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ON THE SYMMETRY ANALYSIS OF THE MICZ PROBLEM ON THE CONE

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Abstract: In this paper we used reduction method of Arunaye [1] to obtain the symmetries of the McIntosh and Cisneros [2]; and Zwanziger [3] (MICZ) problem and present some exact symmetry transformations of the MICZ problem on the cone of motion. We note relationship between the symmetries of the Kepler problem and the MICZ problem, and we obtain a novel nonlocal scaling symmetry of the MICZ problem.

Keywords: symmetry analysis; MICZ problem.

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1. Introduction

Symmetry principles are paramount to understanding the laws of nature. They are known to synthesis the regularities of the laws governing phenomenon that are independent of specific dynamics. Thus, invariance principles of symmetry provide a structure and coherence to laws of nature. Just as the laws of nature structure and coherence to set of events (Arunaye [4], Hill [5]. It is worthy to note that without regularities in the laws of nature every other phenomenon would have exhibited chaotic behavior. In present day applicable mathematics, we recognized that symmetry principles are even more powerful. They dictate the form of the laws of nature (Olver [6]; Bluman and Anco [7]; Ibragimov [8], [9]). In pattern formation, symmetry is a grand nun and central focus for understanding pattern structures. It is remarkable that Kepler problem is the central vehicle for particle Physics. Laplace is the first to have obtained seven first integrals for the Kepler problem, namely three Cartesian components of Laplace-Runge-Lenz vector $\mathbf{J}$, three Cartesian components of the angular momentum $\mathbf{L}$, and what is essentially known as the

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total energy $\mathbf{E}$. But only five of them are independent, Laplace established a relationship between the seven integrals; and show that the conservation of the angular momentum $\mathbf{L}$ implied that the orbit of the motion is planar , and that the orbit equation could be expressed as a conic section (Olver [10], [11]; Leach and Flessas [12]). In 1896, Poincaré identified a conserved vector $\mathbf{P}$ with properties reminiscent of angular momentum for the classical monopole problem (popularly known as the Poincaré vector).

However the 21st century researchers found Ermanno-Bernoulli constants, and algebraic technique that are more powerful for reducing dynamical systems to coupled harmonic oscillators and conservation laws for the calculation of their symmetries (local and nonlocal types). The MICZ system is related to the problem of the asymptotic scattering of two self-dual monopoles by canonical transformation (Cordani [13], Leach and Flessas [12]) which gives the reduced Hamiltonian of a particle in EuclideanTaub-NUT space (Cordani, Fehér, and Horváthy [14]). The motion of a spineless test particle in the field of a Dirac monopole plus Coulomb potential with an additional centrifugal potential is named MIC problem by Mladenov and Tsanov [15] or MICZ problem by Cordani [13], after the studies of this problem by McIntosh and Cisneros [2] and also by Zwanziger [3]. It is common knowledge nowadays that many problems which are relaxation of the Kepler problem were found to posses vector (first integrals) which were sufficient to provide their orbits just as $\mathbf{L}$ and $\mathbf{J}$ provided the orbit of the Kepler problem (Leach and Flessas [12], Leach and Nucci [16]). Leach and Flessas [12] considered the MICZ-Kepler and Generalization of the Kepler problem with specific equations of motion and obtained the symmetry group (local and nonlocal type) using Nucci reduction technique. Section 2 of this paper presents the preliminaries of this research in which we present the algebraic process for reducing the MICZ problem on the cone of motion to coupled oscillator and a conservation law. While in section 3 we present the main results and final concluding remarks.

2.0 Preliminaries

It is important to bring to the fore required information for the proper understanding of the methodology for realizing the objective of the research paper, thus this section presents the basic literature results with which we formally obtain the main results of this paper.

2.1 Reduction of the MICZ problem on the cone

In the following we used the algebraic reduction method of Arunaye [1] for the reduction of the MICZ problem on the cone of motion into a coupled system of oscillator and a conservation law. The general equation of motion for the MICZ dynamical system is given by

$$\ddot{\mathbf{r}} + \frac{\lambda \mathbf{L}}{r^3} + \left(\frac{\mu}{r^3} + \frac{2\nu}{r^4} \right) \mathbf{r} = \mathbf{0}. \tag{2.1}$$

By setting $-\lambda^2 = 2\nu$, Leach and Flessas [12] got the classical equation of motion of the MICZ problem as

$$\ddot{\mathbf{r}} + \frac{\lambda \mathbf{L}}{r^3} + \left(\frac{\mu}{r^2} - \frac{\lambda^2}{r^3} \right) \hat{\mathbf{r}} = \mathbf{0}. \tag{2.2}$$

In which the mass of the particle is taken as unity, λ the strength of the monopole and μ is the strength of the Coulombic field. The radial component equation of the motion (2.2) is

$$\ddot{r} - r\dot{\phi}^2 \sin^2 \theta = -\frac{\mu}{r^2} + \frac{\lambda^2}{r^3}, \tag{2.3}$$

where ϕ and θ are the respective azimuthal and transverse angles of the motion. Taking $L = r^2 \sin \theta \dot{\phi}$ as constant and $(\sin \theta)^{-1} L = \omega^{-2} \dot{\phi}$, where $\omega = r^{-1}$, equation (2.3) becomes

$$\ddot{r} - Lr^{-3} = -\frac{\mu}{r^2} + \frac{\lambda^2}{r^3}. \tag{2.4}$$

So we rewrite (2.2) as

$$\ddot{r} - (\lambda^2 + L^2) r^{-3} = -\mu r^{-2}. \tag{2.5}$$

Thus, we have $\dot{r} = -\omega^{-2}\dot{\omega} = -\omega^{-2}\omega_\phi \dot{\phi}$ and we get by substitution into (2.5)

$$\ddot{r} = -L(\sin \theta)^{-1} \omega_{\phi\phi} \dot{\phi} = -L(\sin \theta)^{-2} \omega_{\phi\phi} \omega^2, \tag{2.6}$$

where $\dfrac{\partial \omega}{\partial \phi} = \omega_\phi$, $\dfrac{\partial^2 \omega}{\partial \phi \phi} = \omega_{\phi\phi}$. Equation (2.5) then implies

$$-L^2 (\sin \theta)^{-2} \omega_{\phi\phi} u^2 - (\lambda^2 + L^2) \omega^3 = -\mu \omega^2. \tag{2.7}$$

If we let $S = \sin \theta$ be a dummy variable, then (2.7) becomes

$$\omega_{\phi\phi} + S^2 (1 + \lambda^2 L^{-2}) \omega = \mu S^2 L^{-2} \ . \tag{2.8}$$

It is hereby noted that the literature had it that magnetic monopoles Poincaré’s vector **P** is termed the total angular momentum which distinguished it from the mechanical angular momentum **L**, called orbital angular momentum. From the Poincaré vector, the identities

$P^2 = L^2 + \lambda^2$, $\mathbf{P} \cdot \mathbf{r} = -\lambda r$, $P \cos\theta = -\lambda$ holds. So that the relations $\cos\theta = -\dfrac{\lambda}{(L^2 + \lambda^2)^{1/2}}$ and $\sin\theta = \dfrac{\lambda}{(L^2 + \lambda^2)^{1/2}}$ hold. Therefore, (2.8) reduced to

$$\omega_{\phi\phi} + \omega = \frac{\mu}{L^2 + \lambda^2}. \tag{2.9}$$

So the MICZ problem on the cone of motion is reduced to a system of coupled oscillator and conservation law

$$\begin{aligned} &\omega_{1,\phi\phi} + \omega_1 = 0, \\ &\omega_{2,\phi} = 0, \end{aligned} \tag{2.10}$$

where $\omega_1 = \omega - \dfrac{\mu}{L^2 + \lambda^2}$ and $\omega_2 = L$ is the angular momentum which is constant.

The Lie symmetries of the reduced Kepler problem (Leach and Flessas [12], Leach and Nucci [16]) possess nine point symmetries, they are

$$\begin{aligned} &\gamma_1 = 2\omega_1 \partial_{\omega_1} + \omega_2 \partial_{\omega_2}, \\ &\gamma_2 = \partial_\theta, \\ &\gamma_3 = \omega_1 \partial_{\omega_1}, \\ &\gamma_{4\pm} = e^{\pm 2i\theta} \partial_{\omega_1}, \\ &\gamma_{6\pm} = e^{\pm 2i\theta}[\partial_\theta \pm i\omega_1 \partial_{\omega_1}], \\ &\gamma_{8\pm} = e^{\pm 2i\theta}[\omega_1 \partial_\theta \pm i{\omega_1}^2 \partial_{\omega_1}]. \end{aligned} \tag{2.11}$$

The corresponding symmetries in original variables for (2.10) (Leach and Flessas [12]) obtained via backward transformation

$$\begin{aligned} &X_1 = 3t\partial_t + 2r\partial_r, \\ &X_2 = \partial_\theta, \\ &X_3 = 2[\mu \int r dt - L^2 t]\partial_t + r(\mu r - L^2)\partial_r, \\ &X_{4\pm} = 2[\int r e^{\pm i\theta} dt]\partial_t + r^2 e^{\pm i\theta} \partial_r, \\ &X_{6\pm} = 2[\int (\mu r + 3L^3) e^{\pm i\theta} dt]\partial_t + r(\mu r + 3L^3) e^{\pm i\theta} \partial_r + L^2 e^{\pm 2i\theta} \partial_\theta, \end{aligned} \tag{2.12}$$

$$X_{8\pm} = 2[\int \{2\dot{r}L^3 \pm ir(\mu - r^3\dot{\theta}^2)(\mu + r^3\dot{\theta}^2)\}e^{\pm i\theta}dt]\partial_t + r[2\dot{r}L^3 \pm ir(\mu^2 - r^6\dot{\theta}^4)]\partial_r$$
$$+ L^2(\mu - r^3\dot{\theta}^2)e^{\pm 2i\theta}\partial_\theta .$$

2.2 Symmetry group of the MICZ problem on the cone of motion

We obtain the symmetries of the MICZ problem by adopting the backward transformation method of Leach and Flessas [12], Leach and Nucci [16] as follows. The reduced system of the MICZ problem on the cone implies symmetry in the original variables (t, r, ϕ) becomes symmetry in the reduction variables $(\phi, \omega_1, \omega_2)$ accordingly.

i.e. $$V = \xi^t \partial_t + \xi^r \partial_r + \xi^\phi \partial_\phi \;\rightarrow\; W = \sigma\partial_\phi + \Omega\partial_{\omega_1} + \sum \partial_{\omega_2} . \qquad (2.13)$$

The reduction variables for reducing the MICZ problem to oscillator and conservation law on the cone of motion are

$$\omega_2 = r^2\dot{\phi},\ \omega_1 = \frac{1}{r} - \frac{\mu}{\omega_2{}^2 + \lambda^2}; \qquad (2.14)$$

and utilizing the relations

$$W(\phi) = V(\phi), W(\omega_1) = V(r^{-1} - \frac{\mu}{L^2 + \lambda^2}) \ ; \qquad (2.15)$$

one obtains the infinitesimals

$$\sum = 2\xi^r r\dot{\phi} + r^2(\dot{\sigma} - \dot{\phi}\dot{\xi}^r), \quad \Omega = -\frac{\xi^r}{r^2} + \frac{2\mu\omega_2}{(\omega_2{}^2 + \lambda^2)^2}\cdot\sum \ . \qquad (2.16)$$

3.0 Main Results

The backward transformation of Leach and Nucci [16] and exact symmetry computation technique of Arunaye [17] and White [18] are adopted to be able to obtain the following results.

3.1 Symmetries of the MICZ problem on the cone

If L^2 is replaced by $L^2 + \lambda^2$ in the symmetry representation of the Kepler problem one obtain the complete symmetry group representation of the MICZ problem on its cone of motion as follows except the X_1 which is completely different from the scaling symmetry obtained for the Kepler problem, this is a novel symmetry. They are

$$X_1 = (3t + \int \frac{4\mu\lambda^2 r}{L_1{}^2})\partial_t + (2r + \frac{2\mu\lambda^2}{L_1{}^2}r^2)\partial_r \ ,$$

$$X_2 = \partial_\phi,$$
$$X_3 = 2[\mu\int r dt - L_1{}^2 t]\partial_t + (\mu r - L_1{}^2)\partial r, \tag{3.1}$$
$$X_{4\pm} = 2[\int r e^{\pm i\phi} dt]\partial_t + r^2 e^{\pm i\phi}\partial_r,$$
$$X_{6\pm} = 2[\int (\mu r + 3L_1{}^2) e^{\pm i\phi} dt]\partial_t + r(\mu r + 3L_1{}^2) e^{\pm 2i\phi}\partial_r + L_1{}^2 e^{\pm 2i\phi}\partial_\phi,$$
$$X_{8\pm} = 2[\int \{2\dot{r}L_1{}^3 \pm ir(\mu^2 - r^{-2}L_1{}^2)\} e^{\pm i\phi} dt]\partial_t + r[2\dot{r}L_1{}^3 \pm ir(\mu - r^{-2}L_1)] e^{\pm i\phi}\partial_r$$
$$+ L_1{}^2(\mu - r^{-}L_1) e^{\pm i\phi}\partial_\phi;$$

where $L_1{}^2 = L^2 + \lambda^2$. Note that for $\lambda = 0$, we have the symmetries of the Kepler problem, also in the Kepler problem; the two nonlocal symmetries to be included to form the complete symmetry group are of type $X_{4\pm}$.

3.2 Exact symmetries of the MICZ problem on the cone

In this section we compute the symmetry transformations generated by the vector fields $\alpha\omega_1\partial_{\omega_1}$, $\alpha\omega_2\partial_{\omega_2}$, $\alpha\cos\phi\partial_{\omega_1}$ and $\alpha\omega_2\partial_{\omega_1}$ where α is a constant. The flow f_λ^1 of the vector field $\alpha\omega_1\partial_{\omega_1}$ takes the form $f_\lambda^1(\omega_1,\phi) = (\overline{\omega}_1,\overline{\phi})$ with ω_2, ϕ invariants (hence θ and L are invariants) and the transformation quantities $\bar{r}$, $\bar{t}$ obey the relations

$$\bar{r} = H^{-1}r,\ H = C + \mu r(1-C)(\omega_2{}^2 + \lambda^2)^{-1},$$
$$\frac{d\bar{t}}{dt} = \frac{\bar{r}^2}{r^2} = H^{-2}, \tag{3.2}$$

where $C = e^{\alpha\lambda}$ is a constant and λ is a Lie group parameter.

The flow f_λ^2 of the vector field $\alpha\omega_2\partial_{\omega_2}$ takes the form $f_\lambda^2(\omega_2,\phi) = (\overline{\omega}_2,\overline{\phi})$ with ω_1,ϕ invariants (hence, θ, L are invariants) and the transformation quantities $\bar{r}$, $\bar{t}$ obey the relations

$$\bar{r} = H^{-1}r,\ H = 1 + \mu r L^2(1-C^2)\{(C^2L^2 + \lambda^2)(L^2 + \lambda^2)\}^{-1},\ C \neq \pm 1,$$
$$\frac{d\bar{t}}{dt} = \frac{\bar{r}^2}{Cr^2} = C^{-1}H^{-2}. \tag{3.3}$$

When $C = \pm 1$, the transformation becomes global one, i.e.

$\bar{\mathbf{x}} = \alpha\mathbf{x}$, where α is a unit vector,

$\bar{t} = C^{-1}t + b$

The flow f_λ^3 of the vector field $\alpha\cos\phi\partial_{\omega_1}$ takes the form $f_\lambda^3(\omega_1,\phi)=(\overline{\omega}_1,\overline{\phi})$ with ω_2 , ϕ invariants (hence, θ, L are invariants) and the transformation quantities $\bar{r}$, $\bar{t}$ obey the relations

$$\bar{r}=H^{-1}r,\ H=1+\alpha\lambda r\cos\phi, \qquad \frac{d\bar{t}}{dt}=\frac{\bar{r}^2}{r^2}=H^{-2}. \tag{3.4}$$

The flow f_λ^4 of the vector field $\alpha\omega_2\partial_{\omega_1}$ takes the form $f_\lambda^4(\omega_1,\phi)=(\overline{\omega}_1,\overline{\phi})$ with ω_2 , ϕ invariants (hence θ, L are invariants) and the transformation is global defined by

$\bar{\mathbf{x}}=\beta\mathbf{x}$, where β is a unit vector,

$$\bar{t}=t+b, \tag{3.5}$$

where b is a constant.

3.3. Concluding remarks

In Arunaye [17] we exhibited exact symmetry computations of some dynamical systems that are reducible to coupled oscillator and conservation law in both 2-dimensions and 3-dimensions. There, we noted also that the exact symmetry computations of MICZ problem and host of other dynamical systems are much more complicated. This work thus extends that work to include the exact symmetry transformation of the MICZ problem on the cone. The report of White [17] on the computation of symmetry transformations for dynamical systems with Poincaré vector is hereby corroborated as the MICZ problem is one of such dynamical systems referred.

Conflict of Interests

The authors declare that there is no conflict of interests.

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