}xt^3 + x^2 + c_2 x^2 \sin t, \tag{39}$$

which is the exact solution to (30)-(32).

3.4 Example

We consider the following equation [1]

$$u_{tt} - u_{xx} = \alpha t e^x, \quad t > 0, \quad x > 0, \tag{40}$$

subject to,

$$u(x,0) = c\cos x, \qquad u_t(x,0) = 0 \tag{41}$$

and

$$u(0,t) = \alpha(\sinh t - t) + c\cos t, \qquad u_x(0,t) = \alpha(\sinh t - t). \tag{42}$$

for $x, t \in [0,\pi/2]$ and α, c are constants. By using (41), the Laplace transform of (40) is

$$s^2\bar{u}(x,s) - cs\cos x - \frac{d^2\bar{u}}{dx^2} = \frac{ke^x}{s^2}, \tag{43}$$

subject to

$$\bar{u}(0,s) = \alpha\left(\frac{1}{s^2-1} - \frac{1}{s^2}\right) + c\frac{s}{s^2+1}, \qquad \frac{d\bar{u}}{dx}(0,s) = \alpha\left(\frac{1}{s^2-1} - \frac{1}{s^2}\right). \tag{44}$$

Apply the Differential transform to (43)-(44) yields

$$s^2U(k) - csG(k) - (k+1)(k+2)U(k+2) = \frac{\alpha}{s^2k!} \quad : k \geq 0, \tag{45}$$

and

$$U(0) = \alpha\left(\frac{1}{s^2-1} - \frac{1}{s^2}\right) + c\frac{s}{s^2+1}, \qquad U(1) = \alpha\left(\frac{1}{s^2-1} - \frac{1}{s^2}\right), \tag{46}$$

where $G(k)$ in (45) is the coefficient of the Taylor series of $\cos x$.

By the above recurrence equation (45), and the initials (46), we get

$$\begin{aligned} U(2) &= \frac{\alpha}{2}\frac{1}{s^2-1} - \frac{c}{2}\frac{s}{s^2+1} - \frac{\alpha}{2s^2}, \\ U(3) &= \frac{\alpha}{6}\frac{1}{s^2-1} - \frac{\alpha}{6s^2}, \\ U(4) &= \frac{\alpha}{24}\frac{1}{s^2-1} + \frac{c}{24}\frac{s}{s^2+1} - \frac{\alpha}{24s^2}, \\ &. \\ &. \\ &. \end{aligned} \tag{47}$$

Therefore,

$$\bar{u}(x,s) = U(0) + U(1)x + U(2)x^2 + U(3)x^3 + U(4)x^4 + ... \tag{48}$$

Apply Inverse Laplace transform to (48) yields

$$\begin{aligned} u(x,t) &= \alpha(\sinh t - t) + c\cos t \\ &+ x(\alpha\sinh t - \alpha t) \\ &+ x^2(\frac{\alpha}{2}\sinh t - \frac{c}{2}\cos t - \frac{\alpha}{2}t) \\ &+ x^3(\frac{\alpha}{6}\sinh t - \frac{\alpha}{6}t) \\ &+ x^4\frac{\alpha}{24}\sinh t + \frac{c}{24}\cos t - \frac{\alpha}{24}t, \\ &. \\ &. \\ &. \end{aligned} \tag{49}$$

Equivalently,

$$\begin{aligned} u(x,t) &= \alpha \sinh t(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+...) \\ &+ c\cos t(1-\frac{1}{2}x^2+\frac{1}{24}x^4+...) \\ &- \alpha t(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+...) \\ &= \alpha e^x \sinh t + c\cos t\cos x - \alpha t e^x \\ &= \alpha e^x(\sinh t - t) + c\cos t \cos x. \end{aligned} \tag{50}$$

which is the exact solution to (40)-(42).

4. Higher dimensional problems

In this section, we demonstrate the effectiveness of the LDTM, for higher dimension problems. So, in the next two examples, we consider a heat-like equation and a wave-like equation with variable coefficients obeying a three-dimensional IBVP [12].

4.1. Example

Consider the two-dimensional heat-like equation

$$u_t = \frac{1}{2}(y^2u_{xx}+x^2u_{yy})\ 0<x,y<1,\ t>0, \tag{51}$$

subject to,

$$u(x,y,0) = y^2 \tag{52}$$

In view of the initial condition (52), the Laplace transform of (51) is

$$s\bar{u} - y^2 = \frac{1}{2}y^2\bar{u}_{xx} + \frac{1}{2}x^2\bar{u}_{yy}, \tag{53}$$

where $\bar{u} = \bar{u}(x,y,s)$

Apply the Differential transform to (53) yields

$$\begin{aligned} sU(k,h) - \delta(h-2)\delta(k) &= \frac{1}{2}\sum_{j=0}^{h}\sum_{i=0}^{k}(i+1)(i+2)U(i+2,j)\delta(k-i)\delta(h-j-2) \\ &+ \frac{1}{2}\sum_{j=0}^{h}\sum_{i=0}^{k}(j+1)(j+2)U(i,j+2)\delta(k-i-2)\delta(h-j) \end{aligned} \tag{54}$$

where $U(k,h) = \frac{1}{h!k!}\left(\frac{\partial^{k+h}\bar{u}(x,y,s)}{\partial x^k \partial y^h}\right)$: $\quad k,h = 0,1,2,....$

Setting $k = 0$ in (54) yields

$$sU(0,h) - \delta(h-2) = \sum_{j=0}^{h} \delta(h-j-2)U(2,j), \tag{55}$$

and setting $h = 0$ in (54) yields

$$sU(k,0) = \sum_{i=0}^{k} \delta(k-i-2)U(i,2). \tag{56}$$

Running equations (55), (56) for the values of h, k yields

$$U(0,0) = U(0,1) = U(1,0) = 0 \tag{57}$$

and

$$sU(0,2) - 1 = U(2,0), \qquad sU(2,0) = U(0,2) \tag{58}$$

and

$$\begin{aligned} sU(3,0) &= U(1,2), \qquad sU(0,3) = U(2,1) \\ &. \\ &. \\ &. \end{aligned} \tag{59}$$

Solving the algebraic equations in (58) gives

$$U(2,0) = \frac{1}{s^2-1}, \qquad U(0,2) = \frac{s}{s^2-1}, \tag{60}$$

by the use of (54) and the results in (59), the other coefficients of the DT series are all zero. Therefore,

$$\begin{aligned} \bar{u}(x,y,s) &= U(0,2)y^2 + U(2,0)x^2 \\ &= \frac{s}{s^2-1}y^2 + \frac{1}{s^2-1}x^2, \end{aligned} \tag{61}$$

and therefore, we get the exact solutions as

$$u(x,y,t) = y^2\cosh t + x^2\sinh t. \tag{62}$$

4.2. Example

We finally, consider the two-dimensional wave-like equation

$$u_{tt} = \frac{1}{12}(x^2u_{xx} + y^2u_{yy})\ 0 < x,y < 1,\ t > 0, \tag{63}$$

subject to,

$$u(x,,y,0) = x^4, \qquad u_t(x,y,0) = y^4. \tag{64}$$

In view of the initial condition (64), the Laplace transform of (63) is

$$s^2\bar{u} - sx^4 - y^4 = \frac{1}{12}x^4\bar{u}_{xx} + \frac{1}{12}y^4\bar{u}_{yy}, \tag{65}$$

where $\bar{u} = \bar{u}(x,y,s)$

Apply the Differential transform to (65) yields

$$\begin{aligned} s^2U(k,h) - s\delta(k-4)\delta(h) - \delta(k)\delta(h-4) &= \frac{1}{12}\sum_{j=0}^{h}\sum_{i=0}^{k}(i+1)(i+2)U(i+2,j)\delta(k-i-2)\delta(h-j) \\ &+ \frac{1}{12}\sum_{j=0}^{h}\sum_{i=0}^{k}(j+1)(j+2)U(i,j+2)\delta(k-i)\delta(h-j-2) \end{aligned} \tag{66}$$

where $U(k,h) = \frac{1}{h!k!}\left(\frac{\partial^{k+h}\bar{u}(x,y,s)}{\partial x^k \partial y^h}\right): \quad k,h = 0,1,2,....$

Setting $k = 0$ in (66) yields

$$s^2U(0,h) - \delta(h-4) = \frac{1}{12}\sum_{j=0}^{h}\delta(h-j-2)(j+1)(j+2)U(0,j+2), \tag{67}$$

and setting $h = 0$ in (66) yields

$$s^2U(k,0) - s\delta(k-4) = \frac{1}{12}\sum_{i=0}^{k}\delta(k-i-2)(i+1)(i+2)U(i+2,0). \tag{68}$$

From (67), we obtain the following results

$$\begin{aligned} h &= 0 : U(0,0) = 0 \\ h &= 1 : U(0,1) = 0 \\ h &= 2 : U(0,2) = 0 \\ h &= 3 : U(0,3) = 0 \\ h &= 4 : s^2U(0,4) - 1 = U(0,4) \Rightarrow U(0,4) = \frac{1}{s^2-1} \\ h &\geq 5 : U(0,h) = 0 \end{aligned} \tag{69}$$

From (68) we obtain the following results

$$\begin{aligned}
k &= 0 : U(0,0) = 0 \\
k &= 1 : U(1,0) = 0 \\
k &= 2 : U(2,0) = 0 \\
k &= 3 : U(3,0) = 0 \\
k &= 4 : s^2 U(4,0) - s = U(4,0) \Rightarrow U(4,0) = \frac{s}{s^2-1} \\
k &\geq 5 : U(k,0) = 0.
\end{aligned} \tag{70}$$

Now, by (69), (70) and (66) and the use of properties of $\delta(h), \delta(k)$ we conclude that

$$U(h,k) = 0 : \quad h,k = 1,2,3,... \tag{71}$$

Therefore,

$$\begin{aligned}
\bar{u}(x,y,s) &= U(0,4)y^4 + U(4,0)x^4 \\
&= \frac{1}{s^2-1}y^4 + \frac{s}{s^2-1}x^4,
\end{aligned} \tag{72}$$

Applying the inverse Laplace transform, we arrive at the exact solution

$$u(x,y,t) = y^4 \sinh t + x^4 \cosh t. \tag{73}$$

5. Concluding remarks

A combined form of the Laplace transform method with the DTM is effectively used to handle six examples of non-homogeneous linear PDEs. The exact solution have been obtained even with just the two first terms of the LDTM solution, which indicates that the proposed method LDTM need much less computational work compared with the standard DTM. The proposed scheme can be applied for other nonlinear PDEs.

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