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KANNAN FIXED POINT THEOREM IN A CONE PENTAGONAL METRIC SPACES

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Abstract. In this paper, we prove Kannan's fixed point theorem in cone pentagonal metric space. Our results extend and improve many known results in the literature.

Keywords: Cone pentagonal metric space; Contraction mapping principle; Kannan fixed point.

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1. Introduction

The concept of metric spaces was introduced in 1906 by French mathematician, M. Fŕechet [5]. A metric is a function (satisfying some properties) that takes values in the set of real numbers with its usual ordering.

Let (X,d) be a metric space and $T : X \to X$ be a mapping. Then T is called Banach contraction mapping if there exists $\alpha \in [0,1)$ such that

$$d(Tx,Ty) \leq \alpha d(x,y), \text{ for all } x,y \in X. \tag{1}$$

Banach [4] proved that if X is complete, then every Banach contraction mapping has a fixed point. Thus, Banach contraction principle ensures the existence of a unique fixed point of a Banach contraction on a complete metric space.

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Let (X,d) be a metric space and $T : X \to X$ be a mapping. Then T is called Kannan mapping if there exists $\alpha \in [0,1/2)$ such that

$$d(Tx,Ty) \leq \alpha\big[d(x,Tx)+d(y,Ty)\big], \text{ for all } x,y \in X. \tag{2}$$

Kannan [11] proved that if X is complete, then every Kannan mapping has a fixed point. He further showed that the conditions (1) and (2) are independent of each other (see, [10, 11]). Infact Kannan's fixed point theorem is very important, because Subrahmanyam [12] proved that Kannan's theorem characterizes the metric completeness. That is, a metric space X is complete if and only if every Kannan mapping on X has a fixed point.

In 2007, Huang and Zhang [8] introduced the concept of a cone metric space, they replaced set of real numbers by an ordered Banach space and proved some fixed point theorems for contractive type conditions in cone metric spaces. Later on many authors have (for e.g., [1, 2, 7, 9]) proved fixed point theorems for different contractive types conditions in cone metric spaces.

Recently, Azam et al. [3] introduced the notion of cone rectangular metric space and proved Banach contraction mapping principle in a cone rectangular metric space setting.

In 2009, Jleli and Samet [9] extend the Kannan's fixed point theorem in a cone rectangular metric space.

Very recently, M. Garg and S. Agarwal [6] introduced the notion of cone pentagonal metric space and proved Banach contraction mapping principle in a cone pentagonal metric space.

Motivated by the results of [6, 9], it is our purpose in this paper to continue the study of fixed point theorem in cone pentagonal metric space setting. Our results improve and extend the results of Kannan [11], Jleli and Samet [9], and many others.

2. Preliminaries

In this section, we shall give the notion of cone metric spaces and some related properties introduced in [3, 6, 8], which will be needed in the sequel.

Definition 1.1. Let E be a real Banach space and P a subset of E. P is called a cone if and only if:

(1) P is closed, nonempty, and $P \neq \{0\}$;

(2) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P \Longrightarrow ax + by \in P$;

(3) $x \in P$ and $-x \in P \Longrightarrow x = 0$.

Given a cone $P \subseteq E$, we defined a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of P.

Definition 1.2. A cone P is called normal if there is a number $k \geq 1$ such that for all $x, y \in E$, the inequality

$$0 \leq x \leq y \Longrightarrow \|x\| \leq k\|y\|. \tag{3}$$

The least positive number k satisfying (3) is called the normal constant of P.

In this paper, we always suppose that E is a real Banach space and P is a cone in E with $int(P) \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 1.3. Let X be a nonempty set. Suppose that the mapping $\rho : X \times X \to E$ satisfies:

(1) $0 < \rho(x,y)$ for all $x, y \in X$ and $\rho(x,y) = 0$ if and only if $x = y$;

(2) $\rho(x,y) = \rho(y,x)$ for all $x, y \in X$;

(3) $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$ for all $x, y, z \in X$;

Then ρ is called a cone metric on X, and (X, ρ) is called a cone metric space.

Remark 1.1. Every metric space is a cone metric space. The converse is not necessarily true (e.g., see [8]).

Definition 1.4. Let X be a nonempty set. Suppose that the mapping $\rho : X \times X \to E$ satisfies:

(1) $0 < \rho(x,y)$ for all $x, y \in X$ and $\rho(x,y) = 0$ if and only if $x = y$;

(2) $\rho(x,y) = \rho(y,x)$ for all $x, y \in X$;

(3) $\rho(x,y) \leq \rho(x,w) + \rho(w,z) + \rho(z,y)$ for all $x, y \in X$ and for all distinct points $w, z \in X - \{x, y\}$ [Rectangular property].

Then ρ is called a cone rectangular metric on X, and (X, ρ) is called a cone rectangular metric space.

Remark 1.2. Every cone metric space and so metric space is cone rectangular metric space. The converse is not necessarily true (e.g., see [3]).

Definition 1.5. Let X be a non empty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(1) $0 < d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for $x,y \in X$;

(3) $d(x,y) \le d(x,z) + d(z,w) + d(w,u) + d(u,y)$ for all $x,y,z,w,u \in X$ and for all distinct points $z,w,u, \in X - \{x,y\}$ [Pentagonal property].

Then d is called a cone pentagonal metric on X, and (X,d) is called a cone pentagonal metric space.

Remark 1.3. Every cone rectangular metric space and so cone metric space is cone pentagonal metric space. The converse is not necessarily true (e.g., see [6]).

Definition 1.6. Let (X,d) be a cone pentagonal metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ and that for all $n > n_0$, $d(x_n,x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 1.7. Let (X,d) be a cone pentagonal metric space. Let $\{x_n\}$ be a sequence in X. If for every $c \in E$, with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ such that for all $n,m > n_0$, $d(x_n,x_m) \ll c$, then $\{x_n\}$ is called Cauchy sequence in X.

Definition 1.8. Let (X,d) be a cone pentagonal metric space. If every Cauchy sequence is convergent in X, then X is called a complete cone pentagonal metric space.

Lemma 1.1. [6] *Let (X,d) be a cone pentagonal metric space and P be a normal cone with normal constant k. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ converges to x if and only if $\|d(x_n,x)\| \to 0$ as $n \to \infty$.*

Lemma 1.2. [6] *Let (X,d) be a cone pentagonal metric space and P be a normal cone with normal constant k. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ is a Cauchy sequence if and only if $\|d(x_n,x_{n+m})\| \to 0$ as $n,m \to \infty$.*

Lemma 1.3. *Let (X,d) be a complete cone pentagonal metric space, P be a normal cone with normal constant k. Let $\{x_n\}$ be a Cauchy sequence in X and suppose that there is natural number N such that:*

(1) $x_n \neq x_m$ for all $n,m > N$;

(2) x_n, x are distinct points in X for all $n > N$;

(3) x_n, y are distinct points in X for all $n > N$;

(4) $x_n \to x$ and $x_n \to y$ as $n \to \infty$.

Then $x = y$.

Proof. The proof is similar to the proof of ([9]-Lemma 1.10).

3. Main results

In this section, we prove a fixed point theorem in a cone pentagonal metric space. Our obtained result generalizes well known Kannan's theorem, and result of Jleli and Samet [9].

Theorem 3.1. *Let (X,d) be a complete cone pentagonal metric space, P be a normal cone with normal constant k. Suppose a mapping $S : X \to X$ satisfies the contractive condition:*

$$d(Sx,Sy) \leq \lambda \left[d(Sx,x) + d(Sy,y)\right], \tag{4}$$

for all $x,y \in X$, where $\lambda \in [0,1/2)$. Then

(1) S has a unique fixed point in X.

(2) For any $x \in X$, the iterative sequence $\{S^n x\}$ converges to the fixed point.

Proof. Let $x \in X$. From (4), we have

$$d(Sx,S^2x) \leq \lambda \left[d(x,Sx) + d(Sx,S^2x)\right],$$

i.e,

$$d(Sx,S^2x) \leq \frac{\lambda}{1-\lambda} d(x,Sx).$$

Again

$$d(S^2x,S^3x) \leq \lambda \left[d(Sx,S^2x) + d(S^2x,S^3x)\right],$$

i.e,

$$d(S^2x, S^3x) \leq \frac{\lambda}{1-\lambda} d(Sx, S^2x) \leq \left(\frac{\lambda}{1-\lambda}\right)^2 d(x, Sx).$$

Thus, in general, if n is a positive integer, then

$$\begin{aligned} d(S^n x, S^{n+1}x) &\leq \left(\frac{\lambda}{1-\lambda}\right)^n d(x, Sx) \\ &= r^n d(x, Sx), \end{aligned} \tag{5}$$

where $r = \left(\frac{\lambda}{1-\lambda}\right) \in [0,1)$.

We divide the proof into two cases.

First case :

Let $S^m x = S^n x$ for some $m, n \in \mathbb{N}$, $m \neq n$. Let $m > n$. Then $S^{m-n}(S^n x) = S^n x$, i.e. $S^p y = y$, where $p = m - n,\ y = S^n x$. Now since $p > 1$, we have

$$\begin{aligned} d(y, Sy) &= d(S^p y, S^{p+1} y) \\ &\leq r^n d(y, Sy). \end{aligned}$$

Since $r \in [0,1)$, we obtain $-d(y, Sy) \in P$ and $d(y, Sy) \in P$, which implies that

$$\|(y, Sy)\| = 0.$$

That is, $Sy = y$.

Second case :

Assume that $S^m x \neq S^n x$ for all $m, n \in \mathbb{N}$, $m \neq n$. From (5), and the fact that $0 \leq \lambda < r < 1$, we have

$$\begin{aligned} d(S^n x, S^{n+1}x) &\leq r^n d(x, Sx) \\ &\leq r^n d(x, Sx) + r^{n+1} d(x, Sx) + r^{n+2} d(x, Sx) \\ &\leq r^n (1 + r + r^2 + \cdots) d(x, Sx) \\ &\leq \frac{r^n}{1-r} d(x, Sx), \end{aligned}$$

$$\begin{aligned}
d(S^n x, S^{n+2}x) &\leq \lambda\left[d(S^{n-1}x, S^n x) + d(S^{n+1}x, S^{n+2}x)\right] \\
&\leq \lambda\left[r^{n-1}d(x,Sx) + r^{n+1}d(x,Sx)\right] \\
&\leq r^n d(x,Sx) + r^{n+1}d(x,Sx) + r^{n+2}d(x,Sx) \\
&\leq r^n(1 + r + r^2 + \cdots)d(x,Sx) \\
&= \frac{r^n}{1-r}d(x,Sx),
\end{aligned}$$

and

$$\begin{aligned}
d(S^n x, S^{n+3}x) &\leq \lambda\left[d(S^{n-1}x, S^n x) + d(S^{n+2}x, S^{n+3}x)\right] \\
&\leq \lambda\left[r^{n-1}d(x,Sx) + r^{n+2}d(x,Sx)\right] \\
&\leq r^n d(x,Sx) + r^{n+1}d(x,Sx) + r^{n+2}d(x,Sx) \\
&\leq r^n(1 + r + r^2 + \cdots)d(x,Sx) \\
&= \frac{r^n}{1-r}d(x,Sx).
\end{aligned}$$

Now, if $m > 3$ and $m := 3k+1$, $k \geq 1$ and using the fact that $S^p x \neq S^q x$ for $p, q \in \mathbb{N}$, $p \neq q$, by pentagonal property, we obtain

$$\begin{aligned}
d(S^n x, S^{n+3k+1}x) &\leq d(S^{n+3k+1}x, S^{n+3k}x) + d(S^{n+3k}x, S^{n+3k-1}x) + d(S^{n+3k-1}x, S^{n+3k-2}x) \\
&\quad + d(S^{n+3k-2}x, S^n x) \\
&\leq d(S^{n+3k}x, S^{n+3k+1}x) + d(S^{n+3k-1}x, S^{n+3k}x) + d(S^{n+3k-2}x, S^{n+3k-1}x) \\
&\quad + d(S^{n+3k-2}x, S^{n+3k-3}x) + d(S^{n+3k-3}x, S^{n+3k-4}x) + \cdots \\
&\quad + d(S^{n+2}x, S^{n+1}x) + d(S^{n+1}x, S^n x) \\
&= d(S^n x, S^{n+1}x) + d(S^{n+1}x, S^{n+2}x) + \cdots + d(S^{n+3k-1}x, S^{n+3k}x) \\
&\quad + d(S^{n+3k}x, S^{n+3k+1}x) \\
&\leq r^n d(x,Sx) + r^{n+1}d(x,Sx) + \cdots + r^{n+3k-1}d(x,Sx) + r^{n+3k}d(x,Sx) \\
&\leq r^n(1 + r + r^2 + \cdots)d(x,Sx) \\
&= \frac{r^n}{1-r}d(x,Sx).
\end{aligned}$$

Similarly, if $m > 4$ and $m := 3k+2$, $k \geq 1$ and using the fact that $S^p x \neq S^q x$ for $p, q \in \mathbb{N}$, $p \neq q$, by pentagonal property, we obtain

$$\begin{aligned}
d(S^n x, S^{n+3k+2}x) &\leq d(S^{n+3k+2}x, S^{n+3k+1}x) + d(S^{n+3k+1}x, S^{n+3k}x) + d(S^{n+3k}x, S^{n+3k-1}x) \\
&\quad + d(S^{n+3k-1}x, S^n x) \\
&\leq d(S^n x, S^{n+1}x) + d(S^{n+1}x, S^{n+2}x) + \cdots + d(S^{n+3k-1}x, S^{n+3k}x) \\
&\quad + d(S^{n+3k}x, S^{n+3k+1}x) \\
&\leq r^n d(x, Sx) + r^{n+1} d(x, Sx) + \cdots + r^{n+3k-1} d(x, Sx) + r^{n+3k} d(x, Sx) \\
&\leq r^n (1 + r + r^2 + \cdots) d(x, Sx) \\
&= \frac{r^n}{1-r} d(x, Sx).
\end{aligned}$$

Also, if $m > 5$ and $m := 3k+3$, $k \geq 1$ and using the fact that $S^p x \neq S^q x$ for $p, q \in \mathbb{N}$, $p \neq q$, by pentagonal property, we obtain

$$\begin{aligned}
d(S^n x, S^{n+3k+3}x) &\leq d(S^{n+3k+3}x, S^{n+3k+2}x) + d(S^{n+3k+2}x, S^{n+3k+1}x) + d(S^{n+3k+1}x, S^{n+3k}x) \\
&\quad + d(S^{n+3k}x, S^n x) \\
&\leq d(S^{n+3k+1}x, S^{n+3k+2}x) + d(S^{n+3k}x, S^{n+3k+1}x) + d(S^{n+3k-1}x, S^{n+3k}x) \\
&\quad + d(S^{n+3k-1}x, S^{n+3k-2}x) + d(S^{n+3k-2}x, S^{n+3k-3}x) + \cdots \\
&\quad + d(S^{n+2}x, S^{n+1}x) + d(S^{n+1}x, S^n x) \\
&= d(S^n x, S^{n+1}x) + d(S^{n+1}x, S^{n+2}x) + \cdots + d(S^{n+3k-1}x, S^{n+3k}x) \\
&\quad + d(S^{n+3k}x, S^{n+3k+1}x) \\
&\leq r^n d(x, Sx) + r^{n+1} d(x, Sx) + \cdots + r^{n+3k-1} d(x, Sx) + r^{n+3k} d(x, Sx) \\
&\leq r^n (1 + r + r^2 + \cdots) d(x, Sx) \\
&= \frac{r^n}{1-r} d(x, Sx).
\end{aligned}$$

Thus, combining the above cases, we have

$$d(S^n x, S^{n+m}x) \leq \frac{r^n}{1-r} d(x, Sx), \ \forall m, n \in \mathbb{N}.$$

Since P is a normal cone with normal constant k, therefore, by (3), we have

$$\|d(S^n x, S^{n+m}x)\| \leq \frac{kr^n}{1-r}\|d(x,Sx)\|, \ \forall m,n \in \mathbb{N}.$$

Since

$$\lim_{n\to\infty} \frac{kr^n}{1-r}\|d(x,Sx)\| = 0,$$

we have that

$$\lim_{n\to\infty} \|d(S^n x, S^{n+m}x)\| = 0, \ \forall m,n \in \mathbb{N}. \tag{6}$$

Therefore, by Lemma 1.2, $\{S^n x\}$ is a cauchy sequence in X. By completeness of X, there exists a point $z \in X$ such that

$$\lim_{n\to\infty} S^n x = z. \tag{7}$$

We shall now show that z is a fixed point of S. i.e., $Sz = z$. Without loss of generality, we can assume that $S^r x \neq z, Sz$ for any $r \in \mathbb{N}$. By pentagonal property, we have

$$\begin{aligned} d(z,Sz) &\leq d(z,S^n x) + d(S^n x, S^{n+1}x) + d(S^{n+1}x, S^{n+2}x) + d(S^{n+2}x, Sz) \\ &\leq d(z,S^n x) + d(S^n x, S^{n+1}x) + d(S^{n+1}x, S^{n+2}x) + \lambda\left[d(S^{n+1}x, S^{n+2}x) + d(z,Sz)\right], \end{aligned}$$

which implies that

$$d(z,Sz) \leq \frac{1}{1-\lambda}\left[d(z,S^n x) + d(S^n x, S^{n+1}x) + (1+\lambda)d(S^{n+1}x, S^{n+2}x)\right].$$

Hence,

$$\|d(z,Sz)\| \leq \frac{k}{1-\lambda}\left[\|d(z,S^n x)\| + \|d(S^n x, S^{n+1}x)\| + (1+\lambda)\|d(S^{n+1}x, S^{n+2}x)\|\right].$$

Letting $n \to \infty$, we have $\|d(z,Sz)\| = 0$. Hence, $Sz = z$. i.e., z is a fixed point of S.

Now, we show that z is unique. Suppose z' is another fixed point of S, that is $Sz' = z'$. Therefore,

$$d(z,z') = d(Sz,Sz') \leq \lambda\left[d(z,Sz) + d(z',Sz')\right] = 0,$$

which implies that

$$\|d(z,z')\| = 0.$$

That is, $z = z'$. This completes the proof of the theorem.

To illustrate Theorem 3.1, we give the following example.

Example Let $X=\{1,2,3,4,5\}$, $E=\mathbb{R}^2$ and $P=\{(x,y):x,y\geq 0\}$ is a normal cone in E. Define $d:X\times X\to E$ as follows:

$$d(x,x)=0,\forall x\in X d(1,2)=d(2,1)=(4,16)$$

$$d(1,3)=d(3,1)=d(3,4)=d(4,3)=d(2,3)=d(3,2)=d(2,4)=d(4,2)$$

$$=d(1,4)=d(4,1)=(1,4)$$

$$d(1,5)=d(5,1)=d(2,5)=d(5,2)=d(3,5)=d(5,3)=d(4,5)=d(5,4)=(5,20).$$

Then (X,d) is a complete cone pentagonal metric space, but (X,d) is not a complete cone rectangular metric space because it lacks the rectangular property:

$$(4,16)=d(1,2)>d(1,3)+d(3,4)+d(4,2)$$

$$=(1,4)+(1,4)+(1,4)$$

$$=(3,12) \text{ as } (4,16)-(3,12)=(1,4)\in P.$$

Now, define a mapping $S:X\to X$ as follows

$$Sx=\begin{cases} 4, & \text{if } x\neq 5;\\ 2, & \text{if } x=5.\end{cases}$$

Observe that

$$d(S(1),S(2))=d(S(1),S(3))=d(S(1),S(4))=d(S(2),S(3))$$

$$=d(S(2),S(4))=d(S(3),S(4))=0.$$

And in all other cases $d(S(x),S(y))=(1,4)$, $d(x,y)=(5,20)$.

We remark that S is not a contractive mapping with respect to the standard metric in X, because we have

$$|S5-S3|=2=|5-3|.$$

However, S satisfies

$$d(Sx,Sy)\leq\lambda\left[d(x,Sx)+d(y,Sy)\right],\ \forall x,y\in X,$$

with $\lambda = 1/4$. Applying Theorem 3.1, we obtain that S admits a unique fixed point, that is $z = 4$.

In the above Example, results of Jleli and Samet [9], (or Huang and Zhang [8]) are not applicable to obtained the fixed point of the mapping S on X. Since (X,d) is not a cone rectangular (or cone) metric space.

Corollary 3.2. [9] *Let (X,d) be a complete cone rectangular metric space and P be a normal cone with normal constant k. Suppose the mapping $S: X \to X$ satisfies the contractive condition:*

$$d(Sx,Sy) \leq \lambda \left[d(x,Sx) + d(y,Sy)\right], \tag{8}$$

for all $x,y \in X$, where $\lambda \in [0,1/2)$. Then

(1) S has a unique fixed point in X.

(2) For any $x \in X$, the iterative sequence $\{S^n x\}$ converges to the fixed point.

Proof. This follows from the Remark 1.3 and Theorem 3.1.

Conflict of Interests

The authors declare that there is no conflict of interests.

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