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BLOCK HYBRID TOP ORDER METHODS (BhTOMs)

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Abstract: We consider the construction of a class of sixth and tenth order block hybrid top order methods (BhTOMs). Derivation of the methods was based on the use of power series polynomial as basis function. This approach provides continuous interpolants for dense output (approximations). The suggested block hybrid approach eliminates requirement for starting values, it also reduced computational effort. Stability properties of the methods were investigated and found to have good stability regions suitable for stiff system of ODE's.

Keywords: top order methods; hybrid methods; basis function; stiff ordinary differential equations; continuous formulation.

2010 AMS Subject Classification: 65L04.

1. Introduction

We wish to consider the numerical method for solving first order initial value problem of the form

$$y' = f(x,y),\ y(a) = y_0 \tag{1}$$

where, (1) assumed to be continuous and satisfied the Lipchitz conditions.

The general k-step linear multistep method associated with equation (1) is given in the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j} \tag{2}$$

where,

$\alpha_{j's}$ and $\beta_{j's}$, are unknown coefficients of the method to be uniquely determined.

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h = Step length and $x_{n+i} - x_i = nh$ [9, 10].

Equation (2) generates discrete multistep schemes which are used for solving equation (1).

Solving (1) using (2) requires:

(i) That we provide $k-1$ starting values

(ii) $A-stable$ method if our problem is stiff

(iii) High order method whenever we want to achieve high accuracy.

According to [4], the order of equation (2) cannot exceed $k+1$ (for k odd) and $k+2$ (for k even) for the method to be stable.

Several authors such as [10] and [1, 2] have proposed modified forms of equation (2) which are shown to overcome the Dahlquist barrier theorem. These methods known as hybrid method were obtained by incorporating an off-step point(s) in the derivation process.

It is worth noting that [3], introduced a family of boundary value methods (BVMs) which are said to also overcome the Dahlquist barriers. Among some of the methods they proposed are the Top Order Methods (TOMs) which are categorized among a class of symmetric schemes with odd step number k. According to [3], the main TOMs are required to be combined with some additional methods called the initial additional equations and the final additional equations for the method to be stable and also in its implementation. In this sense, they generated discrete multistep TOMs which they used for solving (1)

Accordingly, [3] proposed a class of TOMs based on (2) as:

$$\sum_{i=0}^{v} \alpha_i (y_{n-v-1+i} - y_{n+v-i}) = h \sum_{i=0}^{v} \beta_i (f_{n-v-1+i} + f_{n+v-i}), n = v+1, ..., N-1 \quad (3)$$

In this paper, we developed a class of sixth and tenth order block hybrid methods using continuous formulation of ETR2s with one off-grid interpolation point. In order to solve (3), we seek an approximation of the exact solution $y(x)$ by assuming a continuous solution $\overline{Y}(x)$ of the form:

$$\overline{Y}(x) = \sum_{j=0}^{r+s-1} \psi_j \varphi_j(x) \quad (4)$$

where, $\varphi_j(x)$ denote a power series polynomial used as basis function and given in the form:

$$\varphi_j(x) = \left(\frac{x - x_n}{h} \right)^j \quad (5)$$

and $\psi_{j's}$ are unknown coefficients of the method with degree $p = 2k$. The number of interpolation points r and the number of distinct collocation points s are uniquely chosen to satisfy $1 < r < k$ and $s > 0$. The interger $k \geq 2$ denotes the step number of the given method. We need to obtain $2k - 1$ equations for a particular class of block hybrid methods.

Equation (4) satisfies the unperturbed ODE

$$\begin{Bmatrix} Y'(x) = f(x, y(x)), x_k \leq x \leq x_{k+p} \\ Y(x_k) = Y_k \end{Bmatrix} \tag{6}$$

Equation (3) is then evaluated at some end point(s) and at some off-step point(s) to obtain the block hybrid top order methods (BhTOMs).

2. Preliminaries

2.1: Sixth order TOMs

Case I: Using equation (4) in (6) with $r = 3$ and $s = 4$, we obtain the following equations

$$\left.\begin{aligned} y_n &= \psi_0 \\ y_{n+1} &= \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6 \\ y_{n+2} &= \psi_0 + 2\psi_1 + 2^2\psi_2 + 2^3\psi_3 + 2^4\psi_4 + 2^5\psi_5 + 2^6\psi_6 \\ hf_n &= \psi_1 \\ hf_{n+1} &= \psi_1 + 2\psi_2 + 3\psi_3 + 4\psi_4 + 5\psi_5 + 6\psi_6 \\ hf_{n+2} &= \psi_1 + 4\psi_2 + 12\psi_3 + 32\psi_4 + 80\psi_5 + 192\psi_6 \\ hf_{n+3} &= \psi_1 + 6\psi_2 + 27\psi_3 + 108\psi_4 + 405\psi_5 + 1458\psi_6 \end{aligned}\right\} \tag{7}$$

Representing equation (7) in matrix form yield:

$$MQ = H \tag{8}$$

where,

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2^0 & 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 4 & 12 & 32 & 80 & 192 \\ 0 & 1 & 6 & 27 & 108 & 405 & 1458 \end{bmatrix}, \quad Q = \begin{bmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ hf_n \\ hf_{n+1} \\ hf_{n+2} \\ hf_{n+3} \end{bmatrix}$$

Solving for $\psi_{j's}$, $j = 0,1,...,6$, we obtain

$$\psi_0 = y_n$$

$$\psi_1 = hf_n$$

$$\psi_2 = \frac{36}{11}y_{n+1} - \frac{109}{33}hf_n - \frac{27}{4}y_n + \frac{153}{44}y_{n+2} - \frac{63}{11}hf_{n+1} - \frac{27}{22}hf_{n+2} + \frac{1}{33}hf_{n+3}$$

$$\psi_3 = \frac{45}{4}y_n + \frac{183}{44}hf_n - \frac{20}{11}y_{n+1} - \frac{415}{44}y_{n+2} + \frac{145}{11}hf_{n+1} + \frac{151}{44}hf_{n+2} - \frac{1}{11}hf_{n+3}$$

$$\psi_4 = \frac{195}{22}y_{n+2} - \frac{82}{33}hf_n - \frac{15}{11}y_{n+1} - \frac{15}{2}y_n - \frac{467}{44}hf_{n+1} - \frac{37}{11}hf_{n+2} + \frac{13}{132}hf_{n+3}$$

$$\psi_5 = \frac{9}{4}y_n + \frac{31}{44}hf_n + \frac{12}{11}y_{n+1} - \frac{147}{44}y_{n+2} + \frac{79}{22}hf_{n+1} + \frac{59}{44}hf_{n+2} - \frac{1}{22}hf_{n+3}$$

$$\psi_6 = \frac{19}{44}y_{n+2} - \frac{5}{66}hf_n - \frac{2}{11}y_{n+1} - \frac{1}{4}y_n - \frac{19}{44}hf_{n+1} - \frac{2}{11}hf_{n+2} + \frac{1}{132}hf_{n+3} \qquad (9)$$

Substituting (9) in (4), we obtain the continuous formulation (cf) of (3) and evaluating at end point(s) and at some off-step points yield the following set of equations which when implemented in block form yields the numerical solution $y_i, i = 1,2,...$ of the sixth order BhTOMs (10).

Table 1: Coefficients of sixth order BhTOMs

p	No. of eqns	η_k	α_0	$\alpha_{\frac{1}{2}}$	α_1	$\alpha_{\frac{3}{2}}$	α_2	α_3		η_k	β_0	$\beta_{\frac{1}{2}}$	β_1	$\beta_{\frac{3}{2}}$	β_2	β_3
	1)	60	-11	-	-27	-	27	11		20	1	-	9	-	9	1
	2)	390	-143	-	-1512	2816	-1161	-		130	12	-	207	-	-90	1
6	3)	1		-	-1	-	1	-		1080	-1	-	189	704	189	-1
	4)	930	189	-2816	1512	-	413	-		310	-56	-	321	-	46	-1
	5)	30	-33	-	36	-	-3	-		900	155	704	75	-	-35	1

(10)

2.2: Tenth Order TOMs

Case 2: Consider the following specifications $r = 5$ and $s = 6$.

Adopting the approach as in subsection (2.1), we obtain (8) where

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2^0 & 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 & 2^7 & 2^8 & 2^9 & 2^{10} \\ 3^0 & 3^1 & 3^2 & 3^3 & 3^4 & 3^5 & 3^6 & 3^7 & 3^8 & 3^9 & 3^{10} \\ 4^0 & 4^1 & 4^2 & 4^3 & 4^4 & 4^5 & 4^6 & 4^7 & 4^8 & 4^9 & 4^{10} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 0 & 1 & 4 & 12 & 32 & 80 & 192 & 448 & 1024 & 2304 & 5120 \\ 0 & 1 & 6 & 27 & 108 & 405 & 1458 & 5103 & 17496 & 59049 & 196830 \\ 0 & 1 & 8 & 48 & 256 & 1280 & 6144 & 28672 & 131072 & 589824 & 2621440 \\ 0 & 1 & 10 & 75 & 500 & 3125 & 18750 & 109375 & 625000 & 3515625 & 19531250 \end{bmatrix},$$

$$Q = [\psi_0 \;,\; \psi_1 \;,\; \psi_2 \;,\; . \; . \; . \; . \;,\; \psi_{10}]^T$$

And

$$H = [y_n \;,\; y_{n+1} \;,\; . \; . \; . \; . \;,\; y_{n+4} \quad hf_n \;,\; hf_{n+1} \;,\; . \; . \; . \; . \;,\; hf_{n+5}]^T$$

Solving for $\psi_{j's}$, $j = 0,1,...,10$, we obtain

$$\psi_0 = y_n, \qquad \psi_1 = hf_n$$

$$\psi_2 = \frac{756}{137} y_{n+2} - \frac{17911}{4110} hf_n - \frac{7340}{411} y_{n+1} - \frac{175}{16} y_n + \frac{2660}{137} y_{n+3} + \frac{24965}{6576} y_{n+4}$$
$$- \frac{2710}{137} hf_{n+1} - \frac{3990}{137} hf_{n+2} - \frac{5120}{411} hf_{n+3} - \frac{565}{548} hf_{n+4} + \frac{6}{685} hf_{n+5}$$

$$\psi_3 = \frac{23155}{864} y_n + \frac{158885}{19728} hf_n + \frac{98326}{1233} y_{n+1} - \frac{3909}{274} y_{n+2} - \frac{283906}{3699} y_{n+3} - \frac{612617}{39456} y_{n+4} + \frac{27299}{411} hf_{n+1}$$
$$+ \frac{15392}{137} hf_{n+2} + \frac{61808}{1233} hf_{n+3} + \frac{27839}{6576} hf_{n+4} - \frac{5}{137} hf_{n+5}$$

$$\psi_4=\frac{21421}{2192}y_{n+2}-\frac{81847}{9864}hf_n-\frac{1930727}{14796}y_{n+1}-\frac{53381}{1728}y_n+\frac{1852423}{14796}y_{n+3}+\frac{6252593}{236736}y_{n+4}$$
$$-\frac{916199}{9864}hf_{n+1}-\frac{586421}{3288}hf_{n+2}-\frac{103064}{1233}hf_{n+3}-\frac{142777}{19728}hf_{n+4}+\frac{209}{3288}hf_{n+5}$$

$$\psi_5=\frac{23975}{1152}y_n+\frac{138559}{26304}hf_n+\frac{181741}{1644}y_{n+1}+\frac{1701}{548}y_{n+2}-\frac{542759}{4932}y_{n+3}-\frac{1284437}{52608}y_{n+4}$$
$$+\frac{234227}{3288}hf_{n+1}+\frac{1002359}{6576}hf_{n+2}+\frac{31009}{411}hf_{n+3}+\frac{177071}{26304}hf_{n+4}-\frac{199}{3288}hf_{n+5}$$

$$\psi_6=\frac{1127945}{19728}y_{n+3}-\frac{281921}{131520}hf_n-\frac{1079785}{19728}y_{n+1}-\frac{15575}{2192}y_{n+2}-\frac{10115}{1152}y_n+\frac{2121875}{157824}y_{n+4}$$
$$-\frac{433337}{13152}hf_{n+1}-\frac{507323}{6576}hf_{n+2}-\frac{16612}{411}hf_{n+3}-\frac{32747}{8768}hf_{n+4}+\frac{2273}{65760}hf_{n+5}$$

$$\psi_7=\frac{1355}{576}y_n+\frac{7375}{13152}hf_n+\frac{9031}{548}y_{n+1}+\frac{513}{137}y_{n+2}-\frac{89023}{4932}y_{n+3}-\frac{39691}{8768}y_{n+4}$$
$$+\frac{15529}{1644}hf_{n+1}+\frac{78337}{3288}hf_{n+2}+\frac{21701}{1644}hf_{n+3}+\frac{16687}{13152}hf_{n+4}-\frac{5}{411}hf_{n+5}$$

$$\psi_8=\frac{33509}{9864}y_{n+3}-\frac{1199}{13152}hf_n-\frac{29341}{9864}y_{n+1}-\frac{2073}{2192}y_{n+2}-\frac{7}{18}y_n+\frac{17993}{19728}y_{n+4}$$
$$-\frac{3603}{2192}hf_{n+1}-\frac{7247}{1644}hf_{n+2}-\frac{1411}{548}hf_{n+3}-\frac{3397}{13152}hf_{n+4}+\frac{17}{6576}hf_{n+5}$$

$$\psi_9=\frac{125}{3456}y_n+\frac{661}{78912}hf_n+\frac{365}{1233}y_{n+1}+\frac{65}{548}y_{n+2}-\frac{1295}{3699}y_{n+3}-\frac{15895}{157824}y_{n+4}$$
$$+\frac{523}{3288}hf_{n+1}+\frac{2951}{6576}hf_{n+2}+\frac{1357}{1932}hf_{n+3}+\frac{253}{8768}hf_{n+4}-\frac{1}{3288}hf_{n+5}$$

$$\psi_{10}=\frac{899}{59184}y_{n+3}-\frac{131}{394560}hf_n-\frac{739}{59184}y_{n+1}-\frac{13}{2192}y_{n+2}-\frac{5}{3456}y_n+\frac{2213}{473472}y_{n+4}$$
$$-\frac{259}{39456}hf_{n+1}-\frac{127}{6576}hf_{n+2}-\frac{61}{4932}hf_{n+3}-\frac{107}{78912}hf_{n+4}+\frac{1}{65760}hf_{n+5} \quad (11)$$

Substituting (11) into (4) and evaluating the (cf) of (3) for the tenth order TOMs at end point(s) and at some off-step points yield the following set of equations for the tenth order BhTOMs (12).

Table 2: Coefficients of the tenth order BhTOMs

p	No of eqns	η_k	α_0	$\alpha_{\frac{1}{2}}$	α_1	$\alpha_{\frac{3}{2}}$	α_2	$\alpha_{\frac{5}{2}}$	α_3	$\alpha_{\frac{7}{2}}$	α_4	α_5
	1)	15120	-274	-	-3250	-	-4000	-	4000	-	3250	274
	2)	1689660	-61239	-	-1091000	-	-8667000	17956864	-7773000	-	-364625	-
10	3)	4524660	147275	-	6790200	-17956864	8687000	-	2073800	-	278589	-
	4)	28060200	3859975	-17956864	1166200	-	4223800	-	7376264	-	1330625	-
	5)	1869000	-	-	-79	-	-5103	-	5103	-	79	-
	6)	3454500	-959	-	-99252	-	102060	-	-1228	-	-621	-
	7)	1808100	12741	-	-27958	-	2538	-	10614	-	2065	-
	8)	12651660	-147275	-	-2025464	-	-4223800	-	-6517000	17956864	-5043325	-
	9)	635460	33565	-	419860	-	532980	-	-4062100	-	3075695	-

No of eqns	η_k	β_0	$\beta_{\frac{1}{2}}$	β_1	$\beta_{\frac{3}{2}}$	β_2	$\beta_{\frac{5}{2}}$	β_3	$\beta_{\frac{7}{2}}$	β_4	β_5
1)	252	1	-	25	-	100	-	100	-	25	1
2)	56322	429	-	14150	-	125100	-	-80400	-	-2975	18
3)	150822	-977	-	-57990	-	170820	-	36560	-	2427	-18
4)	935340	-20237	-	400320	-	392980	-	151312	-	11935	-980
5)	654150000	-27	-	6175	-	367200	1122304	367200	-	6175	-27
6)	120907500	6356	-	667170	2244608	597240	-	-54880	-	-6048	54
7)	189850500	-186543	-1122304	687645	-	1657320	-	713664	-	58815	-497
8)	60246	151	-	4186	-	20580	-	39760	-	-4445	14
9)	317730	-3648	-	-94899	-	-394940	-	-68880	1122304	241710	-497

(12)

3. Analysis of the Sixth and Tenth Order BhTOMs

Following [6] and [9], we define the order and error constants associated with equation (2) to be the linear difference operator L as

$$L[y(x),h] = \sum_{j=0}^{k}\{\alpha_j y(x_{n+j}) - h\beta_j y(x_{n+j})\} \qquad (13)$$

We take the Taylor series expansion about the point x to yield

$$L[y(x),h] = C_0 y(x_n) + C_1 h y'(x_n) + \ldots + C_q h^q y^q(x_n) + \ldots \qquad (14)$$

where, the constant coefficients

$$C_0 = \sum_{j=0}^{k}\alpha_j$$

$$C_1 = \sum_{j=0}^{k} j\alpha_j$$

…

$$C_q = \frac{1}{q!}\sum_{j=1}^{k} j^q \alpha_j - \frac{1}{(q-1)!}\sum_{j=1}^{k} j^{(q-1)}\beta_j \tag{15}$$

The method (2) is said to be of order p if $C_0 = C_1 = ... = C_p = 0, C_{p+1} \neq 0$, C_{p+1} is called the error constant.

Following (13) to (15), the block hybrid TOMs (10) has order and error constants given respectively as:

$p = (6,6,6,6,6)^T$ and $C_7 = \left(-\frac{1}{2800}, -\frac{1}{2800}, \frac{1}{40320}, \frac{61}{260400}, -\frac{31}{302400}\right)^T$

Subsequently, the order and error constants of equation in table 2 are given as:

$p = (10,10,10,10,10,10,10,10,10)^T$ and

$$C_{11} = \begin{pmatrix} -\frac{1}{698544}, -\frac{1}{698544}, \frac{1019}{1393595280}, \frac{431}{529634160}, \frac{23}{23026080000}, \\ -\frac{389}{10639860000}, \frac{7333}{33413688000}, -\frac{157}{238574160}, \frac{5833}{1677614400} \end{pmatrix}^T$$

BhTOMs (10) and (12) have order each greater or equal to one i:e $p \geq 1$, hence consistent [7].

3.1 Convergence of the Methods

3.1.1 Consistency

The block hybrid method (10) can be represented by a matrix finite difference equation in block form

$$A^0 Y_{m+1} = A^1 Y_{m-1} + h[\beta_0 F_{m+1} + \beta_1 F_{m-1}] \tag{16}$$

where,

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1057}{1408} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\beta_0 = \begin{bmatrix} \frac{35}{72} & -\frac{487}{1920} & \frac{49}{360} & -\frac{211}{5760} & \frac{1}{640} \\ \frac{32}{45} & \frac{11}{120} & \frac{8}{135} & -\frac{7}{360} & \frac{1}{1080} \\ \frac{27}{40} & \frac{243}{640} & \frac{13}{40} & -\frac{27}{640} & \frac{1}{640} \\ \frac{32}{45} & \frac{4}{15} & \frac{32}{45} & \frac{7}{45} & 0 \\ 0 & \frac{81}{40} & -\frac{8}{5} & \frac{81}{40} & \frac{11}{40} \end{bmatrix}, \quad \beta_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{959}{5760} \\ 0 & 0 & 0 & 0 & \frac{169}{1080} \\ 0 & 0 & 0 & 0 & \frac{103}{640} \\ 0 & 0 & 0 & 0 & \frac{7}{45} \\ 0 & 0 & 0 & 0 & \frac{11}{40} \end{bmatrix}$$

and

$$Y_{m+1} = \begin{pmatrix} y_{n+\frac{1}{2}} & y_{n+1} & y_{n+\frac{3}{2}} & y_{n+2} & y_{n+3} \end{pmatrix}^T, \quad Y_{m-1} = \begin{pmatrix} y_{n-2} & y_{n-\frac{3}{2}} & y_{n-1} & y_{n-\frac{1}{2}} & y_n \end{pmatrix}^T,$$

$$F_{m+1} = \begin{pmatrix} f_{n+\frac{1}{2}} & f_{n+1} & f_{n+\frac{3}{2}} & f_{n+2} & f_{n+3} \end{pmatrix}^T \text{ and } F_{m-1} = \begin{pmatrix} f_{n-2} & f_{n-\frac{3}{2}} & f_{n-1} & f_{n-\frac{1}{2}} & f_n \end{pmatrix}^T$$

3.1.2 Zero-Stability

Definition 3.1: *A block method is said to be zero stable if as* $h \to 0,$ *the roots* $r_j = 1(2)k$ *of the first characteristics polynomials* $\rho(R)$ *is given by*

$$\rho(R) = \det\left[\sum_{i=0}^{k} A^{(i)} R^{k-i}\right] = 0 \tag{17}$$

satisfies $|R_j| \le 1$, *the multiplicity must not exceed two [6].*

Sustituting A^0 and A^1 into (17), we obtain the first characteristics polynomial of the block hybrid TOMs (10) as

$$\rho(R) = \det\left(RA^0 - A^1\right) = 0$$

$$\rho(R) = \det\left[R\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1057}{1408} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right]$$

$$\rho(R) = \det\begin{bmatrix} R & 0 & 0 & 0 & \frac{1057}{1408} \\ 0 & R & 0 & 0 & -1 \\ 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & R & -1 \\ 0 & 0 & 0 & 0 & R-1 \end{bmatrix}$$

$$\rho(R) = R^4(R-1) = 0$$

This implies that $R_1 = 1, R_2 = R_3 = R_4 = R_5 = 0$. Hence by definition 3.1, the block hybrid TOMs (10) is zero-stable. By [7], the hybrid method is convergence since it is consistent and zero stable. Similarly, the block hybrid method (12) can also be represented as in (16) where A^0 and A^1 are substituted into (17) to obtain its first characteristic polynomial in the form

$$\rho(R) = \det\left(RA^0 - A^1\right) = 0$$

$$\rho(R) = \det\left[R\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right]$$

$$\rho(R) = \det\begin{bmatrix} R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & R & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & R & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & R & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & R & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & R & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & R & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R-1 \end{bmatrix}$$

$$\rho(R) = R^8(R-1) = 0$$

where, $R_1 = 1, R_2 = R_3 = \ldots = R_9 = 0$. This shows that our block hybrid method (12) is zero-stable.

3.1.3 Absolute Stability Regions of the Block Hybrid Top Order Methods

The absolute stability regions of (10) and (12) are plotted by reformulating as general linear methods. The absolute stability regions of the methods are shown in figures (1) and (2). While figure (1) shows the $A(\alpha)-stable$ nature of BhTOMs (10), figure (2) shows that BhTOMs (12) is $A-stable$.

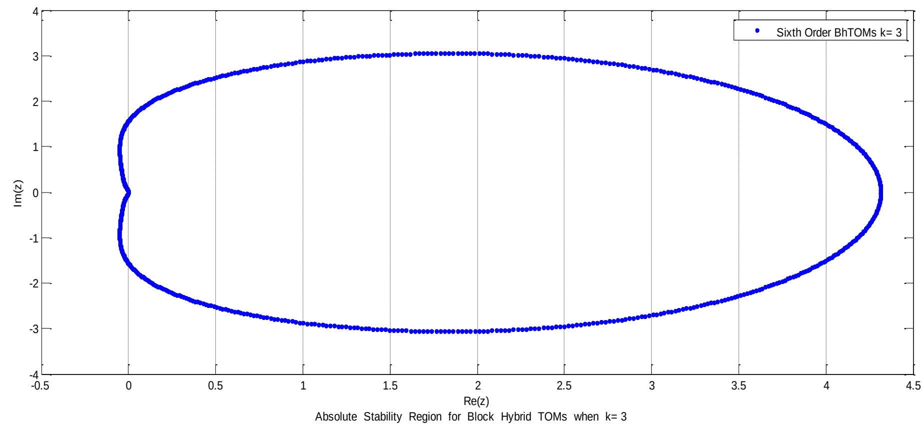

Fig.1: Absolute stability region for BhTOMs (10)

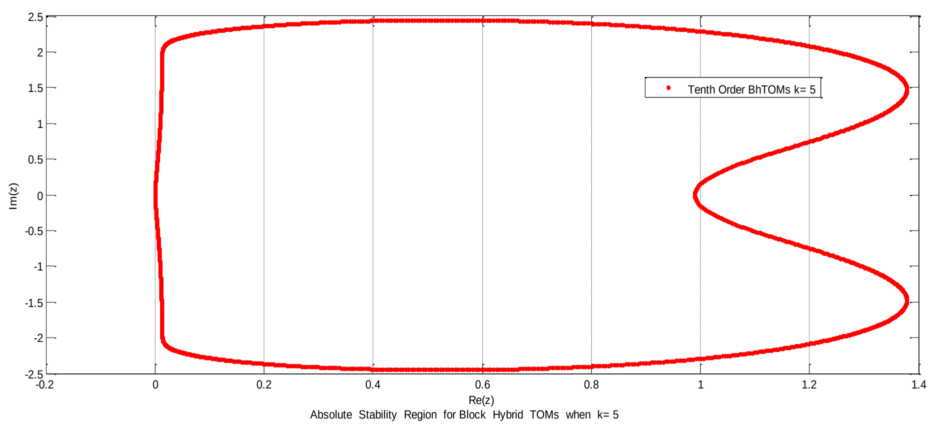

Fig.2: Absolute stability region for BhTOMs (12)

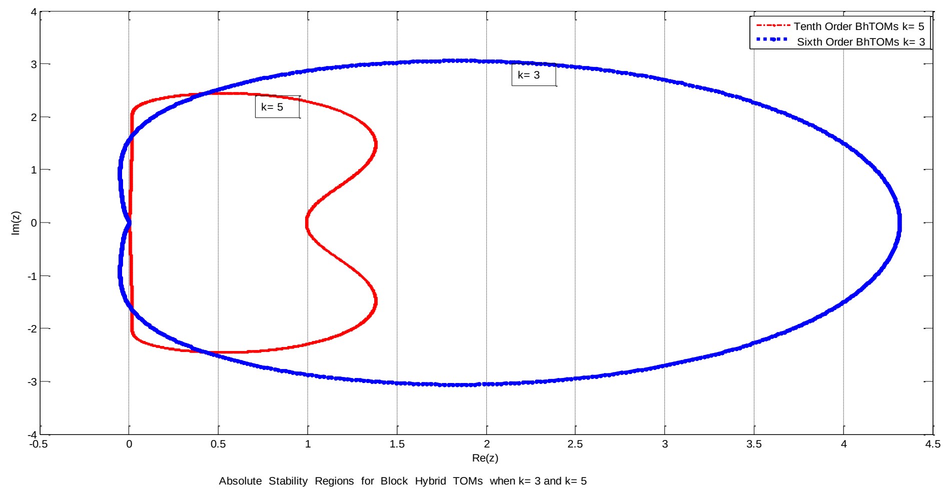

Fig.3: Absolute stability regions for BhTOMs (10 and 12)

4. Numerical Example

In this section, we report four numerical examples taken from several literatures to compare the performance of our methods. The absolute errors obtained from computed results at selected mesh points are also reported in the tables shown. The solution curves of our presented suggest that our block hybrid top order methods (BhTOMs) competes favorably well with the Ode Solver. All computed results are obtained using MATLAB package.

Problem 1:

$y_1' = -10(y-1)^2, \qquad y(0) = 2, \qquad h = 0.01, \qquad 0 \le x \le 1$

The Analytic Solution for the system Source: [12]

$$y(x) = 1 + \frac{1}{(1+10x)}$$

Table 3: Comparism of Absolute Errors of problem 1

Mesh Values	*Method [13]*	*Method [12]*	*BhTOMs (10) P=6*	*BhTOMs (12) P=10*
0.01	*1.07e-003*	5.5272e-005	*2.7447e-004*	4.4887e-004
0.02	*2.38e-003*	7.5203e-005	*4.2156e-004*	2.6395e-004
0.03	*2.21e-003*	7.9907e-005	*4.9607e-004*	1.6636e-004
0.04	*5.36e-003*	7.8033e-005	*5.2837e-004*	1.1329e-004
0.05	*7.53e-003*	7.3464e-005	*5.3590e-004*	8.1815e-005
0.06	*9.00e-003*	6.7983e-005	*5.2894e-004*	6.1752e-005
0.07	*9.98e-003*	6.2411e-005	*5.1365e-004*	4.8220e-005
0.08	*1.06e-002*	5.7112e-005	*4.9382e-004*	3.8678e-005
0.09	*1.10e-002*	5.2232e-005	*4.7175e-004*	3.1703e-005
0.10	*1.12e-002*	4.7811e-005	*4.4887e-004*	2.6453e-005

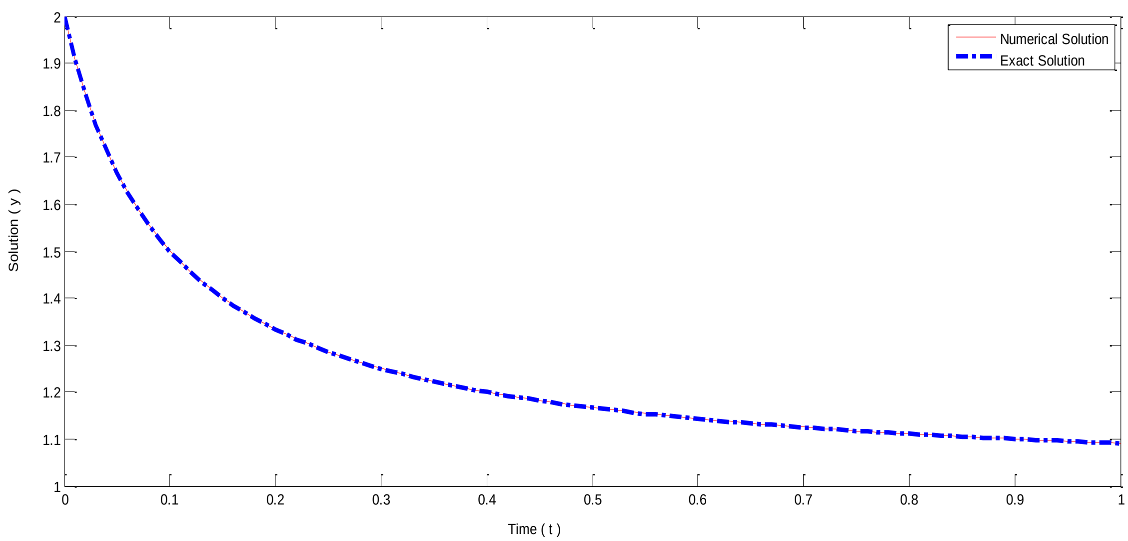

Fig. 4: Solution curve for problem 1 using BhTOMs (10)

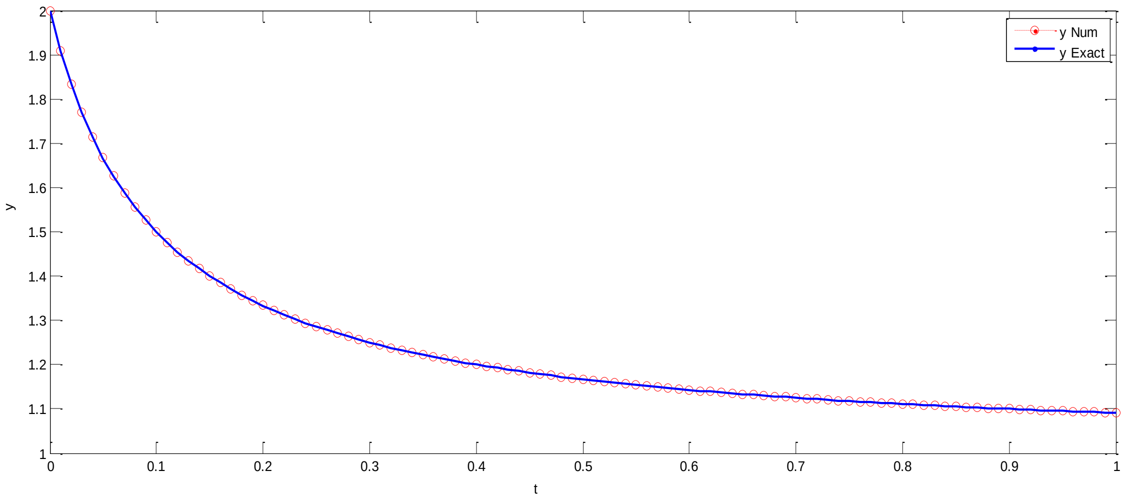

Fig. 5: Solution curve for problem 1 using BhTOMs (12)

Problem 2:

$$y_1' = 998y_1 + 1998y_2, \qquad y_1(0) = 1, \quad h = 0.01, \quad 0 \le x \le 10$$

$$y_2' = -999y_1 - 1999y_2, \qquad y_2(0) = 1$$

The Analytic Solution for the system

$$y_1 = 4e^{-x} - 3e^{-1000x}$$ Source: [8]

$$y_2 = -2e^{-x} + 3e^{-1000x}$$

Table 4: Comparism of Absolute Errors of problem 2

Mesh Values	*Method [8] P=10*	*BhTOMs (12) P=10*
0	*0*	*0*
20	*7.64e-013*	*3.2419e-014*
40	*1.95e-013*	*7.4940e-016*
60	*5.55e-014*	*2.1597e-015*
80	*6.44e-015*	*4.4799e-016*
100	*1.29e-015*	*6.9009e-017*

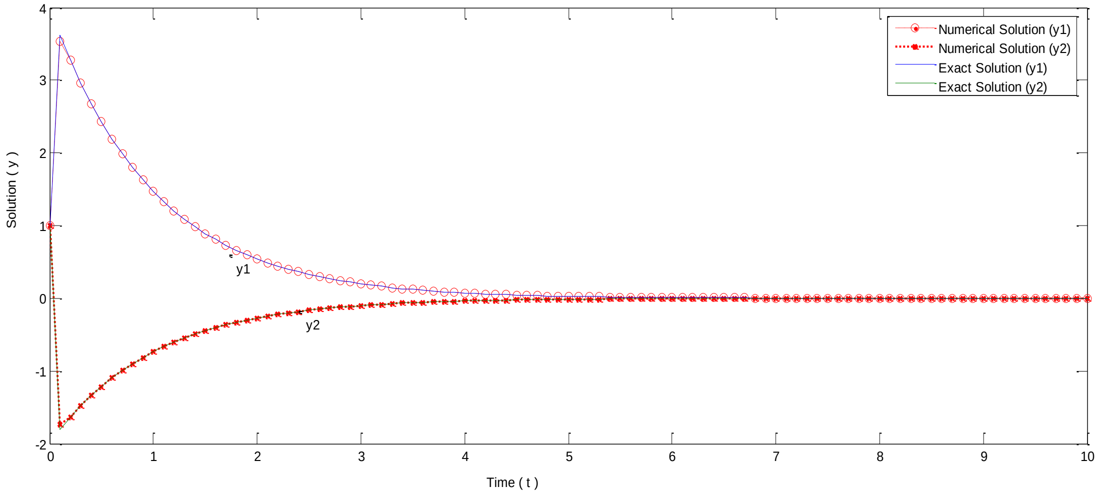

Fig. 6: Solution curve for problem 2 using BhTOMs (12)

Problem 3:

$$y_1'(x) = -8y_1(x) + 7y_2(x), \qquad y_1(0) = 1$$

$$y_2'(x) = 42y_1(x) - 43y_2(x), \qquad y_2(0) = 8$$

$h = 0.1$, $\quad 0 \le x \le 10$ Source: [5 and 8]

Theoretical Solution for the system

$$y_1(x) = 2e^{-x} + e^{-50x} \text{ and } \quad y_2(x) = 2e^{-x} + 6e^{-50x}$$

Table 5: Comparism of Absolute Errors of problem 3

Mesh Values	*Method [5] P=9*	*BhTOMs (10) P=6*	*Method [8] P=10*	*BhTOMs (12) P=10*
10	*1.04e-002*	*2.3896e-008*	-	7.1054e-015
20	*3.81e-003*	*3.2203e-010*	*1.17e-015*	1.9984e-015
30	*1.34e-005*	*3.2552e-012*	-	4.9960e-016
40	*4.74e-006*	*2.8866e-014*	*2.29e-016*	8.3267e-017
50	*1.67e-008*	*6.6613e-016*	-	1.7347e-018
60	*5.90e-009*	*1.5543e-015*	*3.30e-017*	1.2143e-017
70	*2.08e-011*	*1.8874e-015*	-	1.0192e-017
80	*7.33e-012*	*2.3315e-015*	*6.07e-018*	5.6379e-018
90	*2.58e-014*	*2.3315e-015*	-	2.6563e-018
100	*9.12e-015*	*2.4425e-015*	*9.35e-019*	1.2875e-018

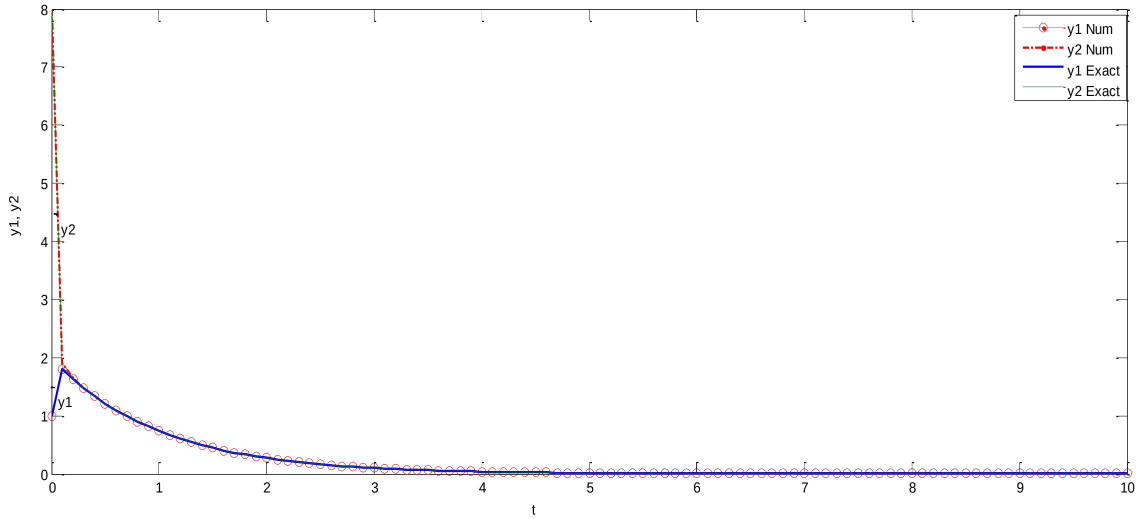

Fig. 7: Solution curve for problem 3 using BhTOMs (10)

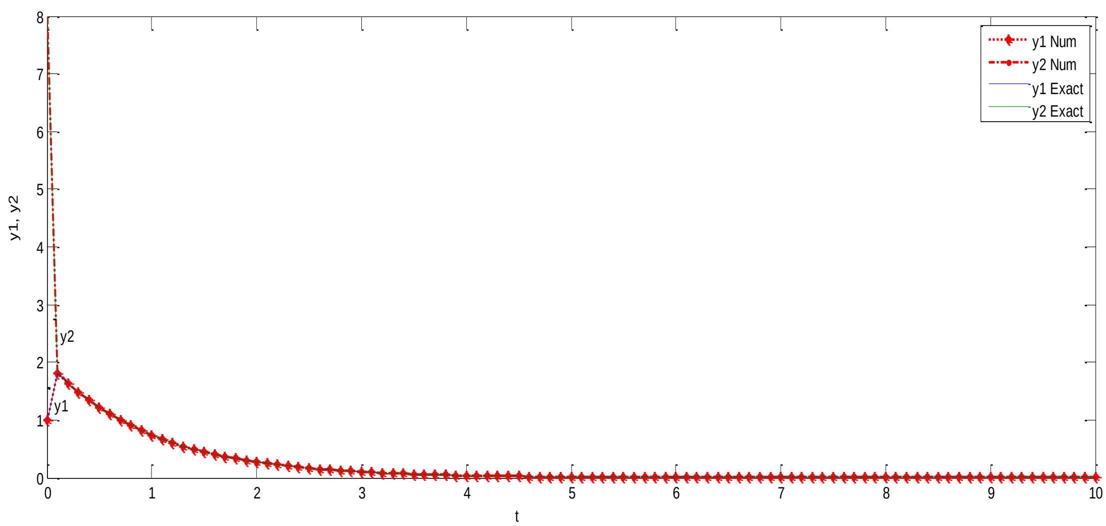

Fig. 8: Solution curve for problem 3 using BhTOMs (12)

Problem 4:

$y_1' = -0.1y_1 - 49.9y_2, \qquad y_1(0) = 1,$

$y_2' = -50y_2, \qquad y_2(0) = 0$

$y_3' = 70y_2 - 120y_3, \qquad y_3(0) = 0$

$h = 0.01, \quad 0 \le x \le 2$ Source: [11]

The Analytic Solution for the system

$y_1 = e^{-0.1x} + e^{-50x}, \quad y_2 = e^{-50x}$ and $y_3 = e^{-50x} + e^{-120x}$

Table 6: Comparism of Absolute Errors of problem 4

h	*y*	*Method [11]*	*BhTOMs (10)* *P=6*	*BhTOMs (12)* *P=10*
0.1	y_1	*2.41e-008*	*1.0436e-014*	2.8089e-014
	y_2	*3.54e-011*	*1.3683e-018*	1.4447e-018
	y_3	*6.93e-009*	*4.7828e-001*	4.7828e-001

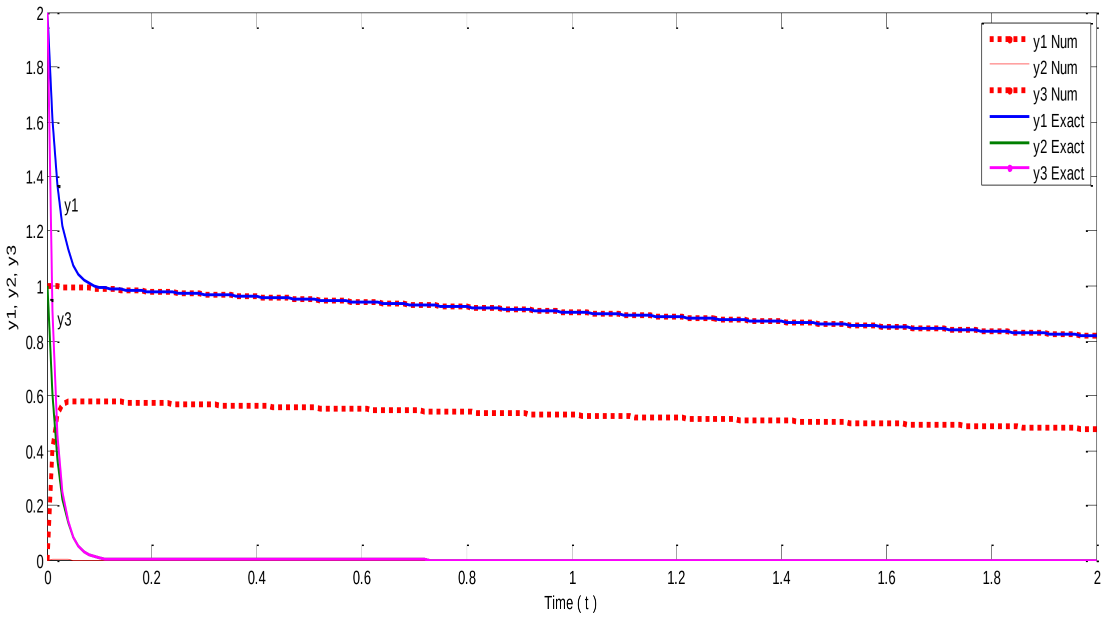

Fig. 9: Solution curve for problem 4 using BhTOMs (10)

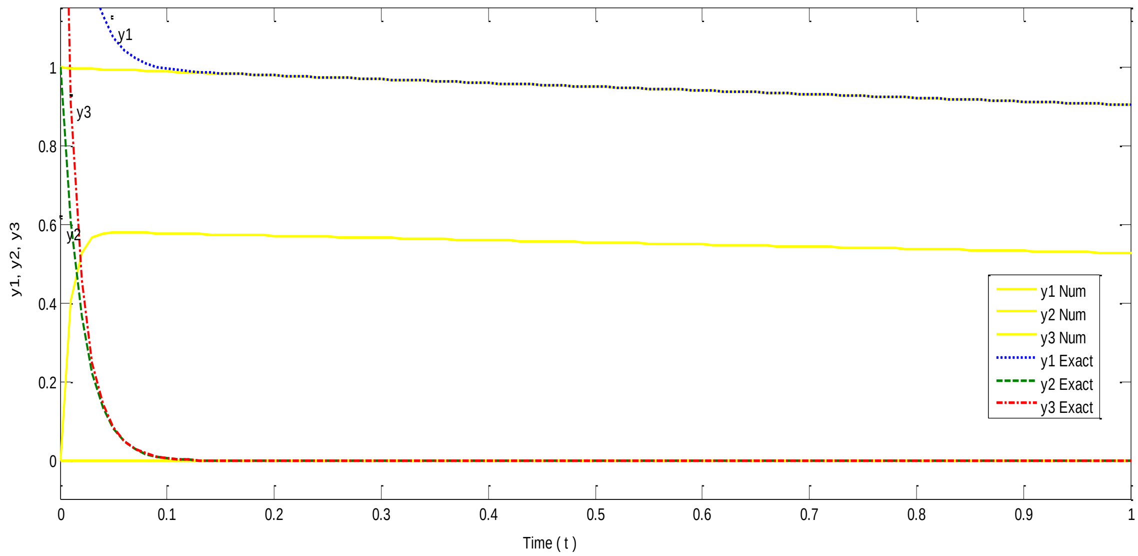

Fig. 10: Solution curve for problem 4 using BhTOMs (12)

5. Conclusion

We developed sixth and Tenth order block hybrid top order methods (BhTOMs). The methods (10) and (12) were derived using trial/basis functions. Graphs of their absolute stability regions show that they are $A(\alpha)-stable$ and $A-stable$ respectively, as such suitable for implementation on stiff system of ODE's (fig. 1-3). The methods proved to be very efficient when tested on four numerical stiff systems and when compared with results obtained by relevant authors (tables 3-6), they also compete well with known ODE solvers (fig. 4-10).

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] Butcher, J.C., A modified multistep method for the numerical integration of ordinary differential equations, J. Assoc. Comput. Math. 12 (1965), 124-135.

[2] Butcher, J.C, Numerical Methods for Ordinary differential systems, John Wiley & Sons, West Sussex, England, 2003.

[3] L. Brugnano and D. Trigiante, Solving Differential Problems by Multistep Initial and Boundary Value Methods, Gordon and Breach Science Publishers, Amsterdam, 1998.

[4] Dahlquist, G., A special stability problem for linear multistep methods, BIT Numer. Math. 3 (1963) 27-43.

[5] Dauda G. Yakubu et-al., High order A-Stable continuous general linear methods for solution of systems of initial value problems in ODE's, Leonardo J. Sci. 21 (2012), 13-32.

[6] Fatunla, S.O., Block methods for second order IVP's, Int. J. Comput. Math. 41 (1991), 55-63.

[7] Henrici, P., Discrete variable methods for ODE's, John Wiley, New York, 1962

[8] Kumleng G. M and Sirisena U.W.W., $A(\alpha)$-Stable order ten second derivative block multistep method for stiff initial value problems, Int. J. Math. Stat. Invent. 2 (2014), 37-43.

[9] Lambert, J.D., Computational methods for ordinary differential equations, John Wiley, New York, 1973

[10] Lambert, J.D., Numerical methods for ordinary differential systems, John Wiley, New York, 1991

[11] M. Mehdizadeh Khalsarael, M. Molayi, The new class of A-Stable hybrid multistep methods for numerical solution of stiff initial value problem, Math. Theory Model. 5 (1) (2015), 97-103.

[12] Momoh A. A., et-al., A new numerical integrator for the solution of stiff first order ordinary differential equations, Eng. Math. Lett., 2014 (2014), Article ID 5.

[13] Okunuga, S.A., Sofoluwe, A.B., Ehigie, J.O., Some block numerical schemes for solving initial value problems in ODE's, J. Math. Sci. 2 (2013): 387-402.