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SOME REFINEMENT OF THE INEQUALITY IN QUASI-2-NORMED SPACES AND QUASI-$(2;p)$-NORMED SPACES

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Abstract. In this paper, we establish a generalization of the triangle inequality and the Dunkl-Williams inequality in quasi-2-normed linear spaces and quasi-$(2;p)$-normed spaces.

Keywords: Quasi-2-normed spaces; 2-inner product spaces; Quasi-$(2;p)$-normed spaces; Dunkl-Williams inequality.

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1. Introduction

The concept of 2-normed spaces was introduced by Gähler [1] in 1965, and has been developed extensively in different subjects by others (see [2]). After that, in 1973 and 1977, Diminnie, Gähler and White introduced the concept of 2-inner product spaces (see [3,4]). In 2006, Choonkil Park [5] introduced the notion of quasi-2-normed spaces. Our aim in this paper is to present the recent results of sharp triangle inequalities and Dunkl-Williams inequality in quasi-2-normed linear spaces and quasi-$(2;p)$-normed spaces.

In 1964, Dunkl and Williams [12] proved that for any two non-zero elements x, y in a normed

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linear space, then

$$\| \frac{x}{\| x \|} - \frac{y}{\| y \|} \| \leq \frac{4 \| x-y \|}{\| x \| + \| y \|}. \tag{1.1}$$

Moreover,they proved that in an inner product space,the constant 4 can be replaced by 2.

This inequality has some important applications in the study of Banach spaces. It is one of the most fundamental inequality in calculus. Many interesting refinements and reverses of this inequality in normed linear spaces have been given (see[6,7]). Kirk and Smiley, 1964, proved that in this inequality with 2 instead of 4 characterizes inner product spaces. Now we extend it in quasi-2-normed linear spaces.

2. Preliminaries

Definition 2.1. [1] Let X be a linear space of dimension greater than 1 on the field $\mathbb{K}$ and let $\| \cdot, \cdot \|: X \times X \longrightarrow [0, +\infty)$be a function satisfying the following conditions:

(1) $\| x,y \| = 0$ if and only if x and y are linearly dependent,

(2) $\| x,y \| = \| y,x \|$,

(3) $\| \alpha x,y \| = |\alpha| \, \| x,y \|$ for all $\alpha \in \mathbb{K}$,

(4) $\| x+y,z \| \leq \| x,z \| + \| y,z \|$,

for all $x,y,z \in X$.$\|\cdot,\cdot\|$ is called a 2-normed and $(X, \|\cdot,\cdot\|)$ is called a linear 2-normed space.

It is known that equality holds for every in definition (1.1) part (4) if and only if x and y are linearly dependent with the same direction. It is easy to show that the 2-norm $\|\cdot,\cdot\|$ is non-negative and $\|x, y+\alpha x\| = \|x,y\|$ for all $x,y \in X$ and $\alpha \in \mathbb{R}$.

Example 2.1. In the set consisting of bounded functions of real numbers in interval [a,b]by

$$\|f,g\| = \sup_{x\in[a,b]} \sup_{y\in[a,b]} | \begin{vmatrix} f(x) & f(y) \\ g(x) & g(y) \end{vmatrix} |$$

a 2-norm is defined.

Definition 2.2. [2,3] Let $\mathbb{K}$ be the symbol of the field $\mathbb{R}$ or $\mathbb{C}$ and X be a linear space on $\mathbb{K}$. Define the $\mathbb{K}$-valued function $(\cdot,\cdot|\cdot)$ on $X \times X \times X$ with the following properties:

(1) $(x,x|y) \geq 0; (x,x|y) = 0$ if and only if x and y are linearly dependent,

(2) $(x,x|y) = (y,y|x)$,

(3) $(x,y|z) = \overline{(y,x|z)}$,

(4) $(\alpha x,y|z) = \alpha(x,y|z)$ for any $\alpha \in \mathbb{K}$,

(5) $(x+x',y|z) = (x,y|z)+(x',y|z)$,

for all $x,x',y,z \in X$. $(\cdot,\cdot|\cdot)$ is called a 2-inner product and $(X,(\cdot,\cdot|\cdot))$ is called a 2-inner product space.

Example 2.2. [10] Let $\mathbf{R}^3$ be a Hilbert space with the usual inner product and $(X,\|\cdot,\cdot\|)$ be a 2-normed space. Then by

$$(x,y|z) = \begin{vmatrix} (x,y) & (x,z) \\ (y,z) & (z,z) \end{vmatrix} \qquad x,y,z \in X$$

a 2-inner product is defined.

Definition 2.3. [5] Let X be a linear space. A quasi-2-norm is a real-valued function on $X \times X$ satisfying the following conditions:

(1) $\| x,y \| = 0$ if and only if x and y are linearly dependent,

(2) $\| x,y \| = \| y,x \|$,

(3) $\| \alpha x,y \| = |\alpha| \, \| x,y \|$ for all $\alpha \in \mathbb{K}$,

(4) There is a constant $K \geq 1$ such that $\| x+y,z \| \leq K(\|x,z\| + \|y,z\|)$ for all $x,y,z \in X$.

The pair $(X,\|\cdot,\cdot\|)$ is called a quasi-2-normed space if $\|\cdot,\cdot\|$ is a quasi-2-norm on X.

It follows from (4) that

$$\|x-y,z\| \geq \frac{1}{K}\|x,z\| - \|y,z\|$$

and

$$\|x-y,z\| \geq \frac{1}{K}\|y,z\| - \|x,z\|$$

so,

$$\|x-y,z\| \geq \max\{\frac{1}{K}\|x,z\| - \|y,z\|, \frac{1}{K}\|y,z\| - \|x,z\|\}. \tag{2.1}$$

Example 2.3. [11] Let X be a linear space with dim $X \geq 2$ and let $\|\cdot,\cdot\|$ be 2-norm on X.

$$\|x,y\|_q = 2\|x,y\|$$

is quasi-2-norm on X, and $(X,\|\cdot,\cdot\|)$ is a quasi-2-normed space.

A quasi-2-norm $\|\cdot,\cdot\|$ is called quasi-(2;p)-norm $(0 < p \leq 1)$ if

$$||x+y,z||^p \leq ||x,z||^p + ||y,z||^p \tag{2.2}$$

for all $x,y,z \in X$. So, Let X be a quasi-$(2;p)$-normed space, then we have

$$|\|x,z\|^p - \|y,z\|^p| \leq \|x-y,z\|^p. \tag{2.3}$$

3. Dunkl-Williams inequality with two elements in quasi-2-normed spaces

Abrishami[7] has refined Dunkl-Williams inequality in quasi-2-normed spaces and prove the following lemma.

Lemma 3.1. [7] For all non-zero vectors x, y and z in a quasi-2-normed space X with $z \notin$ span{x,y}, we have

$$\|\frac{x}{\|x,z\|} - \frac{y}{\|y,z\|}, z\| \leq \frac{K\|x-y,z\| + K|\|y,z\| - \|x,z\||}{\max\{\|x,z\|, \|y,z\|\}} \tag{3.1}$$

$$\|\frac{x}{\|x,z\|} - \frac{y}{\|y,z\|}, z\| \geq \frac{\|x-y,z\| - K|\|y,z\| - \|x,z\||}{K\min\{\|x,z\|, \|y,z\|\}}, \tag{3.2}$$

where $K \geq 1$.

The following result is needed to prove our theorems. It given a equivalent condition of inequality (3.1).

Theorem 3.1. Let X be a quasi-2-normed space with $K \geq 1$. The following statements are equivalent:

(1) For all non-zero vectors $x,y \in X$ and $z \notin$span{x,y} it is true that

$$\|\frac{x}{\|x,z\|} - \frac{y}{\|y,z\|}, z\| \leq \frac{K\|x-y,z\| + K|\|y,z\| - \|x,z\||}{\max\{\|x,z\|, \|y,z\|\}} \tag{3.3}$$

(2) If $x,y \in X$ are such that $\|x,z\| = \|y,z\| = 1$, then

$$\|\frac{x+y}{2}, z\| \leq \frac{K\|(1-t)x+ty,z\| + K|2t-1|}{2\max\{1-t,t\}} \tag{3.4}$$

for each $t \in (0,1)$.

Proof. $(1) \Rightarrow (2)$. Assume that the statement (1) holds. Then (1)implies the following

$$\begin{aligned}
\|\frac{x+y}{2},z\| &= \frac{1-t}{2}(1+\frac{t}{1-t})\|x+y,z\| \\
&= \frac{1-t}{2}(\|x,z\|+\|\frac{t}{1-t}y,z\|)\|\frac{x}{\|x,z\|}-\frac{\frac{t}{t-1}y}{\|\frac{t}{t-1}y,z\|},z\| \\
&\leq \frac{1-t}{2}(\|x,z\|+\|\frac{t}{1-t}y,z\|)\frac{K\|x-\frac{t}{t-1}y,z\|+K|\|\frac{t}{t-1}y,z\|-\|x,z\||}{\max\{\|x,z\|,\|\frac{t}{t-1}y,z\|\}} \\
&= \frac{1-t}{2}(1+\frac{t}{1-t})\frac{K\|x-\frac{t}{t-1}y,z\|+K|\frac{2t-1}{1-t}|}{\max\{1,\frac{t}{1-t}\}} \\
&= \frac{K\|(1-t)x+ty,z\|+K|2t-1|}{2\max\{1-t,t\}}
\end{aligned}$$

i.e. the inequality (3.4) holds true.

$(2) \Rightarrow (1)$. Let assume that the statement (2) holds true. Let x and y be arbitrary non-zero vectors in X. Then, if we take that

$$t=\frac{\|y,z\|-\|x,z\|+2}{4}$$

then by (3.4) we get that

$$\begin{aligned}
\|\frac{x}{\|x,z\|}-\frac{y}{\|y,z\|},z\| &= 2\|\frac{\frac{x}{\|x,z\|}+\frac{-y}{\|y,z\|}}{2},z\| \\
&\leq 2\cdot\frac{K\|(1-t)\frac{x}{\|x,z\|}+t\frac{-y}{\|y,z\|},z\|+K|2t-1|}{2\max\{1-t,t\}} \\
&= \frac{K\|\frac{2-\|y,z\|+\|x,z\|}{4}\cdot\frac{x}{\|x,z\|}+\frac{\|y,z\|-\|x,z\|+2}{4}\cdot\frac{-y}{\|y,z\|},z\|+\frac{1}{2}K|\|y,z\|-\|x,z\||}{\frac{1}{2}\max\{\frac{2-\|y,z\|+\|x,z\|}{2},\frac{\|y,z\|-\|x,z\|+2}{2}\}} \\
&= \frac{K\|x-y,z\|+K|\|y,z\|-\|x,z\||}{\max\{\|x,z\|,\|y,z\|\}}
\end{aligned}$$

i.e. the inequality (3.3) holds true.

Now we are ready to prove the main theorem. A refinement of (1.1) has been obtained by Mercer [14]. Now, we use Mercer's inequality and give a refinement of (3.1) and (3.2).

Theorem 3.2 Let x, y and z be non-zero vectors in a quasi-2-normed space X with $z \notin span\{x,y\}$. We put

$$\beta = \frac{\|x-y,z\|^2 - K^2(\|y,z\| - \|x,z\|)^2}{\|x-y,z\|^2 - (\|y,z\| - \|x,z\|)^2}$$

and β satisfies the condition $\beta > 0$. Then we have

$$\begin{aligned}
&-\frac{1}{2}(K^2+1)\frac{(\|y,z\|-\|x,z\|)^2}{(\|y,z\|+\|x,z\|)^2}\cdot\beta \\
&-\sqrt{(K^2+1)^2(\frac{\|y,z\|-\|x,z\|}{\|y,z\|+\|x,z\|})^4\cdot\beta^2 - 4\beta\cdot\frac{2(K^2+1)(\|y,z\|-\|x,z\|)^2 - 4\|x-y,z\|^2}{(\|y,z\|+\|x,z\|)^2}} \\
&\le \|\frac{x}{\|x,z\|} - \frac{y}{\|y,z\|}, z\| \qquad (3.5)\\
&\le -\frac{1}{2}(K^2+1)\frac{(\|y,z\|-\|x,z\|)^2}{(\|y,z\|+\|x,z\|)^2}\cdot\beta \\
&+\sqrt{(K^2+1)^2(\frac{\|y,z\|-\|x,z\|}{\|y,z\|+\|x,z\|})^4\cdot\beta^2 - 4\beta\cdot\frac{2(K^2+1)(\|y,z\|-\|x,z\|)^2 - 4\|x-y,z\|^2}{(\|y,z\|+\|x,z\|)^2}}.
\end{aligned}$$

Proof. Let $\alpha = \|\frac{x}{\|x,z\|} - \frac{y}{\|y,z\|}, z\|$. From (3.2) we know

$$\alpha \ge \frac{\|x-y,z\| - K|\|y,z\| - \|x,z\||}{K\min\{\|x,z\|,\|y,z\|\}}.$$

Clearly, we have

$$K|\|y,z\| - \|x,z\|| - \|x-y,z\| + 2K\min\{\|x,z\|,\|y,z\|\} = K\|y,z\| + K\|x,z\| - \|x-y,z\|$$

so,

$$K\|y,z\| + K\|x,z\| - \|x-y,z\| \ge K(2-\alpha)\min\{\|x,z\|,\|y,z\|\}.$$

A simple computation shows that

$$\frac{Re(x,y \mid z)}{\|x,z\|\|y,z\|} = 1 - \frac{1}{2}\alpha^2 \qquad \alpha^2 = \frac{\|x-y,z\|^2 - (\|y,z\| - \|x,z\|)^2}{\|x,z\|\|y,z\|}.$$

Therefore,

$$
\begin{aligned}
&\|x-y,z\|^2-\beta\alpha^2\cdot(\frac{\|y,z\|+\|x,z\|}{2})^2\\
&=\frac{4\|x-y,z\|^2\|x,z\|\|y,z\|-(\|y,z\|+\|x,z\|)^2[\|x-y,z\|^2-K^2(\|y,z\|-\|x,z\|)^2]}{4\|x,z\|\|y,z\|}\\
&=\frac{4\|x-y,z\|^2\|x,z\|\|y,z\|-(\|y,z\|+\|x,z\|)^2\|x-y,z\|^2}{4\|x,z\|\|y,z\|}\\
&\quad+\frac{K^2(\|y,z\|-\|x,z\|)^2(\|y,z\|+\|x,z\|)^2}{4\|x,z\|\|y,z\|}\\
&=\frac{-\|x-y,z\|^2(\|y,z\|-\|x,z\|)^2+K^2(\|y,z\|-\|x,z\|)^2(\|y,z\|+\|x,z\|)^2}{4\|x,z\|\|y,z\|}\\
&=\frac{(\|y,z\|-\|x,z\|)^2}{4\|x,z\|\|y,z\|}\cdot[(K\|y,z\|+K\|x,z\|)^2-\|x-y,z\|^2]\\
&=\frac{(\|y,z\|-\|x,z\|)^2}{4\|x,z\|\|y,z\|}[K\|y,z\|+K\|x,z\|-\|x-y,z\|][K\|y,z\|+K\|x,z\|+\|x-y,z\|]\\
&\geq\frac{(\|y,z\|-\|x,z\|)^2}{4\|x,z\|\|y,z\|}K(2-\alpha)\min\{\|x,z\|,\|y,z\|\}(K\|y,z\|+K\|x,z\|\\
&\quad+\max\{\frac{1}{K}\|x,z\|-\|y,z\|,\frac{1}{K}\|y,z\|-\|x,z\|\})\\
&\geq\frac{(\|y,z\|-\|x,z\|)^2}{4\|x,z\|\|y,z\|}K(2-\alpha)\frac{K^2+1}{K}\|x,z\|\|y,z\|\\
&=\frac{K^2+1}{4}(2-\alpha)(\|y,z\|-\|x,z\|)^2.
\end{aligned}
$$

Hence,

$$\|x-y,z\|^2-\beta\alpha^2\cdot(\frac{\|y,z\|+\|x,z\|}{2})^2\geq\frac{K^2+1}{4}(2-\alpha)(\|y,z\|-\|x,z\|)^2$$

Therefore,

$$\alpha^2-\alpha(K^2+1)\frac{(\|y,z\|-\|x,z\|)^2}{(\|y,z\|+\|x,z\|)^2}\beta+\frac{2(K^2+1)(\|y,z\|-\|x,z\|)^2-4\|x-y,z\|^2}{(\|y,z\|+\|x,z\|)^2}\beta\leq 0$$

So, α is between two roots of the quadratic equation

$$\lambda^2-\lambda(K^2+1)\frac{(\|y,z\|-\|x,z\|)^2}{(\|y,z\|+\|x,z\|)^2}\beta+\frac{2(K^2+1)(\|y,z\|-\|x,z\|)^2-4\|x-y,z\|^2}{(\|y,z\|+\|x,z\|)^2}\beta\leq 0$$

Hence, we get (3.5). This completes the proof.

4. Triangle inequality in quasi-(2;p)-normed spaces

In this section ,we refine triangle inequalities in quasi-(2;p)-normed spaces.

Theorem 4.1 *Let X be a quasi-(2;p)-normed spaces on the field* $\mathbb{K}$*. For all* $z,x_1,...,x_n \in X$ *and* $a_1,...,a_n \in \mathbb{K}$*, we have*

$$\|\sum_{i=1}^{n} a_i x_i, z\|^p \le \min_{1\le j\le n}\{|a_j|^p \|\sum_{i=1}^{n} x_i, z\|^p + \sum_{i=1}^{n}|a_i - a_j|^p \|x_i, z\|^p\}, \tag{4.1}$$

$$\|\sum_{i=1}^{n} a_i x_i, z\|^p \ge \max_{1\le j\le n}\{|a_j|^p \|\sum_{i=1}^{n} x_i, z\|^p - \sum_{i=1}^{n}|a_j - a_i|^p \|x_i, z\|^p\}. \tag{4.2}$$

Proof. For a fixed $1 \le i \le n$, we have

$$\begin{aligned}
\|\sum_{i=1}^{n} a_i x_i, z\|^p &= \|\sum_{i=1}^{n} a_j x_i + \sum_{i=1}^{n}(a_i - a_j)x_i, z\|^p \\
&\le \|\sum_{i=1}^{n} a_j x_i, z\|^p + \|\sum_{i=1}^{n}(a_i - a_j)x_i, z\|^p \\
&\le |a_j|^p \|\sum_{i=1}^{n} x_i, z\|^p + \sum_{i=1}^{n}\|(a_i - a_j)x_i, z\|^p \\
&= |a_j|^p \|\sum_{i=1}^{n} x_i, z\|^p + \sum_{i=1}^{n}|(a_i - a_j)|^p \|x_i, z\|^p.
\end{aligned}$$

By taking minimum over $i = 1,...,n$, we obtain (4.1). Now, we have

$$\begin{aligned}
\|\sum_{i=1}^{n} a_i x_i, z\|^p &= \|\sum_{i=1}^{n} a_j x_i - \sum_{i=1}^{n}(a_j - a_i)x_i, z\|^p \\
&\ge |\|\sum_{i=1}^{n} a_j x_i, z\|^p - \|\sum_{i=1}^{n}(a_j - a_i)x_i, z\|^p| \\
&\ge |a_j|^p \|\sum_{i=1}^{n} x_i, z\|^p - \sum_{i=1}^{n}\|(a_j - a_i)x_i, z\|^p \\
&= |a_j|^p \|\sum_{i=1}^{n} x_i, z\|^p - \sum_{i=1}^{n}|(a_j - a_i)|^p \|x_i, z\|^p.
\end{aligned}$$

By taking maximum over $i = 1,...,n$, we obtain (4.2).

Theorem 4.2 *Let X be a quasi-(2;p)-normed space. For all* $z,x_1,...,x_n \in X$ *and* $a_1,...,a_n \in \mathbb{K}$ *the following inequalities hold true*

$$\|\sum_{i=1}^{n} a_i x_i, z\|^p \ge \|x_1, z\|^p \|\sum_{i=1}^{n} \frac{x_i}{\|a_i x_i, z\|}, z\|^p - \sum_{i=1}^{n} |\frac{1}{|a_i|}\|x_1, z\| - a_i\|x_i, z\||^p, \tag{4.3}$$

$$\|\sum_{i=1}^{n} a_i x_i, z\|^p \leq \|x_n, z\|^p \|\sum_{i=1}^{n} \frac{x_i}{\|a_i x_i, z\|}, z\|^p + \sum_{i=1}^{n} |a_i\|x_i, z\| - \frac{1}{|a_i|}\|x_n, z\||^p. \tag{4.4}$$

for $0 < p \leq 1$.

Proof.

$$\begin{aligned}
\|\sum_{i=1}^{n} \frac{x_i}{\|a_i x_i, z\|}, z\|^p &= \|\sum_{i=1}^{n} \frac{a_i x_i}{\|x_1, z\|} + \sum_{i=1}^{n} \frac{x_i}{\|a_i x_i, z\|} - \sum_{i=1}^{n} \frac{a_i x_i}{\|x_1, z\|}, z\|^p \\
&\leq \|\sum_{i=1}^{n} \frac{a_i x_i}{\|x_1, z\|}, z\|^p + \|\sum_{i=1}^{n} \frac{x_i}{\|a_i x_i, z\|} - \sum_{i=1}^{n} \frac{a_i x_i}{\|x_1, z\|}, z\|^p \\
&= \frac{1}{\|x_1, z\|^p}\|\sum_{i=1}^{n} a_i x_i, z\|^p + \frac{1}{\|x_1, z\|^p}\|\sum_{i=1}^{n} \frac{\|x_1, z\| - a_i\|a_i x_i, z\|}{\|a_i x_i, z\|} x_i, z\|^p \\
&\leq \frac{1}{\|x_1, z\|^p}\|\sum_{i=1}^{n} a_i x_i, z\|^p + \frac{1}{\|x_1, z\|^p}\sum_{i=1}^{n} |\frac{1}{|a_i|}\|x_1, z\| - a_i\|x_i, z\||^p
\end{aligned}$$

so,

$$\|\sum_{i=1}^{n} a_i x_i, z\|^p \geq \|x_1, z\|^p \|\sum_{i=1}^{n} \frac{x_i}{\|a_i x_i, z\|}, z\|^p - \sum_{i=1}^{n} |\frac{1}{|a_i|}\|x_1, z\| - a_i\|x_i, z\||^p$$

Further,

$$\begin{aligned}
\|\sum_{i=1}^{n} \frac{a_i x_i}{\|x_n, z\|}, z\|^p &= \|\sum_{i=1}^{n} \frac{a_i x_i}{\|x_n, z\|} + \sum_{i=1}^{n} \frac{x_i}{\|a_i x_i, z\|} - \sum_{i=1}^{n} \frac{x_i}{\|a_i x_i, z\|}, z\|^p \\
&\leq \|\sum_{i=1}^{n} \frac{x_i}{\|a_i x_i, z\|}, z\|^p + \|\sum_{i=1}^{n} \frac{a_i x_i}{\|x_n, z\|} - \sum_{i=1}^{n} \frac{x_i}{\|a_i x_i, z\|}, z\|^p \\
&= \|\sum_{i=1}^{n} \frac{x_i}{\|a_i x_i, z\|}, z\|^p + \frac{1}{\|x_n, z\|^p}\|\sum_{i=1}^{n} \frac{a_i\|a_i x_i, z\| - \|x_n, z\|}{\|a_i x_i, z\|} x_i, z\|^p \\
&\leq \|\sum_{i=1}^{n} \frac{x_i}{\|a_i x_i, z\|}, z\|^p + \frac{1}{\|x_n, z\|^p}\sum_{i=1}^{n} |a_i\|x_i, z\| - \frac{1}{|a_i|}\|x_n, z\||^p
\end{aligned}$$

so,

$$\|\sum_{i=1}^{n} a_i x_i, z\|^p \leq \|x_n, z\|^p \|\sum_{i=1}^{n} \frac{x_i}{\|a_i x_i, z\|}, z\|^p + \sum_{i=1}^{n} |a_i\|x_i, z\| - \frac{1}{|a_i|}\|x_n, z\||^p$$

This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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