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THE PRODUCTS OF SOFT QUASI-UNIFORMITIES AND SOFT TOPOLOGIES

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Abstract. In this paper, we investigate the relations among soft topology, soft closure operators and soft quasi-uniformities in complete residuated lattices. We give their examples.

Keywords: Complete residuated lattices; Soft quasi-uniformities; Soft closure operators; Soft topologies.

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1. Introduction

Hájek [6] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool for algebraic structure of fuzzy contexts [7,11-16,26]. Many researcher introduced the notion of fuzzy uniformities in unit interval [0,1] [3,17], complete distributive lattices [8,32]. Recently, Molodtsov [23] introduced the soft set as a mathematical tool for dealing information as the uncertainty of data in engineering, physics, computer sciences and many other diverse field. Presently, the soft set theory is making progress rapidly [1,5,11-15, 19,23, 30,31]. Pawlak's rough set [24,25] can be viewed as a special case of soft rough sets [5]. The topological structures of soft sets have been developed by many researchers [4,11-15,27,28].

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Kim [15] introduced a fuzzy soft $F : A \to L^U$ as an extension as the soft $F : A \to P(U)$ where L is a complete residuated lattice. Kim [11-15] introduced the soft topological structures, fuzzy quasi-uniformities and soft closure operators in complete residuated lattices.

In this paper, we investigate the relations among soft topology, soft closure operators and soft quasi-uniformities in complete residuated lattices. We give their examples.

2. Preliminaries

Definition 2.1. [2,6.7,26] An algebra $(L, \wedge, \vee, \odot, \to, 0, 1)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a complete lattice with the greatest element 1 and the least element 0;

(C2) $(L, \odot, 1)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \to z$ for $x, y, z \in L$.

In this paper, we assume that $(L, \leq, \odot, \to)$ is a complete residuated lattice and we denote $L_0 = L - \{0\}$.

Lemma 2.2. [2,6.7,26] For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.

(1) $1 \to x = x$, $0 \odot x = 0$,

(2) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$,

(3) $x \odot y \leq x \wedge y \leq x \vee y$,

(4) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$,

(5) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$,

(6) $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$,

(7) $x \to (\bigvee_i y_i) \geq \bigvee_i (x \to y_i)$,

(8) $(\bigwedge_i x_i) \to y \geq \bigvee_i (x_i \to y)$,

(9) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,

(10) $x \odot (x \to y) \leq y$ and $x \to y \leq (y \to z) \to (x \to z)$,

(11) $(x \to y) \odot (z \to w) \leq (x \odot z) \to (y \odot w)$,

(12) $x \to y \leq (x \odot z) \to (y \odot z)$ and $(x \to y) \odot (y \to z) \leq x \to z$.

Definition 2.3. [15] Let X be an initial universe of objects and E the set of parameters (attributes) in X. A pair (F,A) is called a *fuzzy soft set* over X, where $A \subset E$ and $F : A \to L^X$ is a mapping. We denote $S(X,A)$ as the family of all fuzzy soft sets under the parameter A.

Definition 2.4.[15] Let (F,A) and (G,A) be two fuzzy soft sets over a common universe X.

(1) (F,A) is a fuzzy soft subset of (G,A), denoted by $(F,A) \leq (G,A)$ if $F(a) \leq G(a)$, for each $a \in A$.

(2) $(F,A) \wedge (G,A) = (F \wedge G,A)$ if $(F \wedge G)(a) = F(a) \wedge G(a)$ for each $a \in A$.

(3) $(F,A) \vee (G,A) = (F \vee G,A)$ if $(F \vee G)(a) = F(a) \vee G(a)$ for each $a \in A$.

(4) $(F,A) \odot (G,A) = (F \odot G,A)$ if $(F \odot G)(a) = F(a) \odot G(a)$ for each $a \in A$.

(6) $\alpha \odot (F,A) = (\alpha \odot F,A)$ for each $\alpha \in L$.

Definition 2.5. [12] A map $\tau \subset S(X,A)$ is called a soft topology on X if it satisfies the following conditions.

(ST1) $(0_X,A),(1_X,A) \in \tau$, where $0_X(a)(x) = 0, 1_X(a)(x) = 1$ for all $a \in A, x \in X$,

(ST2) If $(F,A),(G,A) \in \tau$, then $(F,A) \odot (G,A) \in \tau$,

(T) If $(F_i,A) \in \tau$ for each $i \in I$, $\bigvee_{i \in I}(F_i,A) \in \tau$.

A map $\tau \subset S(X,A)$ is called a soft cotopology on X if it satisfies (ST1), (ST2) and

(CT) If $(F_i,A) \in \tau$ for each $i \in I$, $\bigwedge_{i \in I}(F_i,A) \in \tau$.

The triple (X,A,τ) is called a soft topological (resp. cotopological) space.

Let (X,A,τ_1) and (X,A,τ_2) be soft fuzzy topological spaces. Then τ_1 is finer than τ_2 if $(F,A) \in \tau_1$, for all $(F,A) \in \tau_2$.

Definition 2.6. [13] A subset $\mathbf{U} \subset S(X \times X,A)$ is called a soft quasi-uniformity on X iff it satisfies the properties.

(SU1) $(1_{X \times X},A) \in \mathbf{U}$.

(SU2) If $(V,A) \leq (U,A)$ and $(V,A) \in \mathbf{U}$, then $(U,A) \in \mathbf{U}$.

(SU3) For every $(U,A),(V,A) \in \mathbf{U}$, $(U,A) \odot (V,A) \in \mathbf{U}$.

(SU4) If $(U,A) \in \mathbf{U}$ then $(1_\triangle,A) \leq (U,A)$ where

$$1_\triangle(a)(x,y) = \begin{cases} 1, & \text{if x = y} \\ \perp, & \text{if } x \neq y, \end{cases}$$

(SU5) For every $(U,A) \in \mathbf{U}$, there exists $(V,A) \in \mathbf{U}$ such that $(V,A) \circ (V,A) \leq (U,A)$ where

$$((V,A) \circ (V,A))(a)(x,y) = (V(a) \circ V(a))(x,y)$$
$$= \bigvee_{z \in X} (V(a)(z,x) \odot V(a)(x,y)), \ \forall\, x,y \in X, a \in A.$$

The triple $(X,A,\mathbf{U})$ is called a soft quasi-uniform space.

A soft quasi-uniformity $\mathbf{U}$ on X is said to be a soft uniformity if

(U) if $(U,A) \in \mathbf{U}$, then $(U^{-1},A) \in \mathbf{U}$ where $U^{-1}(a)(x,y) = U(a)(y,x)$.

Definition 2.7. [8] A mapping $cl : S(X,A) \to S(X,A)$ is called a soft closure operator if it satisfies the following conditions;

(C1) $cl(0_X,A) = (0_X,A)$,

(C2) $cl(F,A) \geq (F,A)$,

(C3) If $(F,A) \leq (G,A)$, then $cl(F,A) \leq cl(G,A)$,

(C4) $cl(cl(F,A)) = (F,A)$,

(C5) $cl((F,A) \odot (G,A)) \leq cl(F,A) \odot cl(G,A)$.

The pair (X,A,cl) is called a soft closure space.

Theorem 2.8. [14] Let $(X,A,\mathbf{U})$ be a soft quasi-uniform space. Define $cl^r_{\mathbf{U}}, cl^l_{\mathbf{U}} : S(X,A) \to S(X,A)$ as follows

$$cl^r_{\mathbf{U}}(F,A)(y) = \bigwedge_{(U,A) \in \mathbf{U}} \Big(\bigvee_{x \in X} (U,A)(y,x) \odot (F,A)(x) \Big),$$

$$cl^l_{\mathbf{U}}(F,A)(y) = \bigwedge_{(U,A) \in \mathbf{U}} \Big(\bigvee_{x \in X} (U,A)(x,y) \odot (F,A)(x) \Big).$$

Then, for $cl \in \{cl^r_{\mathbf{U}}, cl^l_{\mathbf{U}}\}$, we have following properties.

(1) $cl(0_X,A) = (0_X,A)$ and $cl(F,A) \leq cl(G,A)$ for $(F,A) \leq (G,A)$.

(2) $(F,A) \leq cl(F,A)$.

(3) $cl(cl(F,A)) = cl(F,A)$.

(4) If L satisfies $a \odot \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \odot b_i)$, then $cl((F,A) \odot (G,A)) \leq cl(F,A) \odot cl(G,A)$.

Remark 2.9. If $(L,\odot)$ is a continuous t-norm, then $a\odot\bigwedge_{i\in I}b_i=\bigwedge_{i\in I}(a\odot b_i)$.

Theorem 2.10. [13] Let $(X,A,\mathbf{U})$ be a soft quasi-uniform space and $a\odot\bigwedge_{i\in I}b_i=\bigwedge_{i\in I}(a\odot b_i)$ for $a,b_i\in L$. Define $\tau^r_{\mathbf{U}},\tau^l_{\mathbf{U}}\subset S(X,A)$ as follows

$$\tau^r_{\mathbf{U}}=\{(F,A)\in S(X,A)\mid cl^r_{\mathbf{U}}(F,A)=(F,A)\},$$

$$\tau^l_{\mathbf{U}}=\{(F,A)\in S(X,A)\mid cl^l_{\mathbf{U}}(F,A)=(F,A)\}.$$

Then (1) $\tau^r_{\mathbf{U}}$ is a soft topology on X such that $\tau^r_{\mathbf{U}}=\{cl^r_{\mathbf{U}}(F,A)\mid(F,A)\in S(X,A)\}$.
(2) $\tau^l_{\mathbf{U}}$ is a soft topology on X such that $\tau^l_{\mathbf{U}}=\{cl^l_{\mathbf{U}}(F,A)\mid(F,A)\in S(X,A)\}$.

Lemma 2.11. [13] For every $(F,A),(G,A)\in S(X,A)$, we define $(U_F,A)\in S(X\times X,A)$ by

$$U_F(a)(x,y)=F(a)(x)\to F(a)(y).$$

then we have the following statements
(1) $(1_{X\times X},A)=(U_{0_X},A)=(U_{1_X},A)$,
(2) $(1_{\triangle},A)\le(U_F,A)$,
(3) $(U_F,A)\circ(U_F,A)=(U_F,A)$,
(4) $(U_F,A)\odot(U_G,A)\le(U_{F\odot G},A)$.

Theorem 2.12. [13] Let (X,A,τ) be a soft topological space. Define a function $\mathbf{U}_\tau:S(X\times X,A)\to L$ by

$$\mathbf{U}_\tau=\{(U,A)\in S(X\times X,A)\mid\odot_{i=1}^n(U_{G_i},A)\le(U,A),(G_i,A)\in\tau\}$$

where the first $\bigvee$ is taken over every finite family $\{U_{(G_i,A)}\mid i=1,...,n\}$. Then
(1) $\mathbf{U}_\tau$ is a soft quasi-uniformity on X.
(2) $\tau\subset\tau^l_{\mathbf{U}_\tau}$.

3. The products of soft uniformities and soft topologies

Theorem 3.1. Let $\mathbf{U_1}$ and $\mathbf{U_2}$ be soft quasi-uniformities on X. We define

$$\mathbf{U_1} \oplus \mathbf{U_2} = \{(U,A) \in S(X \times X, A) \mid (U_1,A) \odot (U_2,A) \leq (U,A),\ (U_1,A) \in \mathbf{U_1}, (U_2,A) \in \mathbf{U_2}\}.$$

Then we have the following properties.

(1) $\mathbf{U_1} \oplus \mathbf{U_2}$ is the coarsest quasi-uniformity on X which is finer than $\mathbf{U_1}$ and $\mathbf{U_2}$.

(2) If $a \odot \bigwedge_{i\in I} b_i = \bigwedge_{i\in I}(a \odot b_i)$ for $a, b_i \in L$, then

$$cl^r_{\mathbf{U_1}}(F,A) \odot cl^r_{\mathbf{U_2}}(G,A) = cl^r_{\mathbf{U_1}\oplus\mathbf{U_2}}((F,A) \odot (G,A)).$$

(3) If $a \odot \bigwedge_{i\in I} b_i = \bigwedge_{i\in I}(a \odot b_i)$ for $a, b_i \in L$, then $\tau^r_{\mathbf{U_1}} \oplus \tau^r_{\mathbf{U_2}} = \tau^r_{\mathbf{U_1}\oplus\mathbf{U_2}}$ where

$$\tau^r_{\mathbf{U_1}} \oplus \tau^r_{\mathbf{U_2}} = \{(F,A) = (F_1,A) \odot (F_2,A) \mid (F_i,A) \in \tau^r_{\mathbf{U}_i}, i = 1,2\}.$$

(4) If $a \odot \bigwedge_{i\in I} b_i = \bigwedge_{i\in I}(a \odot b_i)$ for $a, b_i \in L$, then $\tau^l_{\mathbf{U_1}} \oplus \tau^l_{\mathbf{U_2}} = \tau^l_{\mathbf{U_1}\oplus\mathbf{U_2}}$ where

$$\tau^l_{\mathbf{U_1}} \oplus \tau^l_{\mathbf{U_2}} = \{(F,A) = (F_1,A) \odot (F_2,A) \mid (F_i,A) \in \tau^l_{\mathbf{U}_i}, i = 1,2\}.$$

(5) If (X, A, τ_1) and (X, A, τ_2) are soft fuzzy topological spaces, then $\mathbf{U}_{\tau_1\oplus\tau_2} \subset \mathbf{U}_{\tau_1} \oplus \mathbf{U}_{\tau_2}$.

Proof. (1) (SU1) $(1_{X\times X},A) \in \mathbf{U_1} \oplus \mathbf{U_2}$ because $(1_{X\times X},A) \odot (1_{X\times X},A) = (1_{X\times X},A)$ for $(1_{X\times X},A) \in U_i, i = 1,2$.

(SU2) If $(V,A) \leq (U,A)$ and $(V,A) \in \mathbf{U_1} \oplus \mathbf{U_2}$, then there exist $(V_i,A) \in \mathbf{U_i}, i = 1,2$, with $(V_1,A) \odot (V_2,A) \leq (V,A) \leq (U,A)$. Thus $(U,A) \in \mathbf{U_1} \oplus \mathbf{U_2}$.

(SU3) For every $(U,A),(V,A) \in \mathbf{U_1} \oplus \mathbf{U_2}$, then there exist $(U_i,A),(V_i,A) \in \mathbf{U_i}, i = 1,2$, with $(U_1,A) \odot (U_2,A) \leq (U,A)$ and $(V_1,A) \odot (V_2,A) \leq (V,A)$. Thus $(U_1,A) \odot (U_2,A) \odot (V_1,A) \odot (V_2,A) \leq (U,A) \odot (V,A)$. Hence $(U,A) \odot (V,A) \in \mathbf{U_1} \oplus \mathbf{U_2}$.

(SU4) If $(U,A) \in \mathbf{U_1} \oplus \mathbf{U_2}$, then there exist $(U_i,A) \in \mathbf{U_i}, i = 1,2$, with $(U_1,A) \odot (U_2,A) \leq (U,A)$. Since $(U_i,A) \in \mathbf{U_i}, i = 1,2$, by (SU4), $(1_\triangle,A) \leq (U_i,A), i = 1,2$. Hence $(1_\triangle,A) \leq (U,A)$.

(SU5) For each $(U,A) \in \mathbf{U_1} \oplus \mathbf{U_2}$, there exist $(U_1,A) \in \mathbf{U_1}$ and $(U_2,A) \in \mathbf{U_2}$ such that $(U_1,A) \odot (U_2,A) \leq (U,A)$. For each $(U_i,A) \in \mathbf{U_i}, i = 1,2$, there exists $(V_i,A) \in \mathbf{U_i}$ such that $(V_i,A) \circ (V_i,A) \leq (U_i,A)$.

$$\begin{aligned}
&(((V_1,A)\odot(V_2,A))\circ((V_1,A)\odot(V_2,A)))(a)(x,y)\\
&=(V_1(a)\odot V_2(a))\circ(V_1(a)\odot V_2(a))(x,y)\\
&=\bigvee_{z\in X}((V_1(a)\odot V_2(a))(x,z)\odot(V_1(a)\odot V_2(a))(z,y))\\
&=\bigvee_{z\in X}((V_1(a)(x,z)\odot V_1(a)(z,y))\odot(V_2(a)(x,z)\odot V_2(a)(z,y)))\\
&\le\bigvee_{z\in X}(V_1(a)(x,z)\odot V_1(a)(z,y))\odot\bigvee_{w\in X}(V_2(a)(x,w)\odot V_2(a)(w,y))\\
&=((V_1,A)\circ(V_1,A))(a)(x,y)\odot((V_2,A)\circ(V_2,A))(a)(x,y)\\
&=(U_1,A)(a)(x,y)\odot(U_2,A)(a)(x,y)\le(U,A)(a)(x,y).
\end{aligned}$$

Thus, there exists $(V_1,A)\odot(V_2,A)\in\mathbf{U_1}\oplus\mathbf{U_2}$ such that $((V_1,A)\odot(V_2,A))\circ((V_1,A)\odot(V_2,A))\le(U,A)$.

If $(U_1,A)\in\mathbf{U_1}$, then $(U_1,A)\odot(1_{X\times X},A)=(U_1,A)$ such that $(U_1,A)\in\mathbf{U_1},(1_{X\times X},A)\in\mathbf{U_2}$. Hence $(U_1,A)\in\mathbf{U_1}\oplus\mathbf{U_2}$;i.e. $\mathbf{U_1}\subset\mathbf{U_1}\oplus\mathbf{U_2}$. Similarly, $\mathbf{U_2}\subset\mathbf{U_1}\oplus\mathbf{U_2}$. If $\mathbf{U_i}\subset\mathbf{V}$ and $\mathbf{V}$ is a soft quasi-uniformity, for $(U,A)\in\mathbf{U_1}\oplus\mathbf{U_2}$, there exists $(U_i,A)\in\mathbf{U_i}$ such that $(U_1,A)\odot(U_2,A)\le(U,A)$. Since $(U_i,A)\in\mathbf{V}$, then $(U_1,A)\odot(U_2,A)\in\mathbf{V}$. Hence $(U,A)\in\mathbf{V}$. So, $\mathbf{U_1}\oplus\mathbf{U_2}\subset\mathbf{V}$.

(2)

$$\begin{aligned}
&cl^r_{\mathbf{U_1}\oplus\mathbf{U_2}}((F,A)\odot(G,A))(y)\\
&=\bigwedge_{U\in\mathbf{U_1}\oplus\mathbf{U_2}}(\bigvee_{x\in X}(U,A)(y,x)\odot(F,A)(x)\odot(G,A)(x))\\
&\ge\bigwedge_{U_1\odot U_2\in\mathbf{U_1}\oplus\mathbf{U_2}}(\bigvee_{x\in X}(U_1,A)(y,x)\odot(U_2,A)(y,x)\odot(F,A)(x)\odot(G,A)(x))\\
&=\bigwedge_{U_1\in\mathbf{U_1},U_2\in\mathbf{U_2}}(\bigvee_{x\in X}(U_1,A)(y,x)\odot(U_2,A)(y,x)\odot(F,A)(x)\odot(G,A)(x))\\
&=\bigwedge_{U_1\in\mathbf{U_1}}(\bigvee_{x\in X}(U_1,A)(y,x)\odot(F,A)(x))\\
&\odot\bigwedge_{U_2\in\mathbf{U_2}}(\bigvee_{x\in X}(U_2,A)(y,x)\odot(G,A)(x))\\
&=cl^r_{\mathbf{U_1}}(F,A)(y)\odot cl^r_{\mathbf{U_2}}(G,A)(y).
\end{aligned}$$

Suppose there exist $(F,A)\in\mathbf{U}_1,(G,A)\in\mathbf{U}_2$ and $y\in X$ such that

$$cl^r_{\mathbf{U_1}}(F,A)(y)\odot cl^r_{\mathbf{U_2}}(G,A)(y)\not\ge cl^r_{\mathbf{U_1}\oplus\mathbf{U_2}}((F,A)\odot(G,A))(y).$$

Then there exist $U_1\in\mathbf{U}_1,U_2\in\mathbf{U}_2$ such that

$$\bigvee_{x\in X}(U_1(y,x)\odot(F,A)(x))\odot\bigvee_{z\in X}(U_2(y,z)\odot(G,A)(z))\not\ge cl^r_{\mathbf{U_1}\oplus\mathbf{U_2}}((F,A)\odot(G,A))(y).$$

It follows

$$\bigvee_{x\in X}((U_1\odot U_2)(y,x)\odot((F,A)\odot(G,A))(x))\not\geq cl^r_{\mathbf{U_1}\oplus\mathbf{U_2}}((F,A)\odot(G,A))(y).$$

It is a contradiction. Hence $cl^r_{\mathbf{U_1}}(F,A)\odot cl^r_{\mathbf{U_2}}(G,A)\geq cl^r_{\mathbf{U_1}\oplus\mathbf{U_2}}((F,A)\odot(G,A))$. Thus the result holds.

(3)

$$\begin{aligned}&(F,A)\in\tau^r_{\mathbf{U_1}}\oplus\tau^r_{\mathbf{U_2}}\\ &\text{iff } (F,A)=(F_1,A)\odot(F_2,A)=cl^r_{\mathbf{U_1}}(F_1,A)\odot cl^r_{\mathbf{U_2}}(F_2,A)\\ &\text{iff } (F,A)=(F_1,A)\odot(F_2,A)=cl^r_{\mathbf{U_1}\oplus\mathbf{U_2}}((F_1,A)\odot(F_2,A))\\ &\text{iff } (F,A)\in\tau^r_{\mathbf{U_1}\oplus\mathbf{U_2}}.\end{aligned}$$

(4) It is similarly proved as (3).

(5) Let $(U,A)\in\mathbf{U}_{\tau_1\oplus\tau_2}$. Then there exist $(F_i,A)\in\tau_i$ such that $\odot_{j=1}^n(U_{F_{j1}\odot F_{j2}},A)\leq(U,A)$. Since $(U_{F_{j1}},A)\odot(U_{F_{j2}},A)\leq(U_{F_{j1}\odot F_{j2}},A)$ from Lemma 2.11(4), we have

$$\odot_{j=1}^n(U_{F_{j1}},A)\odot(\odot_{j=1}^n(U_{F_{j2}},A))\leq\odot_{j=1}^n(U_{F_{j1}\odot F_{j2}},A)\leq(U,A).$$

Since $\odot_{j=1}^n(U_{F_{j1}},A)\in\mathbf{U}_{\tau_1}$, $\odot_{j=1}^n(U_{F_{j2}},A)\in\mathbf{U}_{\tau_2}$, we have $(U,A)\in\mathbf{U}_{\tau_1}\oplus\mathbf{U}_{\tau_2}$.

Theorem 3.2. Let $\mathbf{U}$ be a soft quasi-uniformities on X. We define

$$\mathbf{U^{-1}}=\{(U,A)\in S(X\times X,A)\mid(U^{-1},A)\in\mathbf{U}\}.$$

$$\mathbf{U}\oplus\mathbf{U^{-1}}=\{(U,A)\in S(X\times X,A)\mid(U_1,A)\odot(U_2,A)\leq(U,A),\ (U_1,A)\in\mathbf{U},(U_2,A)\in\mathbf{U^{-1}}\}.$$

Then we have the following properties.

(1) $\mathbf{U^{-1}}$ a soft quasi-uniformities on X.

(2) $\mathbf{U}\oplus\mathbf{U^{-1}}$ is the coarsest uniformity on X which is finer than $\mathbf{U}$ and $\mathbf{U^{-1}}$.

(3) If $a\odot\bigwedge_{i\in I}b_i=\bigwedge_{i\in I}(a\odot b_i)$ for $a,b_i\in L$, then

$$cl^r_{\mathbf{U}}(F,A)=cl^l_{\mathbf{U^{-1}}}(F,A),cl^l_{\mathbf{U}}(F,A)=cl^r_{\mathbf{U^{-1}}}(F,A),$$

$$cl^r_{\mathbf{U}}(F,A)\odot cl^r_{\mathbf{U^{-1}}}(G,A)=cl^r_{\mathbf{U}\oplus\mathbf{U^{-1}}}((F,A)\odot(G,A)).$$

(4) If $a\odot\bigwedge_{i\in I}b_i=\bigwedge_{i\in I}(a\odot b_i)$ for $a,b_i\in L$, then $\tau^r_{\mathbf{U}}=\tau^l_{\mathbf{U^{-1}}}$, $\tau^l_{\mathbf{U}}=\tau^r_{\mathbf{U^{-1}}}$, $\tau^r_{\mathbf{U}}\oplus\tau^r_{\mathbf{U^{-1}}}=\tau^r_{\mathbf{U}\oplus\mathbf{U^{-1}}}=\tau^l_{\mathbf{U}\oplus\mathbf{U^{-1}}}$ where

$$\tau^r_{\mathbf{U}}\oplus\tau^r_{\mathbf{U^{-1}}}=\{(F,A)=(F_1,A)\odot(F_2,A)\mid(F_1,A)\in\tau^r_{\mathbf{U}},(F_2,A)\in\tau^r_{\mathbf{U}^{-1}}\}=\tau^l_{\mathbf{U}}\oplus\tau^l_{\mathbf{U^{-1}}}.$$

Proof. (1) (SU5) For $(U,A)\in\mathbf{U}^{-1}$ iff $(U^{-1},A)\in\mathbf{U}$, there exists $(V,A)\in\mathbf{U}$ such that $(V,A)\circ(V,A)\leq(U^{-1},A)$ iff $(V^{-1},A)\circ(V^{-1},A)\leq(U,A)$. Other cases are easily proved.

(2) $\mathbf{U}\oplus\mathbf{U^{-1}}$ is the coarsest uniformity on X from Theorem 3.1(1) and

$$\begin{aligned}
&(U,A)\in\mathbf{U}\oplus\mathbf{U^{-1}}\\
&\text{iff } (U,A)\geq(U_1,A)\odot(U_2,A),\ (U_1,A)\in\mathbf{U},(U_2,A)\in\mathbf{U}^{-1}\\
&\text{iff } (U^{-1},A)\geq(U_1^{-1},A)\odot(U_2^{-1},A),\ (U_1^{-1},A)\in\mathbf{U}^{-1},(U_2^{-1},A)\in\mathbf{U}\\
&\text{iff } (U^{-1},A)\in\mathbf{U}\oplus\mathbf{U^{-1}}
\end{aligned}$$

(3) It follows from Theorem 3.1(2) and the definition of $cl^r_{\mathbf{U}}$.

(4) By (3), we have $\tau^r_{\mathbf{U}}=\tau^l_{\mathbf{U^{-1}}}$, $\tau^l_{\mathbf{U}}=\tau^r_{\mathbf{U^{-1}}}$ and

$$\begin{aligned}
\tau^r_{\mathbf{U}}\oplus\tau^r_{\mathbf{U^{-1}}}&=\{(F,A)=(F_1,A)\odot(F_2,A)\mid(F_1,A)\in\tau^r_{\mathbf{U}},(F_2,A)\in\tau^r_{\mathbf{U}^{-1}}\}\\
&=\{(F,A)=(F_1,A)\odot(F_2,A)\mid(F_1,A)\in\tau^l_{\mathbf{U}^{-1}},(F_2,A)\in\tau^l_{\mathbf{U}}\}\\
&=\tau^l_{\mathbf{U}}\oplus\tau^l_{\mathbf{U^{-1}}}.
\end{aligned}$$

Example 3.3. Let $X=\{h_i\mid i=\{1,...,4\}\}$ with h_i=house and $E_Y=\{e,b,w,c,i\}$ with e=expensive,b= beautiful, w=wooden, c= creative, i=in the green surroundings.

Let $(L=[0,1],\odot,\rightarrow)$ be a complete residuated lattice defined by

$$x\odot y=x\wedge y,\ \ x\rightarrow y=\begin{cases}1, & \text{if } x\leq y,\\ y, & \text{otherwise.}\end{cases}$$

Let $X=\{x,y,z\}$ be a set and $W_i(e),W_i(b)\in S(X\times X,A)$ such that

$$W_1(e)=\begin{pmatrix}1 & 0.5 & 0.5\\ 0.7 & 1 & 0.8\\ 0.4 & 0.4 & 1\end{pmatrix}W_1(b)=\begin{pmatrix}1 & 0.6 & 0.7\\ 0.4 & 1 & 0.4\\ 0.5 & 0.6 & 1\end{pmatrix}$$

$$W_2(e)=\begin{pmatrix}1&0.4&0.3\\0.4&1&0.3\\0.6&0.5&1\end{pmatrix}W_2(b)=\begin{pmatrix}1&0.3&0.3\\0.6&1&0.7\\0.5&0.4&1\end{pmatrix}$$

$$(W_1\wedge W_2)(e)=\begin{pmatrix}1&0.4&0.3\\0.4&1&0.3\\0.4&0.4&1\end{pmatrix}(W_1\wedge W_2)(b)=\begin{pmatrix}1&0.3&0.3\\0.4&1&0.4\\0.5&0.4&1\end{pmatrix}$$

Define $\mathbf{U}_i=\{(U,A)\in S(X\times X,A)\mid (U,A)\geq (W_i,A)\}$ for $i=1,2$.

(1) Since $W_i(e)\circ W_i(e)=W_i(e)$ and $W_i(b)\circ W_i(b)=W_i(b)$, $\mathbf{U}_i$ is a soft quasi-uniformity on X.

(2) From Theorem 2.10(1), we obtain $\tau^r_{\mathbf{U}_1}=\{cl^r_{\mathbf{U}_1}(F,A)\mid (F,A)\in L^X\}$ where

$$cl^r_{\mathbf{U}_1}(F,A)(e)=\begin{pmatrix}F(e)(x)\vee(0.5\wedge F(e)(y))\vee(0.5\wedge F(e)(z))\\(0.7\wedge F(e)(x))\vee F(e)(y)\vee(0.8\wedge F(e)(z))\\(0.4\wedge F(e)(x))\vee(0.4\wedge F(e)(y))\vee F(e)(z)\end{pmatrix}$$

$$cl^r_{\mathbf{U}_1}(F,A)(b)=\begin{pmatrix}F(b)(x)\vee(0.6\wedge F(b)(y))\vee(0.7\wedge F(b)(z))\\(0.4\wedge F(b)(x))\vee F(b)(y)\vee(0.4\wedge F(b)(z))\\(0.5\wedge F(b)(x))\vee(0.6\wedge F(b)(y))\vee F(b)(z)\end{pmatrix}$$

Also, we have $\tau^l_{\mathbf{U}_2}=\{cl^l_{\mathbf{U}_2}(F,A)\mid (F,A)\in L^X\}$ where

$$cl^l_{\mathbf{U}_2}(F,A)(e)=\begin{pmatrix}F(e)(x)\vee(0.4\wedge F(e)(y))\vee(0.3\wedge F(e)(z))\\(0.4\wedge F(e)(x))\vee F(e)(y)\vee(0.3\wedge F(e)(z))\\(0.6\wedge F(e)(x))\vee(0.5\wedge F(e)(y))\vee F(e)(z)\end{pmatrix}$$

$$cl^l_{\mathbf{U}_2}(F,A)(b)=\begin{pmatrix}F(b)(x)\vee(0.3\wedge F(b)(y))\vee(0.3\wedge F(b)(z))\\(0.6\wedge F(b)(x))\vee F(b)(y)\vee(0.7\wedge F(b)(z))\\(0.5\wedge F(b)(x))\vee(0.4\wedge F(b)(y))\vee F(b)(z)\end{pmatrix}$$

(3) From Theorem 3.3(3), we obtain $\tau^r_{\mathbf{U_1}}\oplus\tau^r_{\mathbf{U_2}}=\tau^r_{\mathbf{U_1}\oplus\mathbf{U_2}}=\{cl^r_{\mathbf{U_1}\oplus\mathbf{U_2}}(F,A)\mid (F,A)\in L^X\}$ as follows:

$$cl^r_{\mathbf{U_1}\oplus\mathbf{U_2}}(F,A)(e)=\begin{pmatrix} F(e)(x)\vee(0.4\wedge F(e)(y))\vee(0.3\wedge F(e)(z)) \\ (0.4\wedge F(e)(x))\vee F(e)(y)\vee(0.3\wedge F(e)(z)) \\ (0.4\wedge F(e)(x))\vee(0.4\wedge F(e)(y))\vee F(e)(z) \end{pmatrix}$$

$$cl^r_{\mathbf{U_1}\oplus\mathbf{U_2}}(F,A)(b)=\begin{pmatrix} F(b)(x)\vee(0.3\wedge F(b)(y))\vee(0.3\wedge F(b)(z)) \\ (0.4\wedge F(b)(x))\vee F(b)(y)\vee(0.4\wedge F(b)(z)) \\ (0.5\wedge F(b)(x))\vee(0.4\wedge F(b)(y))\vee F(b)(z) \end{pmatrix}$$

Similarly, we obtain $\tau^l_{\mathbf{U_1}}\oplus\tau^l_{\mathbf{U_2}}=\tau^l_{\mathbf{U_1}\oplus\mathbf{U_2}}=\{cl^l_{\mathbf{U_1}\oplus\mathbf{U_2}}(F,A)\mid(F,A)\in L^X\}$ as follows:

$$cl^l_{\mathbf{U_1}\oplus\mathbf{U_2}}(F,A)(e)=\begin{pmatrix} F(e)(x)\vee(0.4\wedge F(e)(y))\vee(0.4\wedge F(e)(z)) \\ (0.4\wedge F(e)(x))\vee F(e)(y)\vee(0.4\wedge F(e)(z)) \\ (0.3\wedge F(e)(x))\vee(0.3\wedge F(e)(y))\vee F(e)(z) \end{pmatrix}$$

$$cl^l_{\mathbf{U_1}\oplus\mathbf{U_2}}(F,A)(b)=\begin{pmatrix} F(b)(x)\vee(0.4\wedge F(b)(y))\vee(0.5\wedge F(b)(z)) \\ (0.3\wedge F(b)(x))\vee F(b)(y)\vee(0.4\wedge F(b)(z)) \\ (0.3\wedge F(b)(x))\vee(0.4\wedge F(b)(y))\vee F(b)(z) \end{pmatrix}$$

(4) We obtain a soft quasi-uniformity $\mathbf{U_1^{-1}}=\{(U,A)\in S(X\times X,A)\mid(U,A)\geq(W_1^{-1},A)\}$ where

$$W_1^{-1}(e)=\begin{pmatrix} 1 & 0.7 & 0.4 \\ 0.5 & 1 & 0.4 \\ 0.5 & 0.8 & 1 \end{pmatrix} W_1^{-1}(b)=\begin{pmatrix} 1 & 0.4 & 0.5 \\ 0.6 & 1 & 0.6 \\ 0.7 & 0.4 & 1 \end{pmatrix}$$

From Theorem 3.2 (2), we obtain a soft uniformity $\mathbf{U_1}\oplus\mathbf{U_1^{-1}}=\{(U,A)\in S(X\times X,A)\mid(U,A)\geq(W\wedge W_1^{-1},A)\}$ where

$$W\wedge W_1^{-1}(e)=\begin{pmatrix} 1 & 0.5 & 0.4 \\ 0.5 & 1 & 0.4 \\ 0.4 & 0.4 & 1 \end{pmatrix} W\wedge W_1^{-1}(b)=\begin{pmatrix} 1 & 0.4 & 0.5 \\ 0.4 & 1 & 0.4 \\ 0.5 & 0.4 & 1 \end{pmatrix}$$

(5) Let $\tau_1=\{(0_X,A),(1_X,A),(F_1,A)\}$ and $\tau_2=\{(0_X,A),(1_X,A),(F_2,A)\}$ where

$$F_1(e)=(0.4,0.5.0.6),\;\; F_1(b)=(0.7,0.4.0.9),$$

$$F_2(e)=(0.5,0.1.0.3),\;\; F_2(b)=(0.6,0.7.0.4).$$

$$U_{F_1}(e) = \begin{pmatrix} 1 & 1 & 1 \\ 0.4 & 1 & 1 \\ 0.4 & 0.5 & 1 \end{pmatrix} U_{F_1}(b) = \begin{pmatrix} 1 & 0.4 & 1 \\ 1 & 1 & 1 \\ 0.7 & 0.4 & 1 \end{pmatrix}$$

$$U_{F_2}(e) = \begin{pmatrix} 1 & 0.1 & 0.3 \\ 0.4 & 1 & 1 \\ 0.4 & 0.1 & 1 \end{pmatrix} U_{F_2}(b) = \begin{pmatrix} 1 & 1 & 0.4 \\ 0.6 & 1 & 0.4 \\ 1 & 1 & 1 \end{pmatrix}$$

$$U_{F_1} \wedge U_{F_2}(e) = \begin{pmatrix} 1 & 0.1 & 0.3 \\ 0.4 & 1 & 1 \\ 0.4 & 0.1 & 1 \end{pmatrix} U_{F_1} \wedge U_{F_2}(b) = \begin{pmatrix} 1 & 0.4 & 0.4 \\ 0.6 & 1 & 0.4 \\ 0.7 & 0.4 & 1 \end{pmatrix}$$

$$U_{F_1 \wedge F_2}(e) = \begin{pmatrix} 1 & 0.1 & 0.3 \\ 1 & 1 & 1 \\ 1 & 0.1 & 1 \end{pmatrix} U_{F_1 \wedge F_2}(b) = \begin{pmatrix} 1 & 0.4 & 0.4 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Define $\mathbf{U}_{\tau_i} = \{(U,A) \in S(X \times X, A) \mid (U,A) \geq (U_{F_i}, A)\}$ for $i = 1,2$. Since $(U_{F_i}, A) \circ (U_{F_i}, A) = (U_{F_i}, A)$, $\mathbf{U}_i$ is a soft quasi-uniformity for $i = 1,2$ where

$$\mathbf{U}_{\tau_1} \oplus \mathbf{U}_{\tau_2} = \{(U,A) \in S(X \times X, A) \mid (U,A) \geq (U_{F_1} \wedge U_{F_1}, A)\}$$
$$\mathbf{U}_{\tau_1 \oplus \tau_2} = \{(U,A) \in S(X \times X, A) \mid (U,A) \geq (U_{F_1 \wedge F_2}, A)\}.$$

Then $\mathbf{U}_{\tau_1 \oplus \tau_2} \subset \mathbf{U}_{\tau_1} \oplus \mathbf{U}_{\tau_2}$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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