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A WEIGHTED FOURTH ORDER RUNGE-KUTTA METHOD BASED ON CONTRA-HARMONIC MEAN

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Abstract. This paper discusses a weighted fourth order Runge-Kutta method based on contra-harmonic mean to solve ordinary differential equations. The local truncation error and stability analysis for this method are presented. The numerical experiment reveals that the proposed method is suitable for solving the stiff initial value problems.

Keywords: Stiff problem; Runge-Kutta method; Contra-harmonic mean.

2010 AMS Subject Classification: 65L04, 65L06.

1. Introduction

Consider a differential equation with the initial value problem

$$y' = f(x,y), \quad y(x_0) = y_0, \tag{1}$$

which can be expressed in the autonomous form [6, p.43]

$$y' = f(y), \quad y(x_0) = y_0.$$

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The solution of differential equations (1) can be obtained by a numerical method, such as the classical third order Runge-Kutta method as follows

$$y_{n+1} = y_n + \frac{h}{4}(k_1 + 2k_2 + k_3), \tag{2}$$

where

$$k_1 = f(x_n, y_n),$$
$$k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1),$$
$$k_3 = f(x_n + h, y_n - hk_1 + 2k_2).$$

Evans [4] rewrites the equation (2) as

$$y_{n+1} = y_n + \frac{h}{2}\left(\frac{k_1 + k_2}{2} + \frac{k_2 + k_3}{2}\right), \tag{3}$$

so that the equation (2) is also known as the third order Runge-Kutta method based on arithmetic mean. Many reseachers follow this idea see [1, 4, 7] for examples. Wazwaz [7] replaces the arithmetic mean in equation (3) with contra-harmonic mean. Then Abadneh and Rosita [1] proposed a weighted third order Runge-Kutta method based on contra-harmonic mean.

Evans and Yaakub [5] introduces a Fourth order Runge-Kutta method based on contra-harmonic mean (RK4CoM) of the form

$$y_{n+1} = y_n + \frac{h}{3}\left(\frac{k_1^2 + k_2^2}{k_1 + k_2} + \frac{k_2^2 + k_3^2}{k_2 + k_3} + \frac{k_3^2 + k_4^2}{k_3 + k_4}\right) \tag{4}$$

where

$$k_1 = f(x_n, y_n),$$
$$k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1),$$
$$k_3 = f(x_n + \frac{h}{2}, y_n + h\frac{1}{8}k_1 + h\frac{3}{8}k_2),$$
$$k_4 = f(x_n + h, y_n + h\frac{1}{4}k_1 - h\frac{3}{4}k_2 + \frac{3}{2}k_3).$$

In this paper, we follows the idea Abadneh and Rosita [1] for deriving A Weighted Fourth Order Runge-Kutta Method Based on Contra-harmonic (RK4MCHW) and its local truncation error as describe in section 2. In section 3, the stability analysis for the proposed method is presented. We end the presentation by numerical comparisons using two examples.

2. Proposed Method

RK4MCHW is derived by inserting weights w_1,w_2 and w_3 into the formula (4) as follows:

$$y_{n+1} = y_n + h\left(w_1\frac{k_1^2+k_2^2}{k_1+k_2} + w_2\frac{k_2^2+k_3^2}{k_2+k_3} + w_3\frac{k_3^2+k_4^2}{k_3+k_4}\right), \tag{5}$$

where

$$k_1 = f(x_n, y_n) \tag{6}$$

$$k_2 = f(x_n + a_1h, y_n + a_1hk_1) \tag{7}$$

$$k_3 = f(x_n + (a_2+a_3)h, y_n + a_2hk_1 + a_3hk_2) \tag{8}$$

$$k_4 = f(x_n + (a_1+a_2+a_3)h, y_n + a_4hk_1 + a_5hk_2 + a_6hk_3). \tag{9}$$

Now we have to find the appropriate values of w_1, w_2, w_3, a_1, a_2, a_3, a_4, a_5 and a_6 to obtain the fourth order accuracy of the method.

By expanding k_1, k_2, k_3 dan k_4 in (6)–(9) into Taylor series [2], where f is considered as a function of y only to simplify algebraic computations, we have

$$k_1 = f(y) = f, \tag{10}$$

$$k_2 = f + a_1 f f_y h + \frac{1}{2}a_1^2 f^2 f_{yy} h^2 + \frac{1}{6}a_1^3 f^3 f_{yyy} h^3, \tag{11}$$

$$\begin{aligned} k_3 = f &+ (a_3 f f_y + a_2 f f_y)h + \left(\frac{1}{2}a_2^2 f^2 f_{yy} + a_1 a_3 f f_y^2 + a_2 a_3 f^2 f_{yy}\right. \\ &\left.+\frac{1}{2}a_3^2 f^2 f_{yy}\right)h^2 + \left(\frac{1}{2}a_2 a_3^2 f^3 f_{yyy} + a_1 a_2 a_3 f^2 f_y f_{yy} + \frac{1}{2}a_2^2 a_3 f^3 f_{yyy}\right. \\ &\left.+\frac{1}{2}a_1^2 a_3 f^2 f_y f_{yy} + \frac{1}{6}a_3^3 f^3 f_{yyy} + \frac{1}{6}a_2^3 f^3 f_{yyy}\right)h^3, \end{aligned} \tag{12}$$

$$
\begin{aligned}
k_4 = f + (a_4 f f_y + a_5 f f_y + a_6 f f_y) h + \Big(& a_5 a_6 f^2 f_{yy} + \frac{1}{2} a_4^2 f^2 f_{yy} + \frac{1}{2} a_5^2 f^2 f_{yy} \\
& + a_4 a_6 f^2 f_{yy} + a_4 a_5 f^2 f_{yy} + a_1 a_5 f f_y^2 + a_6 a_2 f f_y^2 + a_3 a_6 f f_y^2 \\
& + \frac{1}{2} a_6^2 f^2 f_{yy} \Big) h^2 + \Big(a_3 a_6^2 f^2 f_y f_{yy} + a_2 a_3 a_6 f^2 f_y f_{yy} + \frac{1}{2} a_1^2 a_5 f^2 f_y f_{yy} \\
& + \frac{1}{2} a_6 f_y f_{yy} a_2^2 f^2 + a_1 a_3 a_6 f f_y^3 + \frac{1}{2} a_3^2 a_6 f^2 f_y f_{yy} + a_1 a_5^2 f^2 f_y f_{yy} \\
& + a_2 a_6^2 f^2 f_y f_{yy} + a_4 a_5 a_6 f^3 f_{yyy} + a_1 a_4 a_5 f^2 f_y f_{yy} + a_2 a_4 a_6 f^2 f_y f_{yy} \\
& + a_3 a_4 a_6 f^2 f_y f_{yy} + a_2 a_5 a_6 f^2 f_y f_{yy} + a_3 a_5 a_6 f^2 f_y f_{yy} + a_1 a_5 a_6 f^2 f_y f_{yy} \\
& + \frac{1}{2} a_5^2 a_6 f^3 f_{yyy} + \frac{1}{2} a_4 a_5^2 f^3 f_{yyy} + \frac{1}{2} a_5 a_6^2 f^3 f_{yyy} + \frac{1}{2} a_4^2 a_5 f^3 f_{yyy} \\
& + \frac{1}{2} a_4^2 a_6 f^3 f_{yyy} + \frac{1}{2} a_4 a_6^2 f^3 f_{yyy} + \frac{1}{6} a_4^3 f^3 f_{yyy} \\
& + \frac{1}{6} a_5^3 f^3 f_{yyy} + \frac{1}{6} a_6^3 f^3 f_{yyy} \Big) h^3. \qquad (13)
\end{aligned}
$$

To make it simple, the equation (5) is written in the form

$$y_{n+1} = y_n + \frac{numerator}{denominator}, \qquad (14)$$

where

$$
\begin{aligned}
numerator = \quad & h \Big(w_1 \left(k_1^2 + k_2^2\right)(k_2 + k_3)(k_3 + k_4) \\
& + w_2 \left(k_2^2 + k_3^2\right)(k_1 + k_2)(k_3 + k_4) \\
& + w_3 \left(k_3^2 + k_4^2\right)(k_1 + k_2)(k_2 + k_3) \Big), \qquad (15)
\end{aligned}
$$

and

$$denominator = (k_1 + k_2)(k_1 + k_2)(k_3 + k_4). \qquad (16)$$

Substituting k_1, k_2, k_3 and k_4 in (10)–(13) into equation (15) dan (16), we get respectively

$$\begin{aligned} numerator =& h^4\left(2w_1 f_y^3 a_1^3 f^4 + \cdots + 2w_3 f^6 f_{yyy} a_3^3\right) \\ &+ h^3\left(8w_1 f_y^2 a_1^2 f^4 + \cdots + 6w_3 f^5 f_{yy} a_2^3\right) \\ &+ h^2\left(4w_2 f^4 f_y a_5 + \cdots + 12 w_1 f^4 f_y a_3\right) \\ &+ h\left(8w_1 f^4 + 8w_3 f^4 + 8w_2 f^4\right), \end{aligned} \tag{17}$$

and

$$\begin{aligned} denominator =& h^4\left(\frac{3}{2} f_{yy}^2 a_1^2 f^5 a_2^2 + \cdots + \frac{1}{3} f_{yyy} a_3^3 f^5 f_y a_6\right) \\ &+ h^3\left(7 f_y^3 a_1^2 f^3 a_3 + \cdots + f_{yy} a_2^2 f^4 f_y a_5\right) \\ &+ h^2\left(6 f_y^2 a_1 f^3 a_2 + \cdots + 2 f_y^2 a_3^2 f^3\right) \\ &+ h\left(8 f_y a_1 f^3 + \cdots + 4 f^3 f_y a^5\right) + 8 f^3. \end{aligned} \tag{18}$$

A Taylor expansion of $\overline{y}_{n+1}$ about $x = x_n$ is given by

$$\overline{y}_{n+1} = y_n + M, \tag{19}$$

where

$$\begin{aligned} M = hf + \frac{1}{2} h^2 f f_y + \frac{1}{6} h^3\left(f f_y^2 + f^2 f_{yy}\right) \\ + \frac{1}{24} h^4\left(f^3 f_{yyy} + 4 f_y f_{yy} f^2 + f_y^3 f\right). \end{aligned} \tag{20}$$

Comparing (14) and (19), we obtain

$$\begin{aligned} numerator =& h^4\left(\frac{1}{3} f^4 f_y^3 + \cdots + 5 f^4 f_y^3 a_3^2 a_1\right) + h^3\left(\frac{4}{3} f^5 f_{yy} + \cdots + 6 f^4 f_y^2 a_2 a_6\right) \\ &+ h^2\left(8 f^4 f_y a_2 + \cdots + 4 f^4 f_y a_6\right) + h 8 f^4. \end{aligned} \tag{21}$$

Then, by comparing coefficients h^k in equation (17) with equation (21), we obtain

$$\left.\begin{aligned}
f^6 f_{yyy} &: -\tfrac{1}{3}a_2^3 - \tfrac{1}{3}a_2^3 + \cdots + 4w_1a_4a_5a_6 - \tfrac{4}{3}a_1^3 = \tfrac{1}{3},\\
f^5 f_y f_{yy} &: a_4^2 + a_6^2 + a_5^2 + \cdots + 2a_1^3 + 2a_2^2 + 2a_3^2 = \tfrac{4}{3},\\
f^4 f_y^3 &: a_1^2 + \tfrac{4}{3}a_1 + \tfrac{4}{3}a_2 + \cdots + \tfrac{2}{3}a^6 + a_2^2 + a_3^2 = \tfrac{1}{3},\\
f^5 fyy &: 2a_4^2 + 2a_6^2 + 2a_5^2 + \cdots - 8w_3a_5a_6a^6 + 4a_2^2 + 4a_3^2 = \tfrac{4}{3},\\
f^4 f^2 y &: 2a_1^2 + 4a_1 + 4a_2 + \cdots + 2a_6 + a_2^2 + 2a_3^2 = \tfrac{4}{3},\\
f^4 f_y &: 8a_1 + 8a_2 + 8a_3 + \cdots + 4a_4 + 4a_5 + 4a_6 = 4,\\
f^4 &: 8w_1 + 8w_2 + 8w_3 = 8.
\end{aligned}\right\} \tag{22}$$

Solutions of the system of nonlinear equations (22) are $w_1 = \frac{1}{5}$, $w_2 = \frac{3}{5}$, $w_3 = \frac{1}{5}$, $a_1 = \frac{5}{6}$, $a_2 = \frac{2551}{4620}$, $a_3 = -\frac{337}{1540}$, $a_4 = -\frac{2}{231}$, $a_5 = \frac{2162}{693}$ and $a_6 = -\frac{25}{9}$.

Then subtituting these values into (5) – (9) we obtain RK4MCHW as follows:

$$y_{n+1} = y_n + h\left(\frac{1}{5}\frac{k_1^2+k_2^2}{k_1+k_2} + \frac{3}{5}\frac{k_2^2+k_3^2}{k_2+k_3} + \frac{1}{5}\frac{k_3^2+k_4^2}{k_3+k_4}\right), \tag{23}$$

with

$$\left.\begin{aligned}
k_1 &= f(x_n, y_n)\\
k_2 &= f(x_n + \tfrac{5}{6}h, y_n + \tfrac{5}{6}hk_1)\\
k_3 &= f(x_n + \tfrac{1}{3}h, y_n + \tfrac{2551}{4620}hk_1 - \tfrac{337}{1540}hk_2)\\
k_4 &= f(x_n + \tfrac{1}{3}h, y_n - \tfrac{2}{231}hk_1 + \tfrac{2162}{693}hk_2 - \tfrac{25}{9}hk_3)
\end{aligned}\right\} \tag{24}$$

To obtain the local truncation error of RK4MCHW, we expand k_1, k_2, k_3 and k_4 in the equation (24) into fifth order Taylor series and then substitute into (23), we have

$$\begin{aligned}
y_{n+1} = y_n + hf + \frac{1}{2}h^2 f f_y + \frac{1}{6}h^3\left(f f_y^2 + f^2 f_{yy}\right) + \frac{1}{24}h^4\left(f^3 f_{yyy}\right.\\
\left. + 4 f_y f_{yy} f^2 + f_y^3 f\right) + \frac{1}{120}h^5\left(-\frac{649}{31104}f^4 f_{yyyy} - \frac{410425}{1596672}f^3 f_{yy}^2\right.\\
\left. - \frac{424409}{1197504}f^3 f_y f_{yyy} - \frac{60131833}{20490624}f^2 f_y^2 f_{yy} - \frac{263156359}{61471872}f f_y^4\right).
\end{aligned} \tag{25}$$

A fifth order Taylor expansion of $\bar{y}_{n+1}$ around $x = x_n$ is given by

$$\begin{aligned}\bar{y}_{n+1} = y_n + hf + \frac{1}{2}h^2 ff_y + \frac{1}{6}h^3\left(ff_y^2 + f^2 f_{yy}\right) + \frac{1}{24}h^4\left(f^3 f_{yyy}\right.\\ \left. + 4f_y f_{yy} f^2 + f_y^3 f\right) + \frac{1}{120}h^5\left(f^4 f_{yyyy} + 4f^3 f_{yy}^2\right.\\ \left. + 7f^3 f_y f_{yyy} + 7f^2 f_y^2 f_{yy} + ff_y^4\right).\end{aligned} \tag{26}$$

Comparing y_{n+1} in equation(25) and $\bar{y}_{n+1}$ in equation (26) we obtain The local truncation error of RKMCHW as follows:

$$\begin{aligned}LTE = \left(-\frac{4541}{155520}f^4 f_{yyyy} - \frac{2471317}{5987520}f^3 f_y f_{yyy} - \frac{306635597}{102453120}f^2 f_y^2 f_{yy}\right.\\ \left. - \frac{2318237}{7983360}f^3 f_{yy}^2 - \frac{346935763}{307359360}ff_y^4\right)h^5.\end{aligned}$$

3. Stability Analysis

To do the stability analysis of proposed method, we solve a differential equation $y' = \lambda y$ as suggested in Dahlquist [3, p.374]. Substituting y' to equation (24) yields

(27)

$$\left.\begin{aligned} k_1 &= \lambda y_n \\ k_2 &= \lambda y_n\left(1 + \tfrac{5}{6}h\lambda\right) \\ k_3 &= \lambda y_n\left(1 + \tfrac{2551}{4620}h\lambda - \tfrac{337}{1540}h\lambda\left(1 + \tfrac{5}{6}h\lambda\right)\right) \\ k_4 &= \lambda y_n\left(1 - \tfrac{2}{231}h\lambda + \tfrac{2162}{693}h\lambda\left(1 + \tfrac{5}{6}h\lambda\right) - \tfrac{25}{9}h\lambda\left(1 + \tfrac{2551}{4620}h\lambda - \tfrac{337}{1540}h\lambda\left(1 + \tfrac{5}{6}h\lambda\right)\right)\right) \end{aligned}\right\}$$

Substituting (27) to equation (23) and letting $z = h\lambda$ to simplify the equation, we end up with

$$\begin{aligned}\frac{y_{n+1}}{y_n} = 1 + z\left(\frac{1}{5}\frac{2 + \frac{5}{3}z + \frac{25}{36}z^2}{2 + \frac{5}{6}z} + \frac{3}{5}\frac{2 + \frac{7}{3}z + \frac{611}{1386}z^2 - \frac{337}{2772}z^3 + \frac{113569}{3415104}z^4}{2 + \frac{6401}{4620}z - \frac{337}{1540}z\left(1 + \frac{5}{6}z\right)}\right.\\ \left. + \frac{1}{5}\frac{2 + \frac{4}{3}z + \frac{8885}{2772}z^2 + \frac{8347}{4158}z^3 + \frac{10835569}{3415104}z^4 + \frac{2443250}{1440747}z^5 + \frac{70980625}{276623424}z^6}{2 + \frac{837}{1540}z + \frac{40207}{13860}z\left(1 + \frac{5}{6}z\right) - \frac{25}{9}z\left(1 + \frac{2551}{4620}z - \frac{337}{1540}z\left(1 + \frac{5}{6}z\right)\right)}\right).\end{aligned} \tag{28}$$

Ploting the Polynomial (28) and the polynomial obtained from RK4CoM, we see that the stability region of RK4MCHW is wider than RK4CoM as shown in Figure 1.

(a) RK4MCHW (b) RK4CoM

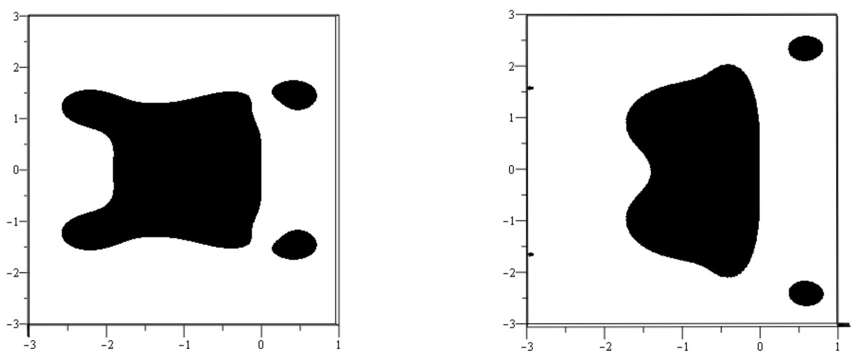

FIGURE 1. Stability region of RK4MCHW and RK4CoM

4. Numerical Comparison

To performs numerical comparisons, RK4MCHW and RK4CoM are applied into two examples using two grid sizes $N = 64$ and 128 as follows:

(1) **Example 1**: Stiff differential equation: $y' = -100y + e^{-2x}$, with initial condition $y(0) = 0$ and exact solution is $y = \frac{1}{98}\left(e^{-2x} - 8e^{-100x}\right)$ on $[0,1]$.

(2) **Example 2**: Ordinary differential equation: $y' = \frac{1}{y}$, with initial condition $y(0) = 1$ and exact solution is $y = \sqrt{2x+1}$ on $[0,1]$.

TABLE 1. The approximated solution and LTE of RK4MCHW and RK4CoM for Example 1 using 64 nodes

i	x_i	y_i			LTE	
		exact	RK4MCHW	RK4CoM	RK4MCHW	RK4CoM
0	0.000000	0.000000000	0.000000000	0.000000000	0.000000e+000	0.000000e+000
1	0.015625	0.007751243	0.010206620	-0.055899072	2.455376e−003	6.365032e−002
2	0.031250	0.009137512	0.009624114	-0.687585896	4.866024e−004	6.967234e−001
3	0.046875	0.009196946	0.009309313	-8.247005698	1.123671e−004	8.256203e+000
4	0.062500	0.008985372	0.009015448	-98.84334631	3.007576e−005	9.885233e+001
5	0.078125	0.008723885	0.008734630	-1184.615946	1.074499e−005	1.184625e+003
⋮	⋮	⋮	⋮	⋮	⋮	⋮
60	0.937500	0.001564847	0.001565359	-250112e+058	5.128626e−007	2.501125e+062
61	0.953125	0.001516701	0.001517198	-299753e+057	4.970835e−007	2.997531e+063
62	0.968750	0.001470037	0.001470519	-359246e+058	4.817899e−007	3.592460e+064
63	0.984375	0.001424809	0.001425276	-430546e+059	4.669667e−007	4.305468e+065
64	1.000000	0.001380972	0.001381425	-515998e+060	4.525997e−007	5.159987e+066

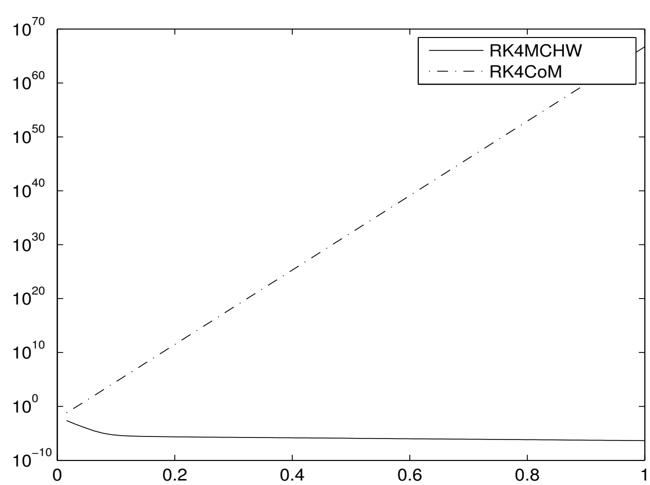

FIGURE 2. LTE of RK4MCHW and RK4CoM for Example 1

TABLE 2. The approximated solution and LTE of RK4MCHW and RK4CoM for Example 1 using 128 nodes

i	x_i	y_i			LTE	
		exact	RK4MCHW	RK4CoM	RK4MCHW	RK4CoM
0	0.000000	0.000000000	0.000000000	0.000000000	0.000000e+000	0.000000e+000
1	0.007813	0.005374113	0.005723556	0.005496198	3.494431e−004	1.220850e−004
2	0.015625	0.007751243	0.008071658	0.007868463	3.204151e−004	1.172195e−004
3	0.023438	0.008757547	0.008986076	0.008846359	2.285289e−004	8.881190e−005
4	0.031250	0.009137512	0.009298957	0.009205335	1.614458e−004	6.782391e−005
5	0.039063	0.009231970	0.009684607	0.009457327	4.526377e−004	2.253576e+004
⋮	⋮	⋮	⋮	⋮	⋮	⋮
124	0.968750	0.001470037	0.001470078	0.001470070	4.082185e−008	1.545776e−007
125	0.976563	0.001447247	0.001447287	0.001447279	4.018897e−008	1.507611e−007
126	0.984375	0.001424809	0.001424849	0.001424841	3.956590e−008	1.470388e−007
127	0.992188	0.001402719	0.001402758	0.001402751	3.895249e−008	1.434084e−007
128	1.000000	0.001380972	0.001381011	0.001381003	3.834858e−008	1.398676e−007

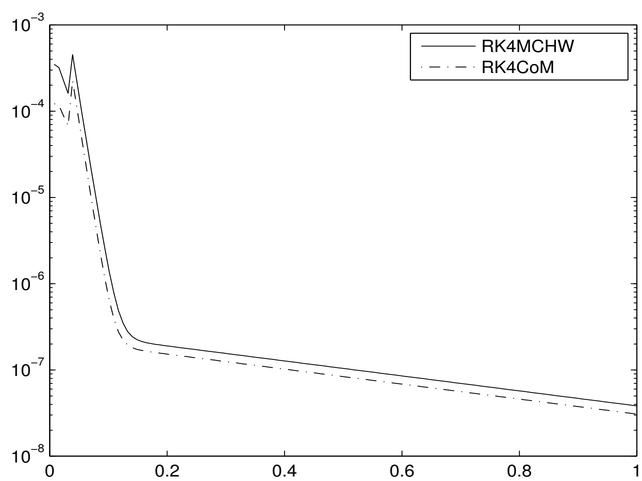

FIGURE 3. LTE of RK4MCHW and RK4CoM for Example 1

.

TABLE 3. The approximated solution and LTE of RK4MCHW and RK4CoM for Example 2 using 64 nodes

i	x_i	y_i			LTE	
		exact	RK4MCHW	RK4CoM	RK4MCHW	RK4CoM
0	0.000000	1.000000000	1.000000000	1.000000000	0.000000e+000	0.000000e+000
1	0.015625	1.015504801	1.015504801	1.015504801	2.819056e−010	5.504930e−011
2	0.031250	1.030776406	1.030776407	1.030776407	5.236007e−010	1.022449e−010
3	0.046875	1.045825033	1.045825034	1.045825033	7.313772e−010	1.428158e−010
4	0.062500	1.060660172	1.060660173	1.060660172	9.104146e−010	1.777740e−010
5	0.078125	1.075290658	1.075290659	1.075290659	1.064999e−009	2.079565e−010
⋮	⋮	⋮	⋮	⋮	⋮	⋮
60	0.937500	1.695582496	1.695582498	1.695582496	1.826477e−009	3.565708e−010
61	0.953125	1.704772712	1.704772714	1.704772712	1.819156e−009	3.551415e−010
62	0.968750	1.713913650	1.713913652	1.713913650	1.811860e−009	3.537168e−010
63	0.984375	1.723006094	1.723006096	1.723006094	1.804592e−009	3.522977e−010
64	1.000000	1.732050808	1.732050809	1.732050808	1.797356e−009	3.508849e−010

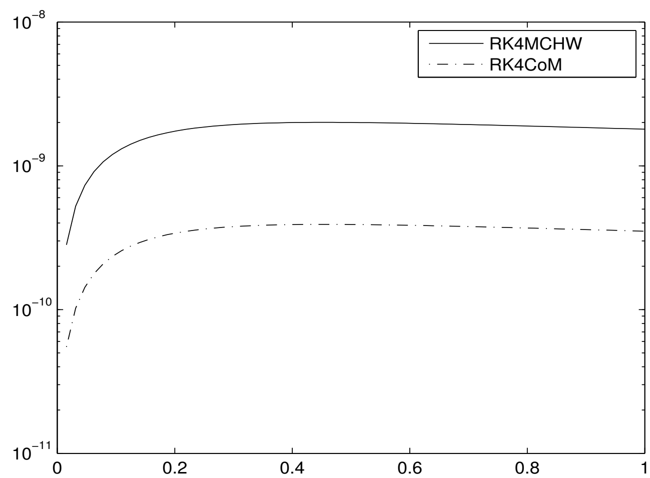

FIGURE 4. Comparing LTE of RK4MCHW and RK4CoM for Example 2

TABLE 4. The approximated solution and LTE of RK4MCHW and RK4CoM for Example 2 using 128 nodes

i	x_i	y_i			LTE	
		exact	RK4MCHW	RK4CoM	RK4MCHW	RK4CoM
0	0.000000	1.000000000	1.000000000	1.000000000	0.000000e+000	0.000000e+000
1	0.007813	1.007782219	1.007782219	1.007782219	9.063417e−012	1.768807e−012
2	0.015625	1.015504801	1.015504801	1.015504801	1.745071e−011	3.405498e−012
3	0.023438	1.023169096	1.023169097	1.023169096	2.521805e−011	4.921397e−012
4	0.031250	1.030776406	1.030776406	1.030776406	3.241629e−011	6.326051e−012
5	0.039063	1.038327983	1.038327983	1.038327983	3.909140e−011	7.628564e+012
⋮	⋮	⋮	⋮	⋮	⋮	⋮
124	0.968750	1.713913650	1.713913650	1.713913650	1.123881e−010	2.192846e−011
125	0.976563	1.718465886	1.718465886	1.718465886	1.121627e−010	2.188449e−011
126	0.984375	1.723006094	1.723006094	1.723006094	1.119380e−010	2.184075e−011
127	0.992188	1.727534370	1.727534370	1.727534370	1.117135e−010	2.179701e−011
128	1.000000	1.732050808	1.732050808	1.732050808	1.114897e−010	2.175349e−011

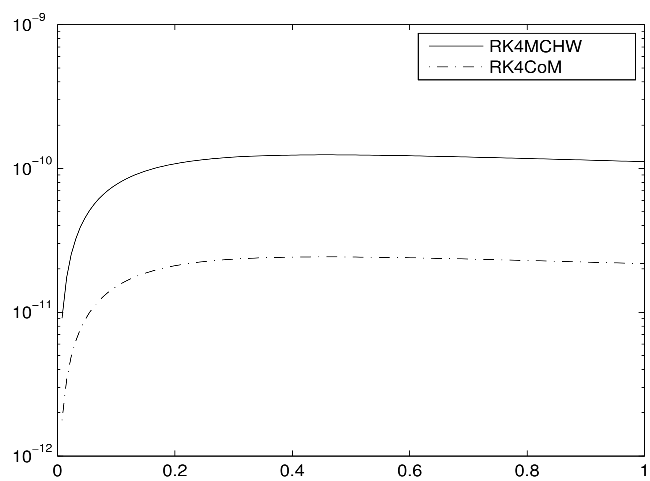

FIGURE 5. Comparing LTE of RK4MCHW and RK4CoM for Example 2

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From Table 1–4 we show that RK4MCHW is better for solving a stiff differential equation compare to RK4CoM. For the non-stiff differential equation RK4MCHW is not as good as RK4CoM.

4. Conclusion

We have shown how to get the formula of the fourth order Runge-Kutta method based on contra-harmonic by adding w_1,w_2 dan w_3 as weight into the formula of RK4CoM. The local truncation error of both methods are almost the same, however the stability region of RK4MCHW is wider than that of RK4CoM. RK4MCHW is better for solving a stiff differential equation compare to RK4CoM. Generally, approximation and stability region show that RK4MCHW gives the similar results with RK4CoM.

Conflict of Interests

The authors declare that there is no conflict of interests.

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