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DIFFERENTIAL EQUATIONS WITH NON-MONOTONE ARGUMENTS: ITERATIVE OSCILLATION RESULTS

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Abstract. In this paper, we present sufficient conditions, involving $\limsup$, in an explicit iterative form which guarantee the oscillation of all solutions of a differential equation with variable non-monotone argument and non-negative coefficients. Corresponding differential equations of both delay and advanced type are studied. Our conditions essentially improve all the known results in the literature. Examples illustrating the significance of the results are also given.

Keywords: differential equation; non-monotone argument; oscillatory solutions; nonoscillatory solutions.

2010 AMS Subject Classification: 34K11, 34K06

1. Introduction

In this paper we consider the differential equation with variable deviating arguments of either delay

(E) $$x'(t) + p(t)x(\tau(t)) = 0, \ \ t \geq t_0$$

or advanced type

(E′) $$x'(t) - q(t)x(\sigma(t)) = 0, \ \ t \geq t_0.$$

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Equations (E) and (E$'$) are studied under the following assumptions: everywhere $p(t) \geq 0$, $q(t) \geq 0$ and $\tau(t)$, $\sigma(t)$ are functions of positive real numbers such that

$$\tau(t) < t \quad \text{for} \quad t \geq t_0 \qquad \text{and} \qquad \lim_{t\to\infty} \tau(t) = \infty \tag{1.1}$$

and

$$\sigma(t) > t \quad \text{for} \quad t \geq t_0, \tag{1.1$'$}$$

respectively.

By a solution of (E) we mean a continuously differentiable function defined on $[\tau(T_0), \infty]$ for some $T_0 \geq t_0$ and such that (E) is satisfied for $t \geq T_0$. Such a solution is called *oscillatory* if it has arbitrarily large zeros. Otherwise, it is called *nonoscillatory*.

The problem of establishing sufficient conditions for the oscillation of all solutions of equations (E) and (E$'$) has been the subject of many investigations. See, for example, [2$-$5, 7$-$19, 21$-$28] and the references cited therein. For the general oscillation theory of differential equations the reader is referred to the monographs [1, 6, 20].

1.1. **Delay differential equations.** The first systematic study for the oscillation of all solutions of equation (E) was made by Myshkis. In 1950 [22] he proved that every solution oscillates if

$$\limsup_{t\to\infty}[t-\tau(t)] < \infty \qquad \text{and} \qquad \liminf_{t\to\infty}[t-\tau(t)] \liminf_{t\to\infty} p(t) > \frac{1}{e}. \tag{1.2}$$

In 1972, Ladas, Lakshmikantham and Papadakis [17] proved that if, τ is non-decreasing and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 1, \tag{1.3}$$

then all solutions of (E) oscillate.

In 1982, Koplatadze and Canturija [12] improved (1.2) to

If

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}. \tag{1.4}$$

Conserning the constant $\frac{1}{e}$ in (1.4), it is to be point out that if the inequality

$$\int_{\tau(t)}^{t} p(s)ds \leq \frac{1}{e}$$

holds eventually, then, according to a result in [12], (E) has a nonoscillatory solution.

It is obvious that there is a gap between the conditions (1.3) and (1.4) when the limit

$$\lim_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds$$

does not exist. How to fill this gap is an interesting problem which has been investigated by several authors.

Assume that the argument $\tau(t)$ is non-monotone. Set

$$h(t) := \sup_{s\le t}\tau(s), \quad t\ge t_0. \tag{1.5}$$

Clearly, $h(t)$ is nondecreasing, and $\tau(t)\le h(t)<t$ for all $t\ge t_0$.

In 2011, Braverman and Karpuz [4] and in 2014, Stavroulakis [24], proved that if

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p(u)du\right)ds > 1, \tag{1.6}$$

or

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p(u)du\right)ds > 1-\frac{1-a-\sqrt{1-2a-a^2}}{2} \tag{1.7}$$

respectively, where $\alpha=\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds$, then all solutions of (E) oscillate.

1.2. **Advanced differential equations.** By Theorem 2.4.3 [20], if, σ is non-decreasing and

$$\limsup_{t\to\infty}\int_{t}^{\sigma(t)} q(s)ds > 1, \tag{1.8}$$

then all solutions of (E$'$) oscillate.

In 1983, Fukagai and Kusano [8] proved that if

$$\liminf_{t\to\infty}\int_{t}^{\sigma(t)} q(s)ds > \frac{1}{e}, \tag{1.9}$$

then all solutions of (E$'$) oscillate, while if

$$\int_{t}^{\sigma(t)} q(s)ds \le \frac{1}{e} \quad \text{for all sufficiently large } t,$$

then Eq. (E$'$) has a nonoscillatory solution.

Assume that the argument $\sigma(t)$ is non-monotone. Set

$$\rho(t) = \inf_{s\ge t}\sigma(s), \quad t\ge t_0. \tag{1.10}$$

Clearly, the function $\rho(t)$, is non-decreasing and $\sigma(t)\ge\rho(t)>t$ for all $t\ge t_0$.

In 2015, Chatzarakis and Ocalan [5], proved that if

$$\limsup_{t\to\infty}\int_t^{\rho(t)} q(s)\exp\left(\int_{\rho(t)}^{\sigma(s)} q(u)du\right)ds > 1, \tag{1.11}$$

then all solutions of (E′) oscillate.

The consideration of non-monotone arguments other than the pure mathematical interest, it approximates the natural phenomena described by equation of the type (E) or (E′). That is because there are always natural disturbances (e.g. noise in communication systems) that affect all the parameters of the equation and therefore the fair (from a mathematical point of view) monotone arguments become non-monotone almost always. In view of this, an interesting question arising in the case where the arguments $\tau(t)$ and $\sigma(t)$ are non-monotone, is whether we can state further oscillation criteria which essentially improve all the known results in the literature.

In the present paper a positive answer to the above question is given.

2. Main results

2.1. **Delay differential equations.** We study further (E) and derive a new sufficient oscillation condition, involving lim sup, which essentially improves all the previous results.

Set $p_0(t) = p(t)$ and

$$p_j(t) = p(t)\left[1+\int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p_{j-1}(u)du\right)ds\right], \quad j \geq 1. \tag{2.1}$$

Theorem 1. *Assume that (1.1) holds. If for some* $j \in \mathbb{N}$

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p_j(u)du\right)ds > 1, \tag{2.2}$$

where $h(t)$ *is defined by (1.5) and* p_j *by (2.1), then all solutions of (E) oscillate.*

Proof. Assume, for the sake of contradiction, that there exists a nonoscillatory solution $x(t)$ of (E). Since $-x(t)$ is also a solution of (E), we can confine our discussion only to the case where the solution $x(t)$ is eventually positive. Then there exists $t_1 > t_0$ such that $x(t),\ x(\tau(t)) > 0$, for all $t \geq t_1$. Thus, from (E) we have

$$x'(t) = -p(t)x(\tau(t)) \leq 0, \quad \text{for all } t \geq t_1,$$

which means that $x(t)$ is an eventually nonincreasing function of positive numbers.

In view of this and taking into accout that $\tau(t) < t$, (E) implies

$$x'(t) + p(t)x(t) \le 0. \tag{2.3}$$

Applying the Grönwall inequality, we obtain

$$x(s) \ge x(t)\exp\left(\int_s^t p(u)du\right), \quad 0 \le s \le t. \tag{2.4}$$

Integrating (E) from $\tau(t)$ to t, we have

$$x(t) - x(\tau(t)) + \int_{\tau(t)}^t p(s)x(\tau(s))\,ds = 0. \tag{2.5}$$

Since $\tau(s) \le h(s) \le h(t)$, (2.4) and (2.5) give

$$x(t) - x(\tau(t)) + x(h(t))\int_{\tau(t)}^t p(s)\exp\left(\int_{\tau(s)}^{h(t)} p(u)du\right)ds \le 0.$$

Multiplying the last inequality by $p(t)$, we take

$$p(t)x(t) - p(t)x(\tau(t)) + p(t)x(h(t))\int_{\tau(t)}^t p(s)\exp\left(\int_{\tau(s)}^{h(t)} p(u)du\right)ds \le 0,$$

which, in view of (E), becomes

$$x'(t) + p(t)x(t) + p(t)x(h(t))\int_{\tau(t)}^t p(s)\exp\left(\int_{\tau(s)}^{h(t)} p(u)du\right)ds \le 0.$$

Hence

$$x'(t) + p(t)x(t) + p(t)x(t)\int_{\tau(t)}^t p(s)\exp\left(\int_{\tau(s)}^{h(t)} p(u)du\right)ds \le 0,$$

or

$$x'(t) + p(t)\left[1 + \int_{\tau(t)}^t p(s)\exp\left(\int_{\tau(s)}^{h(t)} p(u)du\right)ds\right]x(t) \le 0.$$

Therefore

$$x'(t) + p_1(t)x(t) \le 0,$$

where

$$p_1(t) = p(t)\left[1 + \int_{\tau(t)}^t p(s)\exp\left(\int_{\tau(s)}^{h(t)} p(u)du\right)ds\right].$$

Repeating the above argument leads to a new estimate

$$x'(t) + p_2(t)x(t) \le 0,$$

where

$$p_2(t) = p(t)\left[1+\int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p_1(u)du\right)ds\right].$$

Continuing by induction, we get

$$x'(t) + p_j(t)x(t) \le 0,$$

where

$$p_j(t) = p(t)\left[1+\int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p_{j-1}(u)du\right)ds\right].$$

Integrating (E) from $h(t)$ to t, we have

$$x(t) - x(h(t)) + x(h(t))\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p_j(u)du\right)ds \le 0.$$

The strict inequality is valid if we omit $x(t) > 0$ in the left-hand side:

$$-x(h(t)) + x(h(t))\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p_j(u)du\right)ds < 0.$$

This implies

$$x(h(t))\left[\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p_j(u)du\right)ds - 1\right] < 0,$$

i.e.,

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p_j(u)du\right)ds \le 1,$$

which contradicts (2.2).

The proof of the theorem is complete. ∎

Example 1. *(taken and adapted from [4]) Consider the delay differential equation*

(2.6)
$$x'(t) + \frac{11}{50}x(\tau(t)) = 0, \quad t \ge 0,$$

with

$$\tau(t) = \begin{cases} t-1, & \text{if } t \in [3k, 3k+1] \\ -3t+12k+3, & \text{if } t \in [3k+1, 3k+2] \\ 5t-12k-13, & \text{if } t \in [3k+2, 3k+3] \end{cases}, \quad k \in \mathbb{N}_0,$$

where $\mathbb{N}_0$ is the set of non-negative integers.

By (1.5), we see that

$$h(t) := \sup_{s\le t}\tau(s) = \begin{cases} t-1, & \text{if } t\in[3k,3k+1] \\ 3k, & \text{if } t\in[3k+1,3k+2.6] \\ 5t-12k-13, & \text{if } t\in[3k+2.6,3k+3] \end{cases}, \quad k\in\mathbb{N}_0.$$

Observe that the function $F_j : \mathbb{R}_0 \to \mathbb{R}_+$ defined as

$$F_j(t) = \int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p_j(u)du\right)ds,$$

attains its maximum at $t = 3k+2.6$, $k\in\mathbb{N}_0$, for every $j\in\mathbb{N}$. Specifically, by using an algorithm on MATLAB software, we obtain

$$\begin{aligned} F_1(t &= 3k+2.6) = \int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p_1(u)du\right)ds \\ &= \int_{h(t)}^{t} p(s)\exp\left\{\int_{\tau(s)}^{h(t)} p(u)\left[1+\int_{\tau(u)}^{u} p(v)\exp\left(\int_{\tau(v)}^{h(u)} p(w)dw\right)dv\right]du\right\}ds \\ &= \int_{3k}^{3k+2.6} \frac{11}{50}\exp\left\{\int_{\tau(s)}^{3k} \frac{11}{50}\left[1+\int_{\tau(u)}^{u} \frac{11}{50}\exp\left(\int_{\tau(v)}^{h(u)} \frac{11}{50}dw\right)dv\right]du\right\}ds \\ &\simeq 1.0148. \end{aligned}$$

Thus

$$\limsup_{t\to\infty} F_1(t) = \limsup_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p_1(u)du\right)ds \simeq 1.0148 > 1.$$

That is, condition (2.2) of Theorem 1 is satisfied for $j=1$. Therefore, all solutions of (2.6) oscillate.

On the other hand, we see that

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)ds = \limsup_{t\to\infty}\int_{3k}^{3k+2.6}\frac{11}{50}ds = \frac{11}{50}\cdot 2.6 = 0.572 < 1,$$

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds = \frac{11}{50}\cdot\liminf_{t\to\infty}(t-\tau(t)) = 0.22 < \frac{1}{e},$$

$$\begin{aligned} \limsup_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p(u)du\right)ds &= \limsup_{t\to\infty}\int_{3k}^{3k+2.6}\frac{11}{50}\exp\left(\int_{\tau(s)}^{3k}\frac{11}{50}du\right)ds \\ = \frac{11}{50}\cdot\limsup_{t\to\infty} &\left[\begin{array}{c} \int_{3k}^{3k+1}\exp\left(\frac{11}{50}\int_{s-1}^{3k}du\right)ds \\ +\int_{3k+1}^{3k+2}\exp\left(\frac{11}{50}\int_{-3s+12k+3}^{3k}du\right)ds \\ +\int_{3k+2}^{3k+2.6}\exp\left(\frac{11}{50}\int_{5s-12k-13}^{3k}du\right)ds \end{array}\right] \simeq 0.7446 < 1, \end{aligned}$$

$$0.7446 < 1 - \frac{1-a-\sqrt{1-2a-a^2}}{2} \simeq 0.9676,$$

that is, none of conditions (1.3), (1.4), (1.6) and (1.7) is satisfied.

Notation. *It is worth noting that the improvement of condition (2.2) to the corresponding condition (1.3) is significant, approximately 77.4%, if we compare the values on the left-side of these conditions. Also, the improvement compared to condition (1.6) is very satisfactory, around 36.3%.*

2.2. **Advanced differential equations.** Similar oscillation theorem for the (dual) advanced differential equation (E$'$) can be derived easily. The proof of this theorem is omitted, since it is quite similar to the delay equation.

Set $q_0(t) = q(t)$ and

$$q_j(t) = q(t)\left[1 + \int_t^{\sigma(t)} q(s)\exp\left(\int_{\rho(t)}^{\sigma(s)} q_{j-1}(u)du\right)ds\right], \quad j \geq 1. \tag{2.7}$$

Theorem 2. *Assume that (1.1$'$) holds. If for some $j \in \mathbb{N}$*

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s)\exp\left(\int_{\rho(t)}^{\sigma(s)} q_j(u)du\right)ds > 1, \tag{2.8}$$

where $\rho(t)$ is defined by (1.10) and q_j by (2.7), then all solutions of (E$'$) oscillate.

Example 2. *Consider the advanced differential equation*

$$x'(t) - \frac{7}{25}x(\sigma(t)) = 0, \quad t \geq 1, \tag{2.9}$$

where

$$\sigma(t) = \begin{cases} 3t-4k+1, & \text{if } t \in [2k, 2k+1] \\ -t+4k+5, & \text{if } t \in [2k+1, 2k+2] \end{cases}, \quad k \in \mathbb{N}_0.$$

By (1.10), we see that

$$\rho(t) := \inf_{t \leq s} \sigma(s) = \begin{cases} 3t-4k+1, & \text{if } t \in [2k, 2k+2/3] \\ 2k+3 & \text{if } t \in [2k+2/3, 2k+2] \end{cases}, \quad k \in \mathbb{N}_0.$$

Observe, that the function $F_j : \mathbb{R}_0 \to \mathbb{R}_+$ defined as

$$F_j(t) = \int_t^{\rho(t)} q(s)\exp\left(\int_{\rho(t)}^{\sigma(s)} q_j(u)du\right)ds$$

attains its maximum at $t = 2k + 2/3$, $k \in \mathbb{N}_0$, *for every* $j \in \mathbb{N}$. *Specifically, by using an algorithm on MATLAB software, we obtain*

$$\begin{aligned} F_1(t &= 2k+2/3) = \int_t^{\rho(t)} q(s)\exp\left(\int_{\rho(t)}^{\sigma(s)} q_1(u)du\right)ds \\ &= \int_t^{\rho(t)} q(s)\exp\left\{\int_{\rho(t)}^{\sigma(s)} q(u)\left[1+\int_u^{\sigma(u)} q(v)\exp\left(\int_{\rho(u)}^{\sigma(v)} q(w)dw\right)dv\right]du\right\}ds \\ &= \int_{2k+2/3}^{2k+3} \frac{7}{25}\exp\left\{\int_{2k+3}^{\sigma(s)} \frac{7}{25}\left[1+\int_u^{\sigma(u)} \frac{7}{25}\exp\left(\int_{\rho(u)}^{\sigma(v)} \frac{7}{25}dw\right)dv\right]du\right\}ds \\ &\simeq 1.0309 \end{aligned}$$

and therefore

$$\limsup_{t\to\infty} F_1(t) = \limsup_{t\to\infty} \int_t^{\rho(t)} q(s)\exp\left(\int_{\rho(t)}^{\sigma(s)} q_1(u)du\right)ds \simeq 1.0309 > 1.$$

That is, condition (2.8) of Theorem 2 is satisfied and therefore all solutions of (2.9) oscillate.

Observe, however, that

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s)ds = \limsup_{t\to\infty} \int_{2k+2/3}^{2k+3} \frac{7}{25}ds = 0.6533 < 1,$$

$$\liminf_{t\to\infty} \int_t^{\sigma(t)} q(s)ds = \liminf_{t\to\infty} \int_{2k+2}^{2k+3} \frac{7}{25}ds = 0.28 < \frac{1}{e},$$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s)\exp\left(\int_{\rho(t)}^{\sigma(s)} q(u)du\right)ds = \limsup_{t\to\infty} \int_{2k+2/3}^{2k+3} \frac{7}{25}\exp\left(\frac{7}{25}\int_{2k+3}^{\sigma(s)} du\right)ds$$

$$= \frac{7}{25}\cdot\limsup_{t\to\infty}\left[\begin{array}{l} \int_{2k+2/3}^{2k+1}\exp\left(\frac{7}{25}\int_{2k+3}^{3s-4k+1} du\right)ds \\ +\int_{2k+1}^{2k+2}\exp\left(\frac{7}{25}\int_{2k+3}^{-s+4k+5} du\right)ds \\ +\int_{2k+2}^{2k+3}\exp\left(\frac{7}{25}\int_{2k+3}^{3s-4k-3} du\right)ds \end{array}\right] \simeq 0.8696 < 1$$

and therefore none of conditions (1.8), (1.9) and (1.11) is satisfied.

Notation. *It is worth noting that the improvement of condition (2.8) to the corresponding condition (1.8) is significant, approximately 57.8%, if we compare the values on the left-side of these conditions. Also, the improvement compared to condition (1.11) is very satisfactory, around 18.55%.*

2.3. **Differential inequalities.** A slight modification in the proofs of Theorems 1 and 2 leads to the following results about differential inequalities.

Theorem 3. *Assume that all the conditions of Theorem 1 hold. Then*

(i) the delay differential inequality

$$x'(t)+p(t)x(\tau(t))\leq 0,\ t\geq t_0$$

has no eventually positive solutions;

(ii) the delay differential inequality

$$x'(t)+p(t)x(\tau(t))\geq 0,\ t\geq t_0$$

has no eventually negative solutions.

Theorem 4. *Assume that all the conditions of Theorem 2 hold. Then*

(i) the advanced differential inequality

$$x'(t)-q(t)x(\sigma(t))\geq 0,\ t\geq t_0,$$

has no eventually positive solutions;

(ii) the advanced differential inequality

$$x'(t)-q(t)x(\sigma(t))\leq 0,\ t\geq t_0,$$

has no eventually negative solutions.

Conflict of Interests

The author declares that there is no conflict of interests.

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