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THE ADOMIAN DECOMPOSITION METHOD FOR NUMERICAL SOLUTION OF FIRST-ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper, the Adomian decomposition method (ADM) is a powerful method which considers the approximate solution of a non-linear equation as an infinite series which usually converges to the exact solution. In this paper, this method is proposed to solve some first-order differential equations. It is shown that the series solutions converge to the exact solution for each problem. It is observed that the method is particularly suited for initial value problems with oscillatory and exponential solutions.

Keywords: Adomian decomposition method; Initial value problem; Convergent series.

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1. Introduction

The Adomian decomposition method (ADM) was firstly introduced by George Adomian in 1981 and developed in [1]. This method has been applied to solve differential and integral equations of linear and non-linear problems in mathematics, physics, biology and chemistry and up to now a large number of research papers have been published to show the feasibility of the decomposition method.

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The main advantage of this method is that it can be applied directly to all types of differential and integral equations, linear or non-linear, homogeneous or inhomogeneous, with constant or variable coefficients. Another important advantage is that, the method is capable of greatly reducing the size of computational work while still maintaining high accuracy of the numerical solution [2]. The ADM decomposes a solution into an infinite series which converges rapidly to the exact solution. The convergence of the ADM has been investigated by a number of authors [3, 4].

The non-linear problems are solved easily and elegantly without linearising the problem by using ADM. It also avoids linearisation, perturbation and discretization unlike other classical techniques [5].

2. The Adomian decomposition method

Consider the differential equation

$$Ly+Ry+Ny=g(x), \tag{1}$$

where N is a non-linear operator, L is the highest order derivative which is assumed to be invertible and R is a linear differential operator of order less than L. Making Ly subject of the formula, we get

$$Ly=g(x)-Ry-Ny. \tag{2}$$

By solving (2) for Ly, since L is invertible, we can write

$$L^{-1}Ly=L^{-1}g(x)-L^{-1}Ry-L^{-1}Ny. \tag{3}$$

For initial value problems we conveniently define L^{-1} for $L=\dfrac{d^n}{dx^n}$ as the $n-$fold definite integration from 0 to x. If L is a second-order operator, L^{-1} is a two fold integral and so by solving (3) for y, we get

$$y=A+Bx+L^{-1}g(x)-L^{-1}Ry-L^{-1}Ny, \tag{4}$$

where A and B are constants of integration and can be found from the initial or boundary conditions.

The Adomian method consists of approximating the solution of (1) as an infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{5}$$

and decomposing the non-linear operator N as

$$N(y) = \sum_{n=0}^{\infty} A_n, \tag{6}$$

where A_n are Adomian polynomials [6, 7] of $y_0, y_1, y_2, \ldots, y_n$ given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots.$$

Substituting (5) and (6) into (4) yields

$$\sum_{n=0}^{\infty} y_n = A + Bx + L^{-1}g(x) - L^{-1}R\left(\sum_{n=0}^{\infty} y_n\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right).$$

The recursive relationship is found to be

$$\begin{aligned} y_0 &= g(x), \\ y_{n+1} &= -L^{-1}Ry_n - L^{-1}A_n. \end{aligned}$$

Using the above recursive relationship, we can construct the solution y as

$$y = \lim_{n\to\infty} \Phi_n(y), \tag{7}$$

where

$$\Phi_n(y) = \sum_{i=0}^{n} y_i. \tag{8}$$

3. Application to first-order differential equations

1. Problem I

Consider the system

$$\frac{dy}{dx} = y, \quad y(0) = 1,$$

with the theoretical solution given as

$$y(x) = e^x.$$

The equation can be written as

$$\begin{aligned} Ly &= y, \\ y(0) &= 1, \end{aligned}$$

where $L = \dfrac{d}{dx}$ is the differential operator. Operating on both sides with the inverse operator of L (namely $L^{-1}[\cdot] = \int_0^x [\cdot]\, dx$) to get

$$y(x) = y(0) + L^{-1}(y).$$

Applying the ADM technique yields

$$\sum_{n=0}^{\infty} y_n = y(0) + L^{-1}\left(\sum_{n=0}^{\infty} y_n\right).$$

Thus we obtain

$$\begin{aligned} y_0 &= 1, \\ y_{n+1} &= L^{-1}(y_n), \quad n = 0, 1, 2, \ldots. \end{aligned}$$

Therefore we have

$$\begin{aligned} y_1 &= L^{-1}(y_0) = \int_0^x dx = x. \\ y_2 &= L^{-1}(y_1) = \int_0^x x\, dx = \frac{x^2}{2}. \\ y_3 &= L^{-1}(y_2) = \int_0^x \frac{x^2}{2}\, dx = \frac{x^3}{6}. \\ y_4 &= L^{-1}(y_3) = \int_0^x \frac{x^3}{6}\, dx = \frac{x^4}{24}, \end{aligned}$$

and so on. Considering these components, the solution can be approximated as

$$y(x) = \Phi_n(x) = \sum_{i=0}^{\infty} y_i(x),$$

with the following expansions

$$\begin{aligned}
\Phi_1 &= 1+x. \\
\Phi_2 &= 1+x+\frac{x^2}{2}. \\
\Phi_3 &= 1+x+\frac{x^2}{2}+\frac{x^3}{6}. \\
\Phi_4 &= 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}.
\end{aligned}$$

For application purposes, only few terms of the series will be computed. Table 1 compares the ADM result with the theoretical solution.

x	Analytic	Adomian
0	1	1
0.1	1.1052	1.1052
0.2	1.2214	1.2214
0.3	1.3499	1.3498
0.4	1.4918	1.4917
0.5	1.6487	1.6484
0.6	1.8221	1.8214
0.7	2.0138	2.0122
0.8	2.2255	2.2224
0.9	2.4596	2.4538

TABLE 1. Analtyic versus Adomian

2. PROBLEM II

Let us consider the differential equation

$$\frac{dP}{dt} = kP, \quad P(0) = 4454,$$

which gives the world population at mid-year. In general, the modelled population growth has the law of exponential change, that is,

$$\frac{dP}{dt} = kP, \quad P(0) = P_0$$

where P is the population at time t, $k > 0$ is a constant growth rate, and P_0 is the size of the population at time $t = 0$. The solution for this is

$$P = P_0 e^{kt}.$$

Given that $P(0) = 4454$ and $k = 0.017$, by the ADM, we have

$$LP = kP.$$

By finding the inverse operator and imposing the initial condition, we have

$$\begin{aligned} P(t) &= 4454 + kL^{-1}(P), \\ P_0(t) &= 4454, \\ P_{n+1} &= kL^{-1}(P_n). \end{aligned}$$

Hence

$$\begin{aligned} P_1(t) &= k4454t. \\ P_2(t) &= k^2 \frac{4454t^2}{2!}. \\ P_3(t) &= k^3 \frac{4454t^3}{3!}. \\ P_n(t) &= k^n \frac{4454t^n}{n!}. \end{aligned}$$

Considering these components, the solution can be approximated as $P(t) = \Phi_n(t) = \sum_{i=0}^{n} P_i(t)$, with the following expansions

$$\begin{aligned} \Phi_1 &= 4454 + 4454tk. \\ \Phi_2 &= 4454 + 4454tk + k^2 \frac{4454t^2}{2!}. \\ \Phi_3 &= 4454 + 4454tk + k^2 \frac{4454t^2}{2!} + k^3 \frac{4454t^3}{3!}. \end{aligned}$$

For application purposes, only a few terms will be computed. Table 2 compares the results obtained using ADM with that of the analytic solution.

t	Analytic	Adomian
0	4454	4454
0.5	4492	4492
1	4530	4530
1.5	4569	4569
2	4608	4608
2.5	4647	4647
3	4687	4687
3.5	4727	4727
4	4767	4767
4.5	4808	4808

TABLE 2. Analtyic versus Adomian for Population Model

4. Conclusion

The results obtained from the two given examples, have shown that ADM is a powerful and efficient technique in finding an approximate solution of both linear and non-linear first order ordinary initial value problems which occur most often in biology. Table 1 shows small errors whilst Table 2 shows it equals the exact solution. Therefore increasing the number of terms in ADM, makes the approximate solution tend to the exact solution.

Conflict of Interests

The authors declare that there is no conflict of interests.

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