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CR-SUBMANIFOLDS OF A NEARLY HYPERBOLIC SASAKIAN MANIFOLD WITH A SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. CR-submanifolds of nearly hyperbolic Sasakian manifold with a semi-symmetric metric connection are studied. We obtain ξ −horizontal and ξ −vertical CR- submanifolds of a nearly hyperbolic Sasakian manifold with a semi-symmetric metric connection. Parallel distributions on CR-submanifolds of nearly hyperbolic Sasakian manifold with semi-symmetric metric connection are calculated.

Keywords: CR-submanifolds; nearly hyperbolic Sasakian manifold; semi-symmetric metric connection; parallel distribution.

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1. Introduction

Let ∇ be a linear connection in an n-dimensional differential manifold $\bar{M}$. The connection ∇ is metric connection if there is a Riemannian metric g in $\bar{M}$ such that $\nabla g = 0$, otherwise it is non-metric. Friedmann and Schouten [11] introduced the concept of semi-symmetric linear connection. A linear connection ∇ is said to be semi-symmetric connection if its torsion tensor T is of the form [13]

$$T(X,Y) = \eta(Y)X - \eta(X)Y,$$

where η is 1-form. Some properties of semi-symmetric metric connection are studies in [2], [4], [13], [18].

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A. Bejancu introduced the concept of CR-submanifolds of Kaehler manifold as a generalization of invariant and anti-invariant submanifolds [5]. Since then, several papers on Kaehler manifolds were published. CR-submanifolds of Sasakian manifold was studied by C.J. Hsu in [14] and M. Kobayashi in [17]. Yano and Kon [22] studied contact CR-submanifolds. Later, several geometers (see, [3], [4], [6], [20]) enriched the study of CR-submanifolds of almost contact manifolds. The almost hyperbolic (f, ξ, η, g)-structure was defined and studied by Upadhyay and Dube [21]. CR-submanifolds of trans-hyperbolic Sasakian manifold studied by Bhatt and Dube [8]. On the other hand, S. Golab [12] introduced the idea of semi-symmetric and quarter symmetric connections. The first author and S.K. Lovejoy Das [10] studied CR-submanifolds of LP-Sasakian manifold with semi-symmetric non-metric connection. CR-submanifolds of a nearly hyperbolic Sasakian manifold admitting a semi-symmetric semi-metric connection were studied by M.D. Siddiqi and S. Rizvi [3]. Motivated by studies [1, 2, 3, 9, 16, 18], in this paper we study some properties of CR-submanifolds of a nearly hyperbolic Sasakian manifold with a semi-symmetric metric connection.

The paper is organized as follows. In section 2, we give a brief description of nearly hyperbolic Sasakian manifold with a semi-symmetric metric connection. In section 3, some properties of CR-submanifolds of nearly hyperbolic Sasakian manifold are investigated. In section 4, some results on parallel distribution on $\xi-$horizontal and $\xi-$vertical CR- submanifolds of a nearly Sasakian manifold with a semi-symmetric metric connection are obtained.

2. Preliminaries

Let $\bar{M}$ be an n-dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure$(\emptyset, \xi, \eta, g)$, where a tensor $\emptyset$ of type$(1,1)$, a vector field ξ called structure vector field, η the dual 1-form of ξ and g is Riemannian metric satisfying the followings:

$$\emptyset^2 X = X + \eta(X)\xi, \qquad g(X, \xi) = \eta(X), \tag{2.1}$$

$$\eta(\xi) = -1, \quad \emptyset(\xi) = 0, \quad \eta o \emptyset = 0, \tag{2.2}$$

$$g(\emptyset X, \emptyset Y) = -g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for any X, Y tangent to $\bar{M}$ [7]. In this case

$$g(\emptyset X, Y) = -g(\emptyset Y, X). \tag{2.4}$$

An almost hyperbolic contact metric structure $(\emptyset, \xi, \eta, g)$ on $\bar{M}$ is called hyperbolic Sasakian manifold if and only if

$$(\nabla_X \emptyset)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.5}$$

$$\nabla_X \xi = \emptyset X \tag{2.6}$$

for all tangent vectors X, Y and a Riemannian metric g and Riemannian connection ∇ on manifold $\bar{M}$. Further, as a consequence of (2.5), an almost hyperbolic contact metric manifold $\bar{M}$ with $(\emptyset, \xi, \eta, g) -$ structure is called a nearly hyperbolic Sasakian manifold if

$$(\nabla_X \emptyset)Y + (\nabla_Y \emptyset)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X. \tag{2.7}$$

Now, Let M be a submanifold immersed in $\bar{M}$, the Riemannian metric g induced on M. Let TM and $T^{\perp}M$ be the Lie algebra of vector fields tangential to M and normal to M respectively and ∇^* be the induced Levi-Civita connection on M, then the Gauss and Weingarten formulae are given respectively by

$$\nabla_X Y = \nabla^*{}_X Y + h(X,Y), \tag{2.8}$$

$$\nabla_X N = -A_N X + \nabla_X^{\perp} N \tag{2.9}$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ is a connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_N is the Weingarten map associated with N as

$$g(h(X,Y), N) = g(A_N X, Y). \tag{2.10}$$

Any vector X tangent to M is given as

$$X = PX + QX, \tag{2.11}$$

where $PX \in D$ and $QX \in D^{\perp}$.

For any N normal to M, we have

$$\emptyset N = BN + CN, \tag{2.12}$$

where BN $(\text{resp.} CN)$ is the tangential component (resp. normal component) of $\emptyset N$.

Now, we define a semi-symmetric metric connection

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi \tag{2.13}$$

such that $(\bar{\nabla}_X g)(Y,Z) = 0$.

From (2.13) and (2.7), we have

$$(\bar{\nabla}_X \emptyset)Y + \emptyset(\bar{\nabla}_X Y) = (\nabla_X \emptyset)Y + \emptyset(\nabla_X Y) - g(X, \emptyset Y)\xi.$$

Interchanging X and Y, we have

$$(\bar{\nabla}_Y \emptyset)X + \emptyset(\bar{\nabla}_Y X) = (\nabla_Y \emptyset)X + \emptyset(\nabla_Y X) - g(Y, \emptyset X)\xi.$$

Adding above two equations, we get

$$(\overline{\nabla}_X \emptyset)Y + (\overline{\nabla}_Y \emptyset)X + \emptyset(\overline{\nabla}_X Y - \nabla_X Y) + \emptyset(\overline{\nabla}_Y X - \nabla_Y X) = (\nabla_X \emptyset)Y + (\nabla_Y \emptyset)X$$
$$-g(X, \emptyset Y)\xi - g(Y, \emptyset X)\xi.$$

Using equation (2.2), (2.4), (2.7) and (2.13) in above equation, we have

$$(\overline{\nabla}_X \emptyset)Y + (\overline{\nabla}_Y \emptyset)X = 2g(X, Y)\xi - \eta(X)Y$$
$$-\eta(Y)X - \eta(X)\emptyset Y - \eta(Y)\emptyset X. \tag{2.14}$$

From (2.6) and (2.13), we have

$$\overline{\nabla}_X \xi = \emptyset X - X - \eta(X). \tag{2.15}$$

An almost hyperbolic contact metric manifold with almost hyperbolic contact structure $(\emptyset, \xi, \eta, g)$ is called nearly hyperbolic Sasakian manifold with semi-symmetric metric connection if it satisfies (2.14) and (2.15).

In view of (2.8) and (2.9) and (2.13), Gauss and Weingarten formulae for a nearly hyperbolic Sasakian manifold with semi-symmetric metric connection are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.16}$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N. \tag{2.17}$$

Definition 2.1. An m-dimensional submanifold M of an n-dimensional nearly hyperbolic Sasakian manifold $\overline{M}$ is called a CR-submanifold [3] if there exists a differentiable distribution $D: x \to D_x$ on M satisfying the following conditions:

(i) the distribution D is invariant under $\emptyset$, that is $\emptyset D_x \subset D_x$ for each $x \in M$,

(ii) the complementary orthogonal distribution $D^{\perp}$ of D is anti-invariant under $\emptyset$, that is $\emptyset {D^{\perp}}_x \subset T^{\perp}M$ for all $x \in M$.

If $\dim D_x^{\perp} = 0$ $(resp., \dim D_x = 0)$, then the CR-submanifold is called an invariant (resp., anti-invariant) submanifold. The distribution D $(resp., D^{\perp})$ is called the horizontal (resp., vertical) distribution. Also, the pair $(D, D^{\perp})$ is called $\xi -$ horizontal (resp., vertical), if $\xi_X \in D_X$ (resp., $\xi_X \in D_X^{\perp}$).

3. Some Basic Results

Lemma 3.1. If M be a CR-submanifold of a nearly hyperbolic Sasakian manifold $\overline{M}$ with a semi symmetric metric connection. Then

$$2g(X, Y)P\xi - \eta(X)PY - \eta(Y)PX - \eta(X)\emptyset PY - \eta(Y)\emptyset PX + \emptyset P(\nabla_X Y)$$

$$+\emptyset P(\nabla_Y X) = P\nabla_X(\emptyset PY) + P\nabla_Y(\emptyset PX) - PA_{\emptyset QY}X - PA_{\emptyset QX}Y, \quad (3.1)$$

$$2g(X,Y)Q\xi - \eta(X)QY - \eta(Y)QX + 2Bh(X,Y) = Q\nabla_X(\emptyset PY) + Q\nabla_Y(\emptyset PX)$$
$$-QA_{\emptyset QY}X - QA_{\emptyset QX}Y, \quad (3.2)$$

$$-\eta(X)\emptyset QY - \eta(Y)\emptyset QX + \emptyset Q(\nabla_X Y) + \emptyset Q(\nabla_Y X) + 2Ch(X,Y) = h(X,\emptyset PY)$$
$$+h(Y,\emptyset PX) + \nabla_X^{\perp}(\emptyset QY) + \nabla_Y^{\perp}(\emptyset QX) \quad (3.3)$$

for all $X, Y \in TM$.

Proof. From (2.11), we have

$$\emptyset Y = \emptyset PY + \emptyset QY.$$

Differentiating covariantly and using equation (2.16) and (2.17), we have

$$(\overline{\nabla}_X\emptyset)Y + \emptyset(\nabla_X Y) + \emptyset h(X,Y)$$
$$= \nabla_X(\emptyset PY) + h(X,\emptyset PY) - A_{\emptyset QY}X + \nabla_X^{\perp}(\emptyset QY).$$

Interchanging X and Y in above equation, we have

$$(\overline{\nabla}_Y\emptyset)X + \emptyset(\nabla_Y X) + \emptyset h(Y,X)$$
$$= \nabla_Y(\emptyset PX) + h(Y,\emptyset PX) - A_{\emptyset QX}Y + \nabla_Y^{\perp}(\emptyset QX).$$

Adding above two equations, we obtain

$$(\overline{\nabla}_X\emptyset)Y + (\overline{\nabla}_Y\emptyset)X + \emptyset(\nabla_X Y) + \emptyset(\nabla_Y X) + 2\emptyset h(Y,X)$$
$$= \nabla_X(\emptyset PY) + \nabla_Y(\emptyset PX) + h(X,\emptyset PY) + h(Y,\emptyset PX)$$
$$-A_{\emptyset QY}X - A_{\emptyset QX}Y + \nabla_X^{\perp}(\emptyset QY) + \nabla_Y^{\perp}(\emptyset QX).$$

Adding (2.14) in above equation, we have

$$2g(X,Y)\xi - \eta(X)Y - \eta(Y)X - \eta(X)\emptyset Y - \eta(Y)\emptyset X + \emptyset(\nabla_X Y) + \emptyset(\nabla_Y X)$$
$$+2\emptyset h(X,Y) = \nabla_X\emptyset PY + \nabla_Y\emptyset PX + h(X,\emptyset PY) + h(Y,\emptyset PX) - A_{\emptyset QY}X$$
$$-A_{\emptyset QX}Y + \nabla_X^{\perp}\emptyset QY + \nabla_Y^{\perp}\emptyset QX.$$

Using equations (2.11) and (2.12) in above equation, we have

$$2g(X,Y)P\xi + 2g(X,Y)Q\xi - \eta(X)PY - \eta(X)QY - \eta(Y)PX - \eta(Y)QX$$
$$-\eta(X)\emptyset PY - \eta(X)\emptyset QY - \eta(Y)\emptyset QX + \emptyset P\nabla_X Y + \emptyset Q\nabla_X Y + \emptyset P\nabla_Y X$$
$$+\emptyset Q\nabla_Y X + 2Bh(X,Y) + 2Ch(X,Y) = P\nabla_X\emptyset PY + Q\nabla_X\emptyset PY + P\nabla_Y\emptyset PX$$
$$+Q\nabla_Y\emptyset PX + h(X,\emptyset PY) + h(Y,\emptyset PX) - PA_{\emptyset QY}X - QA_{\emptyset QY}X - PA_{\emptyset QX}Y$$
$$-QA_{\emptyset QX}Y + \nabla_X^{\perp}\emptyset QY + \nabla_Y^{\perp}\emptyset QX. \quad (3.4)$$

Comparing tangential, vertical and normal components in (3.4), we get desired results.

Lemma 3.2. If M be a CR-submanifold of a nearly hyperbolic Sasakian manifold $\bar{M}$ with semi symmetric metric connection. Then

$$2(\bar{\nabla}_X \emptyset)Y = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X - \eta(X)\emptyset Y - \eta(Y)\emptyset X + \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X) - \emptyset[X,Y], \tag{3.5}$$

$$2(\bar{\nabla}_Y \emptyset)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X - \eta(X)\emptyset Y - \eta(Y)\emptyset X + \nabla_Y \emptyset X - \nabla_X \emptyset Y + h(Y,\emptyset X) - h(X,\emptyset Y) + \emptyset[X,Y] \tag{3.6}$$

for all $X, Y \in D$.

Proof. From Gauss formula (2.16), we get

$$\bar{\nabla}_X \emptyset Y - \bar{\nabla}_Y \emptyset X = \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X). \tag{3.7}$$

Also, by covariant differentiation, we have

$$\bar{\nabla}_X \emptyset Y - \bar{\nabla}_Y \emptyset X = (\bar{\nabla}_X \emptyset)Y - (\bar{\nabla}_Y \emptyset)X + \emptyset[X,Y]. \tag{3.8}$$

From (3.7) and (3.8), we get

$$(\bar{\nabla}_X \emptyset)Y - (\bar{\nabla}_Y \emptyset)X = \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X) - \emptyset[X,Y]. \tag{3.9}$$

Adding (3.9) and (2.14), we have

$$2(\bar{\nabla}_X \emptyset)Y = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X - \eta(X)\emptyset Y - \eta(Y)\emptyset X + \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X) - \emptyset[X,Y]$$

Subtracting (3.9) from (2.14), get

$$2(\bar{\nabla}_Y \emptyset)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X - \eta(X)\emptyset Y - \eta(Y)\emptyset X + \nabla_Y \emptyset X - \nabla_X \emptyset Y + h(Y,\emptyset X) - h(X,\emptyset Y) + \emptyset[X,Y]$$

for all $X, Y \in D$.

Corollary 3.3. If M be a $\xi -$ vertical CR-submanifold of a nearly hyperbolic Sasakian manifold $\bar{M}$ with semi-symmetric metric connection. Then

$$2(\bar{\nabla}_X \emptyset)Y = 2g(X,Y)\xi + \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X) - \emptyset[X,Y]$$

$$2(\bar{\nabla}_Y \emptyset)X = 2g(X,Y)\xi + \nabla_Y \emptyset X - \nabla_X \emptyset Y + h(Y,\emptyset X) - h(X,\emptyset Y) + \emptyset[X,Y]$$

for all $X, Y \in D$.

Lemma 3.4. If M be a CR-submanifold of a nearly hyperbolic Sasakian manifold $\bar{M}$ with semi symmetric metric connection. Then

$$2(\bar{\nabla}_X \emptyset)Y = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X - \eta(X)\emptyset Y - \eta(Y)\emptyset X + A_{\emptyset X}Y - A_{\emptyset Y}X + \nabla^{\perp}_X \emptyset Y - \nabla^{\perp}_Y \emptyset X - \emptyset[X,Y] \tag{3.10}$$

$$2(\overline{\nabla}_Y \emptyset)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X - \eta(X)\emptyset Y - \eta(Y)\emptyset X + A_{\emptyset Y}X$$
$$-A_{\emptyset X}Y + \nabla_Y^{\perp}\emptyset X - \nabla_X^{\perp}\emptyset Y + \emptyset[X,Y] \quad (3.11)$$

for all $X, Y \in D^{\perp}$.

Proof. For $X, Y \in D^{\perp}$, from Weingarten formula (2.17), we have

$$\overline{\nabla}_X \emptyset Y - \overline{\nabla}_Y \emptyset X = A_{\emptyset X}Y - A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y^{\perp}\emptyset X \quad (3.12)$$

Comparing equations (3.12) and (3.8), we have

$$(\overline{\nabla}_X \emptyset)Y - (\overline{\nabla}_Y \emptyset)X + \emptyset[X,Y] = A_{\emptyset X}Y - A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y^{\perp}\emptyset X \quad (3.13)$$

$$(\overline{\nabla}_X \emptyset)Y - (\overline{\nabla}_Y \emptyset)X = A_{\emptyset X}Y - A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y^{\perp}\emptyset X - \emptyset[X,Y] \quad (3.14)$$

Adding (3.14) and (2.14), we get

$$2(\overline{\nabla}_X \emptyset)Y = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X - \eta(X)\emptyset Y - \eta(Y)\emptyset X + A_{\emptyset X}Y$$
$$-A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y^{\perp}\emptyset X - \emptyset[X,Y]$$

Subtracting (3.14) from (2.14), we get

$$2(\overline{\nabla}_Y \emptyset)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X - \eta(X)\emptyset Y - \eta(Y)\emptyset X + A_{\emptyset Y}X$$
$$-A_{\emptyset X}Y + \nabla_Y^{\perp}\emptyset X - \nabla_X^{\perp}\emptyset Y + \emptyset[X,Y]$$

for all $X, Y \in D^{\perp}$.

Corollary 3.5. If M be a $\xi -$ horizontal CR-submanifold of a nearly hyperbolic Sasakian manifold $\overline{M}$ with semi symmetric metric connection. Then

$$2(\overline{\nabla}_X \emptyset)Y = 2g(X,Y)\xi + A_{\emptyset X}Y - A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y^{\perp}\emptyset X - \emptyset[X,Y],$$
$$2(\overline{\nabla}_Y \emptyset)X = 2g(X,Y)\xi + A_{\emptyset Y}X - A_{\emptyset X}Y + \nabla_Y^{\perp}\emptyset X - \nabla_X^{\perp}\emptyset Y + \emptyset[X,Y]$$

for all $X, Y \in D^{\perp}$.

Lemma 3.6. If M be a CR-submanifold of a nearly hyperbolic Sasakian manifold $\overline{M}$ with semi symmetric metric connection. Then

$$2(\overline{\nabla}_X \emptyset)Y = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X - \eta(X)\emptyset Y - \eta(Y)\emptyset X - A_{\emptyset Y}X$$
$$+\nabla_X^{\perp}\emptyset Y - \nabla_Y \emptyset X - h(Y, \emptyset X) - \emptyset[X,Y], \quad (3.15)$$
$$2(\overline{\nabla}_Y \emptyset)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X - \eta(X)\emptyset Y - \eta(Y)\emptyset X + A_{\emptyset Y}Y$$
$$-\nabla_X^{\perp}\emptyset Y + \nabla_Y \emptyset X + h(Y, \emptyset X) + \emptyset[X,Y] \quad (3.16)$$

for all $X \in D$ and $Y \in D^{\perp}$.

Proof. Let $X \in D$ and $Y \in D^{\perp}$, then from Gauss formula (2.16), we have

$$\overline{\nabla}_Y \emptyset X = \nabla_Y \emptyset X + h(Y, \emptyset X).$$

From Weingarten formula (2.17), we have

$$\overline{\nabla}_X \emptyset Y = -A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset Y.$$

Now, from above two equations, we get

$$\overline{\nabla}_X \emptyset Y - \overline{\nabla}_Y \emptyset X = -A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset Y - \nabla_Y \emptyset X - h(Y, \emptyset X). \quad (3.17)$$

Comparing equation (3.17) and (3.8), we have

$$(\overline{\nabla}_X \emptyset) Y - (\overline{\nabla}_Y \emptyset) X + \emptyset[X, Y] = -A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset Y - \nabla_Y \emptyset X - h(Y, \emptyset X)$$
$$-\eta(X) Y.$$
$$(\overline{\nabla}_X \emptyset) Y - (\overline{\nabla}_Y \emptyset) X = -A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset Y - \nabla_Y \emptyset X - h(Y, \emptyset X) - \eta(X) Y$$
$$-\emptyset[X, Y]. \quad (3.18)$$

Adding (3.18) and (2.14), we have

$$2(\overline{\nabla}_X \emptyset) Y = 2g(X, Y)\xi - \eta(X) Y - \eta(Y) X - \eta(X) \emptyset Y - \eta(Y) \emptyset X - A_{\emptyset Y} X$$
$$+\nabla_X^{\perp} \emptyset Y - \nabla_Y \emptyset X - h(Y, \emptyset X) - \emptyset[X, Y].$$

Subtracting (3.18) from (2.14), we find

$$2(\overline{\nabla}_Y \emptyset) X = 2g(X, Y)\xi - \eta(X) Y - \eta(Y) X - \eta(X) \emptyset Y - \eta(Y) \emptyset X + A_{\emptyset Y} X$$
$$-\nabla_X^{\perp} \emptyset Y + \nabla_Y \emptyset X + h(Y, \emptyset X) + \emptyset[X, Y]$$

for all $X \in D$ and $Y \in D^{\perp}$.

Corollary 3.7. If M be a $\xi -$ horizontal CR-submanifold of a nearly hyperbolic Sasakian manifold $\overline{M}$ with semi symmetric metric connection. Then

$$2(\overline{\nabla}_X \emptyset) Y = 2g(X, Y)\xi - \eta(X) Y - \eta(X) \emptyset Y - A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset Y - \nabla_Y \emptyset X$$
$$-h(Y, \emptyset X) - \emptyset[X, Y],$$
$$2(\overline{\nabla}_Y \emptyset) X = 2g(X, Y)\xi - \eta(X) Y - \eta(X) \emptyset Y + A_{\emptyset Y} X - \nabla_X^{\perp} \emptyset Y + \nabla_Y \emptyset X$$
$$+h(Y, \emptyset X) + \emptyset[X, Y]$$

for all $X \in D \; and \; Y \in D^{\perp}$.

Corollary 3.8. If M be a $\xi -$ vertical CR-submanifold of a nearly hyperbolic Sasakian manifold $\overline{M}$ with semi symmetric metric connection. Then

$$2(\overline{\nabla}_X \emptyset) Y = 2g(X, Y)\xi - \eta(Y) X - \eta(Y) \emptyset X - A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset Y - \nabla_Y \emptyset X$$
$$-h(Y, \emptyset X) - \emptyset[X, Y],$$

$$2(\overline{\nabla}_Y\emptyset)X = 2g(X,Y)\xi - \eta(Y)X - \eta(Y)\emptyset X + A_{\emptyset Y}X - \nabla_X^{\perp}\emptyset Y + \nabla_Y\emptyset X$$
$$+h(Y,\emptyset X) + \emptyset[X,Y]$$

for all $X \in D$ and $Y \in D^{\perp}$.

4. Parallel Distributions

Definition 4.1. The horizontal (resp., vertical) distribution D(resp., $D^{\perp}$) is said to be parallel [7] with respect to the connection ∇ on M if $\nabla_X Y \in D$(resp., $\nabla_Z W \in D^{\perp}$) for any vector field $X, Y \in D$ (resp., $W, Z \in D^{\perp}$).

Theorem 4.2. Let M be a $\xi-$vertical CR-submanifold of a nearly hyperbolic Sasakian manifold $\overline{M}$ with semi symmetric metric connection. Then

$$h(X,\emptyset Y) = h(Y,\emptyset X) \tag{4.1}$$

for any $X, Y \in D$.

Proof. Using parallelism of horizontal distribution D, we have

$$\nabla_X\emptyset Y \in D \quad and\ \nabla_Y\emptyset X \in D, \tag{4.2}$$

for all $X, Y \in D$. From (3.2), we have

$$2g(X,Y)Q\xi - \eta(X)QY - \eta(Y)QX + 2Bh(X,Y) = Q\nabla_X(\emptyset PY)$$
$$+Q\nabla_Y(\emptyset PX) - QA_{\emptyset QY}X - QA_{\emptyset QX}Y.$$

As Q is a projection operator on $D^{\perp}$, so we have

$$g(X,Y)\xi + Bh(X,Y) = 0. \tag{4.3}$$

As we know from (2.12), we have

$$\emptyset h(X,Y) = -g(X,Y)\xi + Ch(X,Y). \tag{4.4}$$

Now, from (3.3) we have

$$-\eta(X)\emptyset QY - \eta(Y)\emptyset QX + \emptyset Q(\nabla_X Y) + \emptyset Q(\nabla_Y X) + 2Ch(X,Y)$$
$$= h(X,\emptyset PY) + h(Y,\emptyset PX) + \nabla_X^{\perp}(\emptyset QY) + \nabla_Y^{\perp}(\emptyset QX).$$

As Q is a projection operator on $D^{\perp}$, we have

$$h(X,\emptyset Y) + h(Y,\emptyset X) = 2Ch(X,Y).$$

Using equation (4.4) in above, we have

$$h(X,\emptyset Y) + h(Y,\emptyset X) = 2\emptyset h(X,Y) + 2g(X,Y)\xi. \tag{4.5}$$

Replacing Y by $\emptyset Y$ in (4.5), we have

$$h(X,\emptyset^2 Y) + h(\emptyset Y,\emptyset X) = 2\emptyset h(X,\emptyset Y) + 2g(X,\emptyset Y)\xi.$$

Using (2.1), we have

$$h(X,Y) + h(\emptyset Y, \emptyset X) = 2\emptyset h(X, \emptyset Y) + 2g(X, \emptyset Y)\xi. \tag{4.6}$$

Similarly, replacing $X \ by \ \emptyset X$ in (4.5) and using (2.1), we have

$$h(\emptyset X, \emptyset Y) + h(Y, X) = 2\emptyset h(\emptyset X, Y) + 2g(\emptyset X, Y)\xi. \tag{4.7}$$

Comparing (4.6) and (4.7), we have

$$2\emptyset h(X, \emptyset Y) + 2g(X, \emptyset Y)\xi = 2\emptyset h(\emptyset X, Y) + 2g(\emptyset X, Y)\xi.$$

Appling $\emptyset$ both side, we have

$$\emptyset^2 h(X, \emptyset Y) + g(X, \emptyset Y)\emptyset\xi = \emptyset^2 h(\emptyset X, Y) + g(\emptyset X, Y)\emptyset\xi.$$

Using equation (2.2) in above, we have

$$h(X, \emptyset Y) = h(\emptyset X, Y)$$

for all $X, Y \in D$.

Theorem 4.3. Let M be a $\xi - vertical$ CR-submanifold of a nearly hyperbolic Sasakian manifold $\bar{M}$ with semi symmetric metric connection. If the distribution $D^{\perp}$ is parallel with respect to the connection on M, then

$$A_{\emptyset X}Y + A_{\emptyset Y}X \in D^{\perp} \tag{4.8}$$

for all $X, Y \in D^{\perp}$.

Proof. Let $X, Y \in D^{\perp}$, then from Weingarten formula (2.17), we have

$$(\bar{\nabla}_X \emptyset)Y = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \emptyset(\bar{\nabla}_X Y).$$

Using Gauss equation (2.16) in above, we have

$$(\bar{\nabla}_X \emptyset)Y = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \emptyset(\nabla_X Y) - \emptyset h(X,Y). \tag{4.9}$$

Interchanging X and Y, we have

$$(\bar{\nabla}_Y \emptyset)X = -A_{\emptyset X}Y + \nabla_Y^{\perp}\emptyset X - \emptyset(\nabla_Y X) - \emptyset h(Y,X). \tag{4.10}$$

Adding (4.9) and (4.10), we get

$$(\bar{\nabla}_X \emptyset)Y + (\bar{\nabla}_Y \emptyset)X = -A_{\emptyset Y}X - A_{\emptyset X}Y + \nabla_X^{\perp}\emptyset Y + \nabla_Y^{\perp}\emptyset X - \emptyset(\nabla_X Y)$$
$$-\emptyset(\nabla_Y X) - 2\emptyset h(X,Y). \tag{4.11}$$

Using (2.14) in (4.11), we have

$$2g(X,Y)\xi - \eta(X)Y - \eta(Y)X - \eta(X)\emptyset Y - \eta(Y)\emptyset X = -A_{\emptyset Y}X - A_{\emptyset X}Y$$
$$+\nabla_X^{\perp}\emptyset Y + \nabla_Y^{\perp}\emptyset X - \emptyset(\nabla_X Y) - \emptyset(\nabla_Y X) - 2\emptyset h(X,Y). \tag{4.12}$$

Taking inner product with $Z \in D$ in (4.12), we have

$$2g(X,Y)g(\xi,Z)-\eta(X)g(Y,Z)-\eta(Y)g(X,Z)-\eta(X)g(\emptyset Y,Z)$$
$$-\eta(Y)g(\emptyset X,Z)=-g(A_{\emptyset Y}X,Z)-g(A_{\emptyset X}Y,Z)+g(\nabla_X^{\perp}\emptyset Y,Z)$$
$$+g(\nabla_Y^{\perp}\emptyset X,Z)-g(\emptyset(\nabla_X Y),Z-g(\emptyset(\nabla_Y X),Z)-2g(\emptyset h(X,Y),Z).$$

If $D^{\perp}$ is parallel then $\nabla_X Y\in D^{\perp}$ and $\nabla_Y X\in D^{\perp}$, so that from above equation,

$$0=-g(A_{\emptyset Y}X,Z)-g(A_{\emptyset X}Y,Z).$$
$$g(A_{\emptyset Y}X+A_{\emptyset X}Y,Z)=0. \tag{4.13}$$

Consequently, we have

$$A_{\emptyset Y}X+A_{\emptyset X}Y\in D^{\perp} \tag{4.14}$$

for all $X,Y\in D^{\perp}$.

Definition 4.4. A CR-submanifold is said to be mixed-totally geodesic if

$$h(X,Y)=0, \qquad \text{for all } X\in D \text{ and } Y\in D^{\perp}.$$

Definition 4.5. A normal vector field $N\neq 0$ is called $D-$parallel normal section if $\nabla_X^{\perp}N=0$ for all $X\in D$.

Theorem 4.6. Let M be a mixed totally geodesic $\xi-$vertical CR-submanifold of a nearly hyperbolic Sasakian manifold $\bar{M}$ with semi symmetric metric connection. Then the normal section $N\in\emptyset D^{\perp}$ is $D-$parallel if and only if $\nabla_X\emptyset N\in D$ for all $X\in D$.

Proof. Let $N\in\emptyset D^{\perp}$, for all $X\in D$ and $Y\in D^{\perp}$ then from (3.2), we have

$$2g(X,Y)Q\xi-\eta(X)QY-\eta(Y)QX+2Bh(X,Y)=Q\nabla_X(\emptyset PY)+Q\nabla_Y(\emptyset PX)-QA_{\emptyset QY}X-QA_{\emptyset QX}Y$$

As M is a $\xi-$vertical CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\bar{M}$ with semi symmetric metric connection, so we have from above equation

$$2Bh(X,Y)=Q\nabla_Y(\emptyset X)-QA_{\emptyset Y}X. \tag{4.15}$$

Using definition of mixed geodesic CR-submanifold, we have

$$Q\nabla_Y(\emptyset X)-QA_{\emptyset Y}X=0.$$
$$Q\nabla_Y\emptyset X=QA_{\emptyset Y}X. \tag{4.16}$$

From (3.3), we have

$$-\eta(X)\emptyset QY-\eta(Y)\emptyset QX+\emptyset Q(\nabla_X Y)+\emptyset Q(\nabla_Y X)+2Ch(X,Y)$$
$$=h(X,\emptyset PY)h(Y,\emptyset PX)+\nabla_X^{\perp}(\emptyset QY)+\nabla_Y^{\perp}(\emptyset QX). \tag{4.17}$$

Using (4.16) in (4.17), we have

$$\emptyset Q\nabla_X(\emptyset N) = \nabla^{\perp}_X N. \qquad (4.18)$$

Then by definition of parallelism of $N,$ we have

$$\emptyset Q\nabla_X(\emptyset N) = 0.$$

Consequently, we have

$$\nabla_X(\emptyset N) \in D \qquad (4.19)$$

for all $X \in D$.

Converse part is a easy consequence of (4.19).

Conflict of Interests

The authors declare that there is no conflict of interests.

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