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THE ALEKSANDROV PROBLEM IN QUASI CONVEX 2-NORMED LINEAR SPACES

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Abstract. In this paper, we prove that the Aleksandrov problem holds without the condition "2-Lipschitz mapping" in quasi convex 2-normed linear spaces. Moreover, we show that the Mazur-Ulam theorem holds in quasi convex 2-normed linear spaces.

Keywords: Aleksandrov problem; Mazur-Ulam theorem; Quasi Convex 2-normed spaces.

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1. Introduction

Let E and F be metric spaces. A mapping $f : E \to F$ is called an isometry if f satisfies

$$d_F(f(x), f(y)) = d_E(x, y)$$

for all $x, y \in E$, where $d_E(,)$ and $d_F(,)$ denote the metric in the space E and F, respectively. For some fixed number $r > 0$, suppose that f preserves distance r; ie, for all $x, y \in E$ with $d_E(x, y) = r$, we have $d_F(f(x), f(y)) = r$. Then r is called a conservative distance for the mapping f. The classical Mazur-Ulam theorem states that every surjective isometry between normed spaces is

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a linear mapping up to translation. In 1970, Aleksandrov [1] posed the following question: ”Whether or not a mapping with distance one preserving property is an isometry? ” It is called the *Aleksandrov problem*. The Aleksandrov problem has been investigated in several papers[4]-[13].

Recently, Chu et al. [4] begin to consider the Aleksandrov problem in linear 2-normed space. They introduced the concept of 2-isometry, which is suitable to represent the notion of area preserving mappings in appropriate spaces as 2-normed spaces. Chu [2] proved the Mazur-Ulam theorem holds in 2-normed spaces via this 2-isometry. However, this ideal cannot be used to prove the Mazur-Ulam theorem in quasi convex 2-normed linear space, since the triangle inequality fails in quasi convex 2-normed linear space. Chu et al.[4] proved also that the Rassias and Šemrl theorem holds under some conditions in linear 2-normed spaces as follows:

Theorem 1.1.[4] Let f be a 2-Lipschitz mapping with the 2-Lipschitz constant $K \leq 1$. Assume that if x, y and z are collinear, then $f(x), f(y)$ and $f(z)$ are collinear, and that f satisfies (DOPP). Then f is a 2-isometry.

In this paper, we consider generalized 2-isometries, which is suitable for representing the notion of distance preserving mappings in quasi convex 2-normed linear spaces. We show that every generalized 2-isometries is affine. Also we prove that a mapping preserving the one distance property and collinear between two quasi convex 2-normed linear spaces is an affine generalized 2-isometry.

2. Preliminaries

In the remainder of this introduction, we will recall some definitions and give some Lemmas about them in quasi convex 2-normed linear space.

Definition 2.1.[12] Let E be a real linear space with $\dim E > 1$ and $\|\cdot,\cdot\|$ be a function from $E \times E$ into $\mathbb{R}$. Then $(E, \|\cdot,\cdot\|)$ is called a *quasi convex 2-normed linear space* if

(a) $\|x,y\| = 0 \Leftrightarrow x$ and y are linearly dependent,

(b) $\|x,y\| = \|y,x\|$,

(c) $\|\alpha x,y\| = |\alpha| \, \|x,y\|$,

(d) $\|tx+(1-t)y,z\| \leq max\{\|x,z\|,\|y,z\|\}$,

for any $\alpha \in R, t \in [0,1]$ and $x,y,z \in E$. The function $\|\cdot,\cdot\|$ is called the *quasi convex 2-norm on* E.

From now on, let E and F be quasi convex 2-normed linear space and the mapping $f : E \to F$.

Definition 2.2.[13] A mapping $f : E \to F$ is said to be a *generalized 2-isometry* if it satifies

$$\|f(x)-f(y), f(p)-f(q)\| = \|x-y, p-q\|.$$

for every $x,y,p,q \in E$. In Particular, if $y = q$, the mapping f is said to be a *2-isometry*.

Definition 2.3.[13] A mapping $f : E \to F$ satisfies the *distance one preserving property (briefly DOPP)*, if $\|x-y, p-q\| = 1$ for all $x,z,p,q \in Y$, it follows that

$$\|f(x)-f(z), f(p)-f(q)\| = 1.$$

Definition 2.4.[4] We call f a *2-Lipschitz mapping* if there is a $K \geq 0$ such that

$$\|f(x)-f(y), f(p)-f(q)\| \leq K\|x-y, p-q\|$$

for all $x,y,p,q \in E$. In this case, the constant k is called the 2-Lipschitz constant.

Definition 2.5.[7] A mapping $f : E \to F$ on two real norme space E and F is called an affine mapping if for all $x,y \in E$ and $\lambda \in [0,1]$ satifies

$$f(\lambda x+(1-\lambda)y) = \lambda f(x)+(1-\lambda)f(y).$$

Definition 2.6.[4] The point x,y,z of E are said to be *collinear* if $y-z = t(x-z)$ for some real number t. We say that a mapping $f : E \to F$ *preserves collinearity*, if $x,y,z \in E$ are collinear, then $f(x), f(y), f(z)$ are collinear.

Remark 2.7. Each 2-Lipschitz mapping preserves collinear.

Lemma 2.1. Let E be a quasi convex 2-normed linear space with $dimE > 1$. For $x,y,z \in E$, if x and y are linearly dependent, then $\|x+y,z\| \leq \|x,z\|+\|y,z\|$.

Lemma 2.2. Let E be a quasi convex 2-normed linear space with $dimE > 1$, for $x_i, z \in E, t_i > 0, \sum_{i=1}^{n} t_i = 1 (i = 1, 2, \cdots, n)$, we have

$$\| \sum_{i=1}^{n} t_i x_i, z \| \leq max\{\|x_i, z\| : i = 1, 2, \cdots, n\}.$$

Proof. If $n = 2$, then $\|t_1 x_1 + t_2 x_2, z\| \leq \max\{\|x_1, z\|, \|x_2, z\|\}$.

Assume that

$$\| \sum_{i=1}^{k-1} t_i x_i, z \| \leq max\{\| x_1, z \|, \| x_2, z \|, \cdots, \| x_{k-1}, z \|\}.$$

Let $n = k$, we can obtain

$$\begin{aligned} \| \sum_{i=1}^{k} t_i x_i, z \| &= \| \sum_{i=1}^{k-1} t_i x_i + t_k x_k, z \| \\ &= \| \sum_{i=1}^{k-1} t_i (\frac{\sum_{i=1}^{k-1} t_i x_i}{\sum_{i=1}^{k-1} t_i}) + t_k x_k, z \| \\ &\leq max\{\| \frac{\sum_{i=1}^{k-1} t_i x_i}{\sum_{i=1}^{k-1} t_i}, z \|, \| x_k, z \|\} \\ &\leq max\{\| x_1, z \|, \| x_2, z \|, \cdots, \| x_{k-1}, z \|, \| x_k, z \|\}. \end{aligned}$$

Therefore

$$\| \sum_{i=1}^{n} t_i x_i, z \| \leq max\{\| x_1, z \|, \| x_2, z \|, \cdots, \| x_{n-1}, z \|, \| x_n, z \|\}$$

i.e.

$$\| \sum_{i=1}^{n} t_i x_i, z \| \leq max\{\|x_i, z\| : i = 1, 2, \cdots, n\}.$$

The next result follows easily from [6, Lemma 8].

Lemma 2.3. Let E be a quasi convex 2-normed linear space with $dimE > 2$. Suppose $0 < \|x - y, p - q\| \leq 2r$, for any $r > 0$, and $x, y, p, q \in E$, then there exists $z \in E$ such that $\|x - z, p - q\| = \|z - y, p - q\| = r$.

3. Main results

In this section, let E and F be quasi convex 2-normed linear spaces with dimension greater than 1.

Lemma 3.1. *Let E and F be two quasi convex 2-normed linear spaces. If $f : E \to F$ satisfies (DOPP) and preserves collinearity, then f is injective and for any $x,y \in E$, we have*

$$f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}.$$

Proof. Let $z = \frac{x+y}{2}$ for distinct $x,y \in E$. Then $z-x = y-z = \frac{y-x}{2} \neq 0$. We can choose $p,q \in E$ such that $\|x-y, p-q\| = 1$. Since the mapping f satisfies (DOPP), we have

$$\|f(x)-f(y), f(p)-f(q)\| = 1.$$

This implies $f(x) \neq f(y)$, and thus f is injective. On the other hand,

$$\|z-y, 2p-2q\| = \|z-x, 2p-2q\| = 1.$$

Then

$$\|f(z)-f(y), f(2p)-f(2q)\| = \|f(z)-f(x), f(2p)-f(2q)\| = 1. \tag{1}$$

Since f preserves collinearity, there exists a real number t such that

$$f(z)-f(y) = t(f(z)-f(x)).$$

Because f is injective, and it follows from the equation (1) we conclude that $t = -1$. Thus $f(z)-f(y) = f(x)-f(z)$ and

$$f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}.$$

Theorem 3.1. *Let E and F be two quasi convex 2-normed linear spaces. if $f : E \to F$ is a generalized 2-isometry, then f is affine.*

Proof. Assume that x,y and z are colinear, then f preserves collinearity by the condition that $\|x-z, y-z\| = 0$ implies $\|f(x)-f(z), f(y)-f(z)\| = 0$. Let $g(x) = f(x)-f(0)$. It suffices to prove that the mapping g is linear. Since g satisfies (DOPP) and $g(0) = 0$. From Lemma 3.1, the mapping g is $\mathbb{Q}$-linear. Let $\xi \in R^+$ with $\xi \neq 1$ and $x \in E$. Since $0, x, \xi x$ are collinear, g preserves collinearity and also $g(0) = 0$, so there exists a real number η such that

$$g(\xi x) = \eta g(x).$$

For any $x \in E$ with $x \neq 0$, there exists $y \in E$ such that $\|x, y\| = 1$. Hence we obtain

$$\xi = \|\xi x, y\| = \|g(\xi x), g(y)\| = \|\eta g(x), g(y)\| = |\eta|\|g(x), g(y)\| = |\eta|.$$

Thus $\eta = \pm\xi$. While $\eta = -\xi$, that is to say $g(\xi x) = -\xi g(x)$, it deduces that

$$\begin{aligned} |\ 1 - \xi\ | &= \|x - \xi x, y\| \\ &= \|g(x) - g(\xi x), g(y)\| \\ &= \|g(x) + \xi g(x), g(y)\| \\ &= (1 + \xi)\|g(x), g(y)\| \\ &= 1 + \xi. \end{aligned}$$

So $\xi = 0$, while it conflict with $\xi \in R^+$. Hence we get $\xi = \eta$, that is to say $g(\xi x) = \xi g(x)$. This completes the proof.

Lemma 3.2. *Let E and F be two quasi convex 2-normed linear spaces, if $f : E \to F$ satisfies (DOPP) and preserves collinearity, then f preserves distance $\frac{m}{k}$, for each $m, k \in \mathbb{N}$.*

Proof. We first prove f preserves distance $\frac{1}{k}$. Let $\|x - y, p - q\| = \frac{1}{k}$ with $x, y, p, q \in E$, we define

$$\omega_i = x + i(y - x) \quad \forall i = 0, 1, \cdots, k.$$

Then

$$\omega_i = \frac{\omega_{i-1} + \omega_{i+1}}{2}, \quad \forall i = 1, \cdots, k-1.$$

According to Lemma 3.1, we have

$$f(\omega_i) = \frac{f(\omega_{i-1}) + f(\omega_{i+1})}{2}, \quad \forall i = 1, \cdots, k-1.$$

That is

$$f(\omega_{i+1}) - f(\omega_i) = f(\omega_i) - f(\omega_{i-1}), \quad \forall i = 1, \cdots, k-1.$$

Hence

$$\begin{aligned} f(\omega_k)-f(x) &= f(\omega_k)-f(\omega_{k-1})+f(\omega_{k-1})-f(\omega_{k-2})+\cdots+f(\omega_1)-f(\omega_0) \\ &= k(f(\omega_1)-f(\omega_0))=k(f(y)-f(x)). \end{aligned}$$

Since $\|\omega_k-x,p-q\|=1$,

$$k\|f(y)-f(x),f(p)-f(q)\|=\|f(\omega_k)-f(x),f(p)-f(q)\|=1.$$

Therefore $\|f(y)-f(x),f(p)-f(q)\|=\frac{1}{k}$.

Next, we shall show that f preserves distance $\frac{m}{k}$ for integers m,k. Let $\|x-y,p-q\|=\frac{m}{k}$ with $x,y,p,q\in E$. We define

$$z_i:=x+\frac{i}{m}(y-x), \quad \forall i=0,1,\cdots,k.$$

Then

$$z_i=\frac{z_{i-1}+z_{i+1}}{2}, \quad \forall i=1,\cdots,k-1.$$

By the same method as above,

$$f(y)-f(x)=f(z_m)-f(z_0)=m(f(z_1)-f(z_0)).$$

Note that $\|z_1-z_0,p-q\|=\frac{1}{k}$ and f preserves distance $\frac{1}{k}$,

$$\|f(y)-f(x),f(p)-f(q)\|=\|m(f(z_1)-f(z_0)),f(p)-f(q)\|=\frac{m}{k}.$$

This completes the proof.

Theorem 3.2. *Let E and F be two quasi convex 2-normed linear spaces with* $\dim E>2$. *If $f:E\to F$ satisfies (DOPP) and preserves collinearity, then f is an affine generalized 2-isometry.*

Proof. We first prove that f is a 2-Lipschitz mapping with the constant $K=1$. That is, for any $x,y,p,q\in E$,

$$\|f(x)-f(y),f(p)-f(q)\|\le\|x-y,p-q\|.$$

If $\|x-y, p-q\| = 0$ for some $x, y, p, q \in E$. Then we have $x - y = t(p-q)$ for some real number t. Let $g(x) = f(x) - f(0)$. It follows from Lemma 3.1 that g is additive and preserves collinearity. Thus

$$\|f(x)-f(y), f(p)-f(q)\| = \|g(x)-g(y), g(p)-g(q)\| = 0.$$

On the other hand, let $x, y, p, q \in E$ and $k, m \in N$, such that

$$\frac{m-1}{k} < \|x-y, p-q\| \leq \frac{m}{k}$$

Set

$$\omega_i = x + \frac{i}{k}\frac{y-x}{\|x-y, p-q\|}, \quad i = 0, 1, \cdots, m-2$$

and also define $\omega_m = y$. Then

$$\|\omega_i - \omega_{i-1}, p-q\| = \frac{1}{k}, \quad i = 1, \cdots, m-2.$$

Moreover,

$$\begin{aligned} 0 < \|\omega_m - \omega_{m-2}, p-q\| &= \|\frac{m-2}{k}\frac{y-x}{\|x-y, p-q\|} + (x-y), p-q\| \\ &= \|x-y, p-q\| - \frac{m-2}{k} \\ &\leq \frac{m}{k} - \frac{m-2}{k} = \frac{2}{k}. \end{aligned}$$

From Lemma 2.3, we can choose $\omega_{m-1} \in E$, such that

$$\|\omega_{m-1} - \omega_{m-2}, p-q\| = \|\omega_{m-1} - \omega_m, p-q\| = \frac{1}{k}$$

By Lemma 3.2, f preserves $\frac{1}{k}$ distance. Therefore, for $i = 0, 1, \cdots, m$, we have

$$\|f(\omega_i) - f(\omega_{i-1}), f(p) - f(q)\| = \frac{1}{k}.$$

From Lemma 2.2,

$$\begin{aligned}
\|f(x)-f(y),f(p)-f(q)\| &= \|f(\omega_0)-f(\omega_m),f(p)-f(q)\| \\
&= \|\sum_{i=0}^{m-1}(f(\omega_i)-f(\omega_{i+1})),f(p)-f(q)\| \\
&= m\|\sum_{i=0}^{m-1}\frac{1}{m}(f(\omega_i)-f(\omega_{i+1})),f(p)-f(q)\| \\
&\leq mmax\{\|f(\omega_i)-f(\omega_{i+1}),f(p)-f(q))\| : i=0,1,\cdots,m-1\} \\
&\leq \frac{m}{k}.
\end{aligned}$$

Hence $\|f(x)-f(y),f(p)-f(q)\| \leq \|x-y,p-q\|$.

Next, we will show that f is a generalized 2-isometry. Otherwise, there exists $x,y,p,q \in E$ and $m \in \mathbb{N}$ such that $0 < \|x-y,p-q\| < m$ and

$$\|f(x)-f(y),f(p)-f(q)\| < \|x-y,p-q\|.$$

Set $z := x + \frac{m(y-x)}{\|x-y,p-q\|}$. Then we obtain that

$$\begin{aligned}
\|z-x,p-q\| &= m \\
\|z-y,p-q\| &= m-\|x-y,p-q\|.
\end{aligned}$$

Since f preserves collinearity, there exists a real number t such that

$$f(z)-f(x) = t(f(y)-f(x)).$$

Then $f(z)-f(y) = (t-1)(f(y)-f(x))$. By Lemma 3.2, f preserves distance m. So we have

$$\begin{aligned}
m &= \|f(z)-f(x),f(p)-f(q)\| \\
&= |t|\|f(x)-f(y),f(p)-f(q)\| \\
&\leq |t-1|\|f(x)-f(y),f(p)-f(q)\| + \|f(x)-f(y),f(p)-f(q)\| \\
&= \|f(z)-f(y),f(p)-f(q)\| + \|f(x)-f(y),f(p)-f(q)\| \\
&< m-\|x-y,p-q\| + \|x-y,p-q\| = m,
\end{aligned}$$

which is a contraction. By Theorem 3.1, the proof of the theorem is finished.

Conflict of Interests

The authors declare that there is no conflict of interests.

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