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DIFFERENTIAL GEOMETRY OF SELF-INTERSECTION CURVES OF A PARAMETRIC SURFACE IN $\mathbb{R}^3$

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Abstract. This paper presents algorithms for computing all the differential geometry properties of Frenet apparatus of "self-intersection curves of a parametric surface and the intersection curves of two parametric surfaces" in $\mathbb{R}^3$, for transversal and tangential intersection. Some examples are given and plotted.

Keywords: geometric properties; Frenet frame; Frenet apparatus; self-intersection; surface-surface intersection; transversal intersection; tangential intersection.

2010 AMS Subject Classification: 53A25.

1. Introduction

The intersection (also the self-intersection) problem is a fundamental process needed in modeling complex shapes in CAD/CAM system. It is useful in the representation of the design of complex objects, in computer animation and in NC machining for trimming off the region bounded by the self-intersection curves of offset surfaces. It is also essential to Boolean operations necessary in the creation of boundary representation in solid modeling [18]. Self-intersections

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of a rational polynomial parametric surface are defined by finding pairs of distinct parameter values $(u,v) \neq S(p,q)$ such that $S(u,v) = S(p,q)$ [21]. Once the shape's offset goes beyond the local radius-of-curvature, a self-intersection is bound to occur in the offset. Other applications could benefit from proper self-intersection detection and elimination as well. Another example, demonstrated herein, is the problem of creating a self-intersection-free metamorphosis between freeform curves [31]. The numerical marching method is the most widely used method for computing intersection curves in $\mathbb{R}^3$. The Marching method involves generation of sequences of points of an intersection curve in the direction prescribed by the local differential geometry [5, 9]. Kruppa [1] explained that the tangential direction of the intersection curve at a tangential intersection point corresponds to the direction from the intersection point towards the intersection of the Dupin indicatrices of the two surfaces. Willmore [2] described how to obtain the unit tangent, the unit principal normal, the unit binormal, the curvature and the torsion of the transversal intersection curve of two implicit surfaces. Barnhill et al. [3] compute surface self-intersections by their procedural surface/surface intersection algorithm. Lasser [4] introduces a method to compute all the self-intersection curves of a Bezier surface patch by subdividing the Bezier control net instead of the surface patch itself. Finaly the self-intersection curves are approximated by the polygons resulting from the plane/plane intersections of the refined Bezier control net. Hoffmann [6] listed the intersection problem as one of the most fundamental problems in the integration of geometric and solid modeling systems. The detection of self-intersections in freeform curves and surfaces and more so, the exact computation of the self-intersection locations, are difficult problems that have been addressed by the geometric modeling community, for several decades [6]. Aomura and Uehara [7] presented a approach based on numerical integration starting from random initial points. Nevertheless, this method does not guarantee the detection of all the components of self-intersections. Elber and Cohen [8] detected local self-intersections of offset curves by checking whether the tangent field of the curve and its offset, have opposite directions. It is a non-trivial task to detect and trim all local and global self-intersections in offset curves and surfaces [8, 11]. Wang [12] proposed an algorithm to compute the intersectioncurve between two offset surfaces. The method is based on the concept of normal projection. The intersection is represented in the parameter spaces of the base surfaces and no offset surface approximation is needed. This algorithm can also deal with global self-intersections, but not local self-intersections. Maekawa et al. [14] presented a method for tracing self-intersection loops in the parameter domain. In their method, starting points are computed by solving a system of nonlinear polynomial equations; nonetheless, they are solving five equations in five variables and their algorithm requires special treatment for trivial solutions. The tangent field approach [8] is limited to detecting local self-intersections only. A similar approach was used in [15] to detect and eliminate self-intersections in sweep surfaces. Andersson et al. [16] provide necessary and sufficient conditions to preclude self-intersections of composite Bezier curves and patches. Samoilov and Elber [17] introduced two new methods for eliminating self-intersections in freeform curve metamorphosis. Both their algorithms exploit the matching algorithm of Cohen et al. [13]. Ye and Maekawa [18] presented algorithms for computing all the differential geometry properties of both transversal and tangentially intersection curves of two parametric surfaces. They described how to obtain these properties for two implicit surfaces or parametric-implicit surfaces. They also gave algorithms to evaluate the higher-order derivative of the intersection curves. Ho and Cohen [19] developed a divide-and-conquer algorithm for computing the self-intersection curves of a surface, which is based on a necessary condition for self-intersection that can be tested using the normal and tangent bounding cones of the surface. Wallner et al. [20] considered the problem of computing the maximum offset distance that guarantees no local or global self-intersections. Patrikalakis et al. [21], introduce a method to find all the self-intersection points of a planar rational polynomial parametric curve. Unlike the curve self-intersection case, it is inefficient to solve surface self-intersection problems with the IPP solver. Thomassen [23] discuss how approximate implicit representations of parametric curves and surfaces may be used in algorithms for finding self-intersections. It have also described how to find the implicit representation given a NURBS curve or surface. Galligo and Pavone [24] presented two different contributions to the determination of a self-intersection locus for a B´ezier bicubic surface. The first one uses a specific sparse resultant and produces an implicit equation of a plane projection of this locus. The second one accurately computes the coordinates of critical points on this locus, by solving a system of four polynomial equations in four variables, derived from a previously computed plane projection of the self-intersection locus. Elber [22] and Seong et al. [25] presented a scheme to trim both local and global self-intersections of offset curves and surfaces. The scheme is based on the derivation of an analytic distance map between the original curve/surface and its offset. The computation of self-intersection of a patch or intersection of two patches are important problems in CAGD; they were the main topics of the European project GAIA II [26]. There are many articles presenting methods and algorithms to compute self-intersections loci [4, 10, 14, 23, 27]. Galligo and Pavone [24] presented two algebraic methods for computing a self-intersection locus for a Bézier bicubic surface. Using a specific sparse resultant, the first method produces an implicit equation of a plane projection of this locus. The second one accurately computes the coordinates of critical points on this locus, by solving a system of four polynomial equations in four variables, derived from a previously computed plane projection of the self-intersection locus. Attempts were similarly made to divide a potentially self-intersecting surface into injective maps [27]. Diana Pekerman et al. [28] present an algorithm for global self-intersection detection and elimination in freeform curves and surfaces. Skytt [29] reported an approach that delineates the topology of the self-intersections; however, in many cases, the topology can be quite complex. L. Buse et al. [30] presented a re-formulation of computer algebra problems related to the determination of the self-intersection and intersection loci of polynomial surfaces patch and in the more general case of rational patches used for NURBS. Diana et al. [31] present several algorithms for self-intersection detection, and possible elimination, in freeform planar curves and surfaces. Both local and global self-intersections are eliminated using a binormal-line criterion and a simple direct algebraic elimination procedure that enables the direct solution of the algebraic (self-)intersection constraints. Gershon Elbera et al. [32] presented an algebraic decomposition that reformulates the surface self-intersection problem using an alternative set of constraints, while removing the redundant components. Xiaohong Jia et al. [33] presented an algorithm to compute the self-intersection curves of a rational ruled surface based on the theory of μ-bases. Soliman et al. [34] provide an algorithm for the evaluation of geometry properties for tangential intersections of two surfaces (implicit-parametric) in $\mathbb{R}^3$. Abdel-All et al. [35]

provide an algorithm for the evaluation of geometry properties for tangential intersections of two implicit surfaces in $\mathbb{R}^3$.

In this paper, we study the differential geometry properties of the intersection curves of two parametric surfaces and the Self-intersection curves of a parametric surface in $\mathbb{R}^3$. The intersection can be transversally or tangentially. The type of intersection may vary point to point along the intersection curve. Finally some examples are given and plotted.

2. Geometric preliminaries

Let us first introduce some notations and definitions. Bold letters such as a, R will be used for vectors and vector functions, respectively. The scalar product and cross product of two vectors a and c are expressed as $\langle a,c\rangle$ and $a\times c$, respectively. The length of the vector a is $\|a\| = \sqrt{\langle a,a\rangle}$.

2.1. **Differential geometry of the curves in $\mathbb{R}^3$.** Let $\alpha\colon I\subset\mathbb{R}\longrightarrow\mathbb{R}^3$ be a regular curve in $\mathbb{R}^3$ with arc-length parametrization,

$$\alpha(s) = (x_1(s), x_2(s), x_3(s)) \tag{2.1}$$

The notations for differentiation of the curve α with respect to the arc length s are $\alpha'(s) = \frac{d\alpha}{ds}$, $\alpha''(s) = \frac{d^2\alpha}{ds^2}$, $\alpha'''(s) = \frac{d^3\alpha}{ds^3}$. From the elementary differential geometry, we have

$$\alpha'(s) = t \tag{2.2}$$

$$\alpha''(s) = \kappa n \tag{2.3}$$

$$\kappa^2(s) = \left\langle \alpha'', \alpha'' \right\rangle \tag{2.4}$$

where t is the unit tangent vector and α'' is the curvature vector. The factor κ is the curvature and n is the unit principal normal vector. The unit binormal vector b is defined as

$$b(s) = t\times n \tag{2.5}$$

The Frenet-Serret formulas along α are given by

$$t'(s) = \kappa n, \quad n'(s) = -\kappa t + \tau b, \quad b'(s) = -\tau n \tag{2.6}$$

where τ is the torsion which is given by

$$\tau = \frac{\langle b, \alpha''' \rangle}{\kappa} \tag{2.7}$$

provided that the curvature does not vanish.

2.2. **Differential geometry of the parametric surfaces in $\mathbb{R}^3$.** Assume that $R(u_1, u_2)$ is a regular parametric surface with $R_1 \times R_2 \neq 0$, where $R_r = \dfrac{\partial R}{\partial u_r}$ $(r = 1, 2)$ denote to the partial derivatives of the surface R. The unit normal vector field on the surface R is given by

$$N = \frac{R_1 \times R_2}{\|R_1 \times R_2\|} \tag{2.8}$$

The coefficients of first fundamental form are given by

$$g_{pq} = \left\langle R_p, R_q \right\rangle; \quad p, q = 1, 2 \tag{2.9}$$

The coefficients of second fundamental form are given by

$$L_{11} = \langle R_{11}, N \rangle, \quad L_{12} = \langle R_{12}, N \rangle, \quad L_{22} = \langle R_{22}, N \rangle \tag{2.10}$$

Let $u_r = u_r(s)$, $r = 1, 2$ be functions in the u_1u_2-plane which defines a curve on the surface R as

$$\alpha(s) = R(u_1(s), u_2(s)). \tag{2.11}$$

Then the fourth derivatives of the curve α are given by

$$\alpha' = R_1 u_1' + R_2 u_2', \tag{2.12}$$

$$\alpha'' = R_{11}(u_1')^2 + 2R_{12}u_1'u_2' + R_{22}(u_2')^2 + R_1 u_1'' + R_2 u_2'', \tag{2.13}$$

$$\begin{aligned} \alpha''' = \; & R_{111}(u_1')^3 + 3R_{112}(u_1')^2 u_2' + 3R_{122}u_1'(u_2')^2 + R_{222}(u_2')^3 \\ & + 3(R_{11}u_1'u_1'' + R_{12}(u_1''u_2' + u_1'u_2'') + R_{22}u_2'u_2'') + R_1 u_1''' + R_2 u_2'''. \end{aligned} \tag{2.14}$$

$$\begin{aligned}\alpha^{(4)}(s) = {} & (u_1')^4R_{1111}+(u_2')^4R_{2222}+4(u_1')^3u_2'R_{1112}+6(u_1')^2(u_2')^2R_{1122}\\ & +4u_1'(u_2')^3R_{1222}+6(u_1')^2u_1''R_{111}+6(u_2')^2u_2''R_{222}\\ & +6(2u_1'u_2'u_1''+(u_1')^2u_2'')R_{112}+6(u_1''(u_2')^2+2u_1'u_2'u_2'')R_{122}\\ & +(3(u_1'')^2+4u_1'u_1''')R_{11}+(3(u_2'')^2+4u_2'u_2''')R_{22}\\ & +2(2u_1'''u_2'+3u_1''u_2''+2u_1'u_2''')R_{12}+u_1^{(4)}R_1+u_2^{(4)}R_2\end{aligned} \tag{2.15}$$

The projection of the curvature vector α'', the third order derivative vector α''' and the fourth order derivative vector $\alpha^{(4)}$ onto the unit normal vector of the surface R, respectively are given by

$$\left\langle \alpha'', \frac{R_1\times R_2}{\|R_1\times R_2\|}\right\rangle = L_{11}(u_1')^2+2L_{12}u_1'u_2'+L_{22}(u_2')^2, \tag{2.16}$$

$$\begin{aligned}\langle \alpha''', N\rangle = {} & (u_1')^3\langle R_{111},N\rangle+3(u_1')^2u_2'\langle R_{112},N\rangle+3u_1'(u_2')^2\langle R_{122},N\rangle\\ & +(u_2')^3\langle R_{222},N\rangle+3(u_1'L_{11}+u_2'L_{12})u_1''+3(u_1'L_{12}+u_2'L_{22})u_2'',\end{aligned} \tag{2.17}$$

$$\begin{aligned}\left\langle \alpha^{(4)}, N\right\rangle = {} & (u_1')^4\langle R_{1111},N\rangle+4(u_1')^3u_2'\langle R_{1112},N\rangle+6(u_1')^2(u_2')^2\langle R_{1122},N\rangle\\ & +(u_2')^4\langle R_{2222},N\rangle+4u_1'(u_2')^3\langle R_{1222},N\rangle+6(u_1')^2u_1''\langle R_{111},N\rangle\\ & +6(u_2')^2u_2''\langle R_{222},N\rangle+6(2u_1'u_2'u_1''+(u_1')^2u_2'')\langle R_{112},N\rangle\\ & +6(u_1''(u_2')^2+2u_1'u_2'u_2'')\langle R_{122},N\rangle+3(u_1'')^2L_{11}+6u_1''u_2''L_{12}\\ & +3(u_2'')^2L_{22}+4(u_1'L_{11}+u_2'L_{12})u_1'''+4(u_1'L_{12}+u_2'L_{22})u_2'''.\end{aligned} \tag{2.18}$$

2.3. **Self-intersection of a parametric surface.** Self-intersection point p of a parametric surface $R=R(u_1,u_2)$; $c_1<u_1<c_2$, $c_3<u_2<c_4$ is defined by finding two pairs of distinct parameter values $(\gamma_1,\gamma_2)\neq(\nu_1,\nu_2)$ in the u_1u_2-plane, such that $p=R(\gamma_1,\gamma_2)=R(\nu_1,\nu_2)$. [21].

Consider a surface $R=R(u_1,u_2)$; $c_1<u_1<c_2$, $c_3<u_2<c_4$, which intersect it self at a curve (with arc length parametrization) $\alpha(s)$. Assume that $(v_1(s),v_2(s))$ and $(w_1(s),w_2(s))$ are two distinct paths in the u_1u_2-plane, defines the curve $\alpha(s)$ (see Fig. 2.1), then we can write

$$\alpha(s)=R(v_1(s),v_2(s))=R(w_1(s),w_2(s));\;\;(v_1(s),v_2(s))\neq(w_1(s),w_2(s)) \tag{2.19}$$

We can consider the surface $R(u_1,u_2)$ as two distinct regular surfaces $P(v_1,v_2)$ and $Q(w_1,w_2)$ which intersect at the curve $\alpha(s)$, where

$$P(v_1,v_2)=R(v_1,v_2),\quad Q(w_1,w_2)=R(w_1,w_2). \tag{2.20}$$

Thus the curve $\alpha(s)$ can be viewed as a curve on both surfaces as

$$\begin{aligned} \alpha(s) &= P(v_1(s), v_2(s)) = (P^1, P^2, P^3), \\ \alpha(s) &= Q(w_1(s), w_2(s)) = (Q^1, Q^2, Q^3). \end{aligned} \tag{2.21}$$

According to (2.21), we can write

$$P^j(v_1(s), v_2(s)) = Q^j(w_1(s), w_2(s)); \ j = 1, 2, 3. \tag{2.22}$$

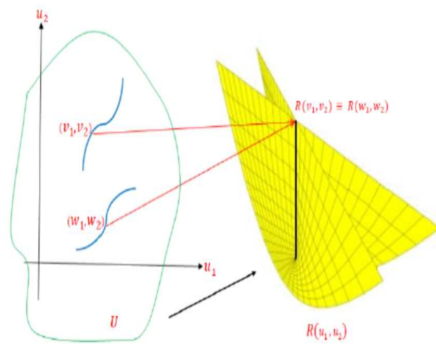

FIGURE 1. Fig 2.1

3. **Transversal intersection**

Assume that the surfaces (2.20) are intersecting transversally at the curve (2.21).

3.1. **The unit tangent vector field.** Differentiation (2.22) with respect to s yields

$$\begin{aligned} P_1^1 v_1' + P_2^1 v_2' &= Q_1^1 w_1' + Q_2^1 w_2', \\ P_1^2 v_1' + P_2^2 v_2' &= Q_1^2 w_1' + Q_2^2 w_2', \\ P_1^3 v_1' + P_2^3 v_2' &= Q_1^3 w_1' + Q_2^3 w_2' \end{aligned} \tag{3.1}$$

Since the surface $Q(w_1(s), w_2(s))$ is regulare, thene without loss of generality, we have

$$\begin{vmatrix} Q_1^l & Q_2^l \\ Q_1^m & Q_2^m \end{vmatrix} \neq 0, \quad \{l, m\} \subset \{1, 2, 3\}$$

The system (3.1) can be written as

$$\begin{aligned} P_1^l v_1' + P_2^l v_2' &= Q_1^l w_1' + Q_2^l w_2', \\ P_1^m v_1' + P_2^m v_2' &= Q_1^m w_1' + Q_2^m w_2', \end{aligned} \tag{3.2}$$

$$P_1^q v_1' + P_2^q v_2' = Q_1^q w_1' + Q_2^q w_2', \quad \{l,m,q\} = \{1,2,3\}. \tag{3.3}$$

where $P_i^j = P_i^j(v_1(s), v_2(s)) = \dfrac{\partial P^j}{\partial u_i}$, $Q_i^j = Q_i^j(w_1(s), w_2(s)) = \dfrac{\partial Q^j}{\partial u_i}$, $i = 1,2$. Solving the coefficients w_1' and w_2' from linear system (3.2) and substituting into (3.3)yields

$$v_1' = v_1', \quad v_2' = -\frac{\eta}{\zeta} v_1'. \tag{3.4}$$

where

$$\eta = Q_1^q A_{12} - Q_2^q A_{11} - B_{12} P_1^q, \quad \zeta = Q_1^q A_{22} - Q_2^q A_{21} - B_{12} P_2^q,$$

$$A_{ij} = \begin{vmatrix} P_i^l & P_i^m \\ Q_j^l & Q_j^m \end{vmatrix}, \qquad B_{12} = \begin{vmatrix} Q_1^l & Q_1^m \\ Q_2^l & Q_2^m \end{vmatrix}, \quad i,j = 1,2. \tag{3.5}$$

Since

$$\sum_{i,j=1}^{2} g_{ij} v_i' v_j' = 1, \quad g_{ij} = \langle P_i, P_j \rangle . \tag{3.6}$$

Substituting (3.4) into (3.6) yields

$$v_1' = \frac{\zeta}{\sqrt{g_{11}\zeta^2 - 2g_{12}\eta\zeta + g_{22}\eta^2}}, \quad v_2' = \frac{-\eta}{\sqrt{g_{11}\zeta^2 - 2g_{12}\eta\zeta + g_{22}\eta^2}}. \tag{3.7}$$

Differentiation (2.21) with respect to s yields

$$t = \alpha'(s) = P_1 v_1' + P_2 v_2' = Q_1 w_1' + Q_2 w_2' \tag{3.8}$$

The unit tangent vector $\alpha'(s)$ can be obtain by substituting (3.7) into (3.8), as follows

$$t = \frac{\zeta P_1 - \eta P_2}{\|\zeta P_1 - \eta P_2\|}. \tag{3.9}$$

Using (3.7), (3.8) and (3.9), we obtain

$$v_1' = \frac{\zeta}{\|\zeta P_1 - \eta P_2\|}, \quad v_2' = \frac{-\eta}{\|\zeta P_1 - \eta P_2\|}; \tag{3.10}$$

Using (3.8), (3.9) and (3.10), we obtain

$$w_1' = \frac{\zeta A_{12} - \eta A_{22}}{B_{12}\|\zeta P_1 - \eta P_2\|}, \quad w_2' = \frac{\eta A_{21} - \zeta A_{11}}{B_{12}\|\zeta P_1 - \eta P_2\|}. \tag{3.11}$$

The tangent vector field of the self-intersection curve of the surface $R = R(u_1,u_2)$ is given by

$$t = \frac{\zeta R_1(v_1,v_2) - \eta R_2(v_1,v_2)}{\|\zeta R_1(v_1,v_2) - \eta R_2(v_1,v_2)\|}. \tag{3.12}$$

Using (3.10) and (3.11) we obtain

$$\begin{aligned} v_1' &= \frac{\zeta}{\|\zeta R_1(v_1,v_2) - \eta R_2(v_1,v_2)\|}, \qquad v_2' = \frac{-\eta}{\|\zeta R_1(v_1,v_2) - \eta R_2(v_1,v_2)\|}, \\ w_1' &= \frac{\zeta A_{12} - \eta A_{22}}{B_{12}\,\|\zeta R_1(v_1,v_2) - \eta R_2(v_1,v_2)\|}, \quad w_2' = \frac{\eta A_{21} - \zeta A_{11}}{B_{12}\,\|\zeta R_1(v_1,v_2) - \eta R_2(v_1,v_2)\|}, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} \eta &= A_{12}R_1^q(w_1,w_2) - A_{11}R_2^q(w_1,w_2) - B_{12}R_1^q(v_1,v_2), \\ \zeta &= A_{22}R_1^q(w_1,w_2) - A_{21}R_2^q(w_1,w_2) - B_{12}R_2^q(v_1,v_2), \\ A_{ij} &= \begin{vmatrix} R_i^l(v_1,v_2) & R_i^m(v_1,v_2) \\ R_j^l(w_1,w_2) & R_j^m(w_1,w_2) \end{vmatrix}, \quad B_{12} = \begin{vmatrix} R_1^l(w_1,w_2) & R_1^m(w_1,w_2) \\ R_2^l(w_1,w_2) & R_2^m(w_1,w_2) \end{vmatrix}. \end{aligned} \tag{3.14}$$

3.2. **Curvature and curvature vector.** Assume that the intersection curve $\alpha(s)$ is given by

$$\alpha(s) = (x_1(s), x_2(s), x_3(s)) \tag{3.15}$$

Then we have

$$\begin{aligned} \alpha'(s) &= (x_1'(s), x_2'(s), x_3'(s)), \qquad \alpha'''(s) = (x_1'''(s), x_2'''(s), x_3'''(s)), \\ \alpha''(s) &= (x_1''(s), x_2''(s), x_3''(s)), \quad \alpha^{(4)}(s) = (x_1^{(4)}(s), x_2^{(4)}(s), x_3^{(4)}(s)). \end{aligned} \tag{3.16}$$

Since the curvature vector is perpendicular on the tangent vector, then we have

$$\begin{bmatrix} x_1' & x_2' & x_3' \end{bmatrix} \begin{bmatrix} x_1'' & x_2'' & x_3'' \end{bmatrix}^T = 0 \tag{3.17}$$

The projection of curvature vector field $\alpha''(s)$ of the curve (2.20) onto the unit normal vector fields $N_1 = (N_1^1, N_1^2, N_1^3)$ and $N_2 = (N_2^1, N_2^2, N_2^3)$ of both surfaces (2.21) are given by

$$\begin{aligned} \begin{bmatrix} N_1^1 & N_1^2 & N_1^3 \end{bmatrix} \begin{bmatrix} x_1'' & x_2'' & x_3'' \end{bmatrix}^T &= (v_1')^2 L_{11}^1 + 2v_1'v_2' L_{12}^1 + (v_2')^2 L_{22}^1, \\ \begin{bmatrix} N_2^1 & N_2^2 & N_2^3 \end{bmatrix} \begin{bmatrix} x_1'' & x_2'' & x_3'' \end{bmatrix}^T &= (w_1')^2 L_{11}^2 + 2w_1'w_2' L_{12}^2 + (w_2')^2 L_{22}^2. \end{aligned} \tag{3.18}$$

Solving the system (3.17)and (3.18) for x_1'', x_2'' and x_3'', we obtain

$$\begin{bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{bmatrix} = \begin{bmatrix} x_1' & x_2' & x_3' \\ N_1^1 & N_1^2 & N_1^3 \\ N_2^1 & N_2^2 & N_2^3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ (v_1')^2 L_{11}^1 + 2v_1'v_2' L_{12}^1 + (v_2')^2 L_{22}^1 \\ (w_1')^2 L_{11}^2 + 2w_1'w_2' L_{12}^2 + (w_2')^2 L_{22}^2 \end{bmatrix} \tag{3.19}$$

The curvature vector field $\alpha''(s)$ can be computed by using (3.12), (3.13) and (3.19). The curvature κ is given by

$$\kappa^2 = \langle \alpha'', \alpha'' \rangle$$

The curvature and curvature vector of self-intersection curves of the surface $R(u_1,u_2)$ are given by replacing the surfaces $P(v_1,v_2)$ and $Q(w_1,w_2)$ with $R(v_1,v_2)$ and $R(w_1,w_2)$, respectively.

3.3. **Torsion and third order derivative.** Since the intersection curve $\alpha(s)$ views as a curve on both surfaces (2.20), then the Eq. (2.13) satisfies on both surfaces thus

$$\alpha'' = (v_1')^2 P_{11} + 2v_1'v_2' P_{12} + (v_2')^2 P_{22} + v_1'' P_1 + v_2'' P_2, \tag{3.20}$$

$$\alpha'' = (w_1')^2 Q_{11} + 2w_1'w_2' Q_{12} + (w_2')^2 Q_{22} + w_1'' Q_1 + w_2'' Q_2. \tag{3.21}$$

Taking the cross product of both hand sides of (3.20) with P_1 and P_2 and projecting the results vector onto the surface normal vector N_1, we obtain

$$\begin{aligned} v_1'' &= \frac{|\alpha'', P_2, N_1|}{\|P_1 \times P_2\|} - (v_1')^2 \frac{|P_{11}, P_2, N_1|}{\|P_1 \times P_2\|} - 2v_1'v_2' \frac{|P_{12}, P_2, N_1|}{\|P_1 \times P_2\|} - (v_2')^2 \frac{|P_{22}, P_2, N_1|}{\|P_1 \times P_2\|}, \\ v_2'' &= \frac{|P_1, \alpha'', N_1|}{\|P_1 \times P_2\|} - (v_1')^2 \frac{|P_1, P_{11}, N_1|}{\|P_1 \times P_2\|} - 2v_1'v_2' \frac{|P_1, P_{12}, N_1|}{\|P_1 \times P_2\|} - (v_2')^2 \frac{|P_1, P_{22}, N_1|}{\|P_1 \times P_2\|}. \end{aligned} \tag{3.22}$$

Taking the cross product of both hand sides of (3.21) with Q_1 and Q_2 and projecting the results vector onto the surface normal vector N_2, we obtain

$$\begin{aligned} w_1'' &= \frac{|\alpha'', Q_2, N_2|}{\|Q_1 \times Q_2\|} - (w_1')^2 \frac{|Q_{11}, Q_2, N_2|}{\|Q_1 \times Q_2\|} - 2w_1'w_2' \frac{|Q_{12}, Q_2, N_2|}{\|Q_1 \times Q_2\|} - (w_2')^2 \frac{|Q_{22}, Q_2, N_2|}{\|Q_1 \times Q_2\|}, \\ w_2'' &= \frac{|Q_1, \alpha'', N_2|}{\|Q_1 \times Q_2\|} - (w_1')^2 \frac{|Q_1, Q_{11}, N_2|}{\|Q_1 \times Q_2\|} - 2w_1'w_2' \frac{|Q_1, Q_{12}, N_2|}{\|Q_1 \times Q_2\|} - (w_2')^2 \frac{|Q_1, Q_{22}, N_2|}{\|Q_1 \times Q_2\|}. \end{aligned} \tag{3.23}$$

Differentiation (2.3) with respect to s and using (2.6)we obtain,

$$\alpha'''(s) = -\kappa^2 t + \kappa' n + \kappa\tau b,$$

then we have

$$\langle \alpha', \alpha''' \rangle = -\kappa^2$$

Which can be written in the matrix form as

$$\begin{bmatrix} x_1' & x_2' & x_3' \end{bmatrix} \begin{bmatrix} x_1''' & x_2''' & x_3''' \end{bmatrix}^T = -\kappa^2 \tag{3.24}$$

The projection of the third order derivative $\alpha'''(s)$ of the intersection curve $\alpha(s)$ onto the unit normal vector fields $N_1 = (N_1^1, N_1^2, N_1^3)$ and $N_2 = (N_2^1, N_2^2, N_2^3)$ of both surfaces $P(v_1, v_2)$ and $Q(w_1, w_2)$ are given by

$$\begin{aligned} \begin{bmatrix} N_1^1 & N_1^2 & N_1^3 \end{bmatrix} \begin{bmatrix} x_1''' & x_2''' & x_3''' \end{bmatrix}^T &= \psi_1, \\ \begin{bmatrix} N_2^1 & N_2^2 & N_2^3 \end{bmatrix} \begin{bmatrix} x_1''' & x_2''' & x_3''' \end{bmatrix}^T &= \psi_2, \end{aligned} \tag{3.25}$$

where

$$\begin{aligned} \psi_1 = {} & (v_1')^3 \langle P_{111}, N_1 \rangle + 3(v_1')^2 v_2' \langle P_{112}, N_1 \rangle + 3v_1'(v_2')^2 \langle P_{122}, N_1 \rangle \\ & + (v_2')^3 \langle P_{222}, N_1 \rangle + 3v_1' v_1'' L_{11}^1 + 3(v_1'' v_2' + v_1' v_2'') L_{12}^1 + 3v_2' v_2'' L_{22}^1, \\ \psi_2 = {} & (w_1')^3 \langle Q_{111}, N_2 \rangle + 3(w_1')^2 w_2' \langle Q_{112}, N_2 \rangle + 3w_1'(w_2')^2 \langle Q_{122}, N_2 \rangle \\ & + (w_2')^3 \langle Q_{222}, N_2 \rangle + 3w_1' w_1'' L_{11}^2 + 3(w_1'' w_2' + w_1' w_2'') L_{12}^2 + 3w_2' w_2'' L_{22}^2. \end{aligned} \tag{3.26}$$

Solving the system (3.25) and (3.25), we obtain

$$\begin{bmatrix} x_1''' \\ x_2''' \\ x_3''' \end{bmatrix} = \begin{bmatrix} x_1' & x_2' & x_3' \\ N_1^1 & N_1^2 & N_1^3 \\ N_2^1 & N_2^2 & N_2^3 \end{bmatrix}^{-1} \begin{bmatrix} -\kappa^2 \\ \psi_1 \\ \psi_2 \end{bmatrix}. \tag{3.27}$$

The third order derivative $\alpha'''(s)$ can be computed by using (3.12), (3.18), (3.26) and (3.27). The torsion τ is given by (2.7). The torsion and third order derivative of self-intersection curves of the surface $R(u_1, u_2)$ are given by replacing the surfaces $P(v_1, v_2)$ and $Q(w_1, w_2)$ to $R(v_1, v_2)$ and $R(w_1, w_2)$, respectively.

4. Tangentially Intersection curves

Assume that the surfaces (2.20) are intersecting tangentially at a point p on the curve (2.21), then the unit surface normal vector fields of both surfaces are parallel to each other. in other words

$$\frac{P_1 \times P_2}{\|P_1 \times P_2\|} = \delta \frac{Q_1 \times Q_2}{\|Q_1 \times Q_2\|}, \quad \delta = \pm 1. \tag{4.1}$$

4.1. **Tangential direction.** Differentiation (2.22), with respect to s yields

$$P_1^j v_1' + P_2^j v_2' = Q_1^j w_1' + Q_2^j w_2', \quad j = 1,2,3 \tag{4.2}$$

where $P_i^j = P_i^j(v_1(s), v_2(s)) = \dfrac{\partial P^j}{\partial u_i}$, $Q_i^j = Q_i^j(w_1(s), w_2(s)) = \dfrac{\partial Q^j}{\partial u_i}$, $i = 1,2$

since the unit surface normal vector fields of both surfaces are parallel to each other, then the system (4.2) reduced to only two Eqs.. Projecting the curvature vector $\alpha''(s)$ onto the two unit normal vector fields of both surfaces and using (4.1), we obtain

$$\left\langle \alpha'', \frac{Q_1 \times Q_2}{\|Q_1 \times Q_2\|} \right\rangle = \delta \left\langle \alpha'', \frac{P_1 \times P_2}{\|P_1 \times P_2\|} \right\rangle$$

Using (2.16) and (4.1), we obtain

$$\sum_{i,j=1}^{2} L_{ij}^2 w_i' w_j' = \delta \sum_{i,j=1}^{2} L_{ij}^1 v_i' v_j' \tag{4.3}$$

where

$$L_{ij}^1 = \langle P_{ij}, N_1 \rangle, \quad L_{ij}^2 = \langle Q_{ij}, N_2 \rangle,$$

Assume that the system (4.2) reduced to

$$\begin{aligned} P_1^l v_1' + P_2^l v_2' &= Q_1^l w_1' + Q_2^l w_2', \\ P_1^m v_1' + P_2^m v_2' &= Q_1^m w_1' + Q_2^m w_2'; \quad \{l,m\} \subset \{1,2,3\}, \end{aligned} \tag{4.4}$$

then we have

$$\begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \frac{1}{B_{12}} \begin{bmatrix} A_{12} & A_{22} \\ -A_{11} & -A_{21} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix}, \; B_{12} \neq 0 \tag{4.5}$$

where

$$A_{ij} = \begin{vmatrix} P_i^l & P_i^m \\ Q_j^l & Q_j^m \end{vmatrix}, \qquad B_{12} = \begin{vmatrix} Q_1^l & Q_1^m \\ Q_2^l & Q_2^m \end{vmatrix}, \quad i,j = 1,2. \tag{4.6}$$

Substituting (4.5) into (4.3) yields

$$a_{11}(\frac{v_2'}{v_1'})^2+2a_{12}(\frac{v_2'}{v_1'})+a_{22}=0; \quad v_1'\neq 0, \tag{4.7}$$

where

$$\begin{aligned}
a_{11}&=(A_{12})^2L_{11}^2-2A_{12}A_{11}L_{12}^2+(A_{11})^2L_{22}^2-\delta(B_{12})^2L_{11}^1\\
a_{12}&=A_{12}A_{22}L_{11}^2-(A_{11}A_{12}+A_{12}A_{21})L_{12}^2+A_{11}A_{21}L_{22}^2-\delta(B_{12})^2L_{12}^1\\
a_{22}&=(A_{22})^2L_{11}^2-2A_{21}A_{22}L_{12}^2+(A_{21})^2L_{22}^2-\delta(B_{12})^2L_{22}^1
\end{aligned} \tag{4.8}$$

Solving (4.7) yield

$$\frac{v_2'}{v_1'}=\frac{-a_{12}\pm\sqrt{(a_{12})^2-a_{11}a_{22}}}{a_{11}}. \tag{4.9}$$

In other words

$$v_2'=\lambda v_1'; \;\; \lambda=\frac{-a_{12}\pm\sqrt{(a_{12})^2-a_{11}a_{22}}}{a_{11}} \tag{4.10}$$

Since $\alpha'(s)$ is the unit tangent vector of the curve $\alpha(s)$ on the surface $P(v_1,v_2)$, then we have

$$\sum_{i,j=1}^{2} g_{ij}v_i'v_j'=1, \quad g_{ij}=\langle P_i,P_j\rangle \tag{4.11}$$

Substituting (4.10) into (4.11) yields

$$\begin{aligned}
v_1'&=\frac{1}{\sqrt{g_{11}+2g_{12}\lambda+g_{22}\lambda^2}},\\
v_2'&=\frac{\lambda}{\sqrt{g_{11}+2g_{12}\lambda+g_{22}\lambda^2}}.
\end{aligned} \tag{4.12}$$

The unit tangent vector of the tangential intersection curves of the parametric surfaces $P(v_1,v_2)$ and$Q(w_1,w_2)$ can be obtained by

$$t=\frac{P_1+\lambda P_2}{\|P_1+\lambda P_2\|}. \tag{4.13}$$

From the previous formulas, it is easy to see that, there are four distinct cases for the solution of (4.7) depending upon the discriminant $\Delta=(a_{12})^2-a_{11}a_{22}$, these cases are as the following:

Lemma 1. *The point p is a branch point of the intersection curve, if $\Delta > 0$ and there is another intersection branch crossing the intersection curve at that point.*

Lemma 2. *The surfaces $P(v_1, v_2)$ and $Q(w_1, w_2)$ intersect at the point p and at its neighborhood, if $\Delta = 0$ and $(a_{11})^2 + (a_{12})^2 + (a_{22})^2 \neq 0$. (Tangential intersection curve).*

Lemma 3. *The point p is an isolated contact point of the surfaces $P(v_1, v_2)$ and $Q(w_1, w_2)$, if $\Delta < 0$.*

Lemma 4. *The surfaces $P(v_1, v_2)$ and $Q(w_1, w_2)$ have contact of at least second order at the point p, if $a_{11} = a_{12} = a_{22} = 0$. (Higher-order contact point).*

Using (4.5),(4.12) and (4.13), we obtain

$$v_1' = \frac{1}{\|P_1 + \lambda P_2\|}, \quad v_2' = \frac{\lambda}{\|P_1 + \lambda P_2\|}; \tag{4.14}$$

Using (4.34) and (4.43), we obtain

$$w_1' = \frac{A_{12} + \lambda A_{22}}{B_{12}\,\|P_1 + \lambda P_2\|}, \quad w_2' = \frac{-A_{11} - \lambda A_{21}}{B_{12}\,\|\zeta P_1 - \eta P_2\|}. \tag{4.15}$$

Then the tangent vector field of the tangential self-intersection curves of the surface $R = R(u_1, u_2)$ is given by

$$t = \frac{R_1(v_1, v_2) + \lambda R_2(v_1, v_2)}{\|R_1(v_1, v_2) + \lambda R_2(v_1, v_2)\|}, \tag{4.16}$$

$$\begin{aligned} v_1' &= \frac{1}{\|R_1(v_1, v_2) + \lambda R_2(v_1, v_2)\|}, \qquad v_2' = \frac{\lambda}{\|R_1(v_1, v_2) + \lambda R_2(v_1, v_2)\|}, \\ w_1' &= \frac{A_{12} + \lambda A_{22}}{B_{12}\,\|R_1(v_1, v_2) + \lambda R_2(v_1, v_2)\|}, \quad w_2' = \frac{-A_{11} - \lambda A_{21}}{B_{12}\,\|R_1(v_1, v_2) + \lambda R_2(v_1, v_2)\|}, \end{aligned} \tag{4.17}$$

where

$$\lambda = \frac{-a_{12} \pm \sqrt{(a_{12})^2 - 4a_{11}a_{22}}}{2a_{11}},$$

$$a_{11} = (A_{12})^2 L_{11}^2 - 2A_{12}A_{11}L_{12}^2 + (A_{11})^2 L_{22}^2 - \delta(B_{12})^2 L_{11}^1,$$

$$a_{12} = A_{12}A_{22}L_{11}^2 - (A_{11}A_{12} + A_{12}A_{21})L_{12}^2 + A_{11}A_{21}L_{22}^2 - \delta(B_{12})^2 L_{12}^1,$$

(4.18)
$$a_{22} = (A_{22})^2 L_{11}^2 - 2A_{21}A_{22}L_{12}^2 + (A_{21})^2 L_{22}^2 - \delta(B_{12})^2 L_{22}^1,$$

$$A_{ij} = \begin{vmatrix} R_i^l(v_1,v_2) & R_i^m(v_1,v_2) \\ R_j^l(w_1,w_2) & R_j^m(w_1,w_2) \end{vmatrix}, \quad L_{ij}^1 = \left\langle R_{ij}(v_1,v_2), N^1 \right\rangle,$$

$$B_{12} = \begin{vmatrix} R_1^l(w_1,w_2) & R_1^m(w_1,w_2) \\ R_2^l(w_1,w_2) & R_2^m(w_1,w_2) \end{vmatrix}, \quad L_{ij}^2 = \left\langle R_{ij}(w_1,w_2), N^2 \right\rangle,$$

4.2. **Curvature and curvature vector.** Since the intersection curve views as a curve on both surfaces, then Eq. (2.13) satisfies on both surfaces thus

(4.19)
$$\begin{aligned} &(v_1')^2 P_{11} + 2v_1'v_2'P_{12} + (v_2')^2 P_{22} + v_1''P_1 + v_2''P_2 \\ &= (w_1')^2 Q_{11} + 2w_1'w_2'Q_{12} + (w_2')^2 Q_{22} + w_1''Q_1 + w_2''Q_2 \end{aligned}$$

Taking the cross product of both hand sides of (4.19) with Q_1 and Q_2 and projecting the resulting equations onto the surface normal vector N_2, we obtain

(4.20)
$$\begin{aligned} w_2'' &= \frac{|Q_1,P_1,N_2|}{\|Q_1 \times Q_2\|} v_1'' + \frac{|Q_1,P_2,N_2|}{\|Q_1 \times Q_2\|} v_2'' + \frac{c_{11}}{\|Q_1 \times Q_2\|}, \\ w_1'' &= \frac{|P_1,Q_2,N_2|}{\|Q_1 \times Q_2\|} v_1'' + \frac{|P_2,Q_2,N_2|}{\|Q_1 \times Q_2\|} v_2'' + \frac{c_{12}}{\|Q_1 \times Q_2\|}. \end{aligned}$$

where

$$\begin{aligned} c_{11} = &\ (v_1')^2 |Q_1, P_{11}, N_2| + 2v_1'v_2' |Q_1, P_{12}, N_2| + (v_2')^2 |Q_1, P_{22}, N_2| \\ & -(w_1')^2 |Q_1, Q_{11}, N_2| - 2w_1'w_2' |Q_1, Q_{12}, N_2| - (w_2')^2 |Q_1, Q_{22}, N_2|, \\ c_{12} = &\ (v_1')^2 |P_{11}, Q_2, N_2| + 2v_1'v_2' |P_{12}, Q_2, N_2| + (v_2')^2 |P_{22}, Q_2, N_2| \\ & -(w_1')^2 |Q_{11}, Q_2, N_2| - 2w_1'w_2' |Q_{12}, Q_2, N_2| - (w_2')^2 |Q_{22}, Q_2, N_2|. \end{aligned} \tag{4.21}$$

Projecting the vector $\alpha'''(s)$ onto N_1 and N_2 then using (4.1), we obtain

$$\begin{aligned} & \delta(v_1'L_{11}^1 + v_2'L_{12}^1)v_1'' + \delta(v_1'L_{12}^1 + 3v_2'L_{22}^1)v_2'' \\ & = (w_1'L_{11}^2 + w_2'L_{12}^2)w_1'' + (w_1'L_{12}^2 + 3w_2'L_{22}^2)w_2'' + \frac{c_{13}}{3}, \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} c_{13} = &\ (w_1')^3 \langle Q_{111}, N_2 \rangle + 3(w_1')^2 w_2' \langle Q_{112}, N_2 \rangle + (w_2')^3 \langle Q_{222}, N_2 \rangle \\ & +3w_1'(w_2')^2 \langle Q_{122}, N_2 \rangle - (v_1')^3 \langle P_{111}, N_2 \rangle - (v_2')^3 \langle P_{222}, N_2 \rangle \\ & -3v_1'(v_2')^2 \langle P_{122}, N_2 \rangle - 3(v_1')^2 v_2' \langle P_{112}, N_2 \rangle. \end{aligned} \tag{4.23}$$

Since the curvature vector is perpendicular on the tangent vector, then we have

$$(v_1'g_{11} + v_2'g_{12})v_1'' + (v_1'g_{12} + v_2'g_{22})v_2'' = - \begin{pmatrix} \langle P_{11}, P_1 \rangle (v_1')^3 + \langle P_{22}, P_2 \rangle (v_2')^3 \\ +(2\langle P_{12}, P_1 \rangle + \langle P_{11}, P_2 \rangle)(v_1')^2 v_2' \\ +(2\langle P_{12}, P_2 \rangle + \langle P_{22}, P_1 \rangle) v_1' (v_2')^2 \end{pmatrix}, \tag{4.24}$$

We can compute v_1'', v_2'', w_1'' and w_2'' by solving (4.20), (4.22) and (4.24).

The curvature vector and the curvature of the tangential intersection curves of the parametric surfaces $P(v_1, v_2)$ and $Q(w_1, w_2)$ can be computed by using (2.13) and (2.4) respectively.

The curvature and curvature vector of the tangential self-intersection curves of the surface $R(u_1, u_2)$ are given by replacing the surfaces $P(v_1, v_2)$ and $Q(w_1, w_2)$ to $R(v_1, v_2)$ and $R(w_1, w_2)$, respectively.

4.3. **Torsion and third order derivative.** The intersection curve views as a curve on both surfaces, then Eq. (2.14) satisfies on both surfaces thus

$$\begin{pmatrix} (v_1')^3P_{111}+3(v_1')^2v_2'P_{112} \\ +3v_1'(v_2')^2P_{122}+(v_2')^3P_{222} \\ +3v_1'v_1''P_{11}+3(v_1''v_2'+v_1'v_2'')P_{12} \\ +3v_2'v_2''P_{22}+v_1'''P_1+v_2'''P_2, \end{pmatrix} = \begin{pmatrix} (w_1')^3Q_{111}+3(w_1')^2w_2'Q_{112} \\ +3w_1'(w_2')^2Q_{122}+(w_2')^3Q_{222} \\ +3w_1'w_1''Q_{11}+3(w_1''w_2'+w_1'w_2'')Q_{12} \\ +3w_2'w_2''Q_{22}+w_1'''Q_1+w_2'''Q_2, \end{pmatrix} \tag{4.25}$$

Taking the cross product of both hand sides of (4.25) with Q_1 and Q_2 and projecting the resulting equations onto the surface normal vector N_2, we obtain

$$\begin{aligned} w_2''' &= \frac{|Q_1,P_1,N_2|}{\|Q_1\times Q_2\|}v_1'''+\frac{|Q_1,P_2,N_2|}{\|Q_1\times Q_2\|}v_2'''+\frac{c_{14}}{\|Q_1\times Q_2\|}, \\ w_1''' &= \frac{|P_1,Q_2,N_2|}{\|Q_1\times Q_2\|}v_1'''+\frac{|P_2,Q_2,N_2|}{\|Q_1\times Q_2\|}v_2'''+\frac{c_{15}}{\|Q_1\times Q_2\|}. \end{aligned} \tag{4.26}$$

where

$$\begin{aligned} c_{14} = {} & (v_1')^3\,|Q_1,P_{111},N_2|+3(v_1')^2v_2'\,|Q_1,P_{112},N_2|+3v_1'(v_2')^2\,|Q_1,P_{122},N_2| \\ & -3w_1'(w_2')^2\,|Q_1,Q_{122},N_2|+3v_1'v_1''\,|Q_1,P_{11},N_2|+3v_2'v_2''\,|Q_1,P_{22},N_2| \\ & +(v_2')^3\,|Q_1,P_{222},N_2|-3w_1'w_1''\,|Q_1,Q_{11},N_2|-3(w_1')^2w_2'\,|Q_1,Q_{112},N_2| \\ & -(w_1')^3\,|Q_1,Q_{111},N_2|-(w_2')^3\,|Q_1,Q_{222},N_2|-3w_2'w_2''\,|Q_1,Q_{22},N_2| \\ & +3(v_1''v_2'+v_1'v_2'')\,|Q_1,P_{12},N_2|-3(w_1''w_2'+w_1'w_2'')\,|Q_1,Q_{12},N_2|, \\ c_{15} = {} & (v_1')^3\,|P_{111},Q_2,N_2|+3(v_1')^2v_2'\,|P_{112},Q_2,N_2|+3v_1'(v_2')^2\,|P_{122},Q_2,N_2| \\ & +(v_2')^3\,|P_{222},Q_2,N_2|+3v_1'v_1''\,|P_{11},Q_2,N_2|-3(w_1')^2w_2'\,|Q_{112},Q_2,N_2| \\ & +3v_2'v_2''\,|P_{22},Q_2,N_2|-3w_1'(w_2')^2\,|Q_{122},Q_2,N_2|-3w_2'w_2''\,|Q_{22},Q_2,N_2| \\ & -(w_1')^3\,|Q_{111},Q_2,N_2|-(w_2')^3\,|Q_{222},Q_2,N_2|-3w_1'w_1''\,|Q_{11},Q_2,N_2| \\ & +3(v_1''v_2'+v_1'v_2'')\,|P_{12},Q_2,N_2|-3(w_1''w_2'+w_1'w_2'')\,|Q_{12},Q_2,N_2|, \end{aligned} \tag{4.27}$$

Projecting the vector $\alpha^{(4)}(s)$ onto the two unit normal vector fields of both surfaces and using (2.18) and (4.1), we obtain

$$\begin{aligned} & (w_1'L_{11}^2+w_2'L_{12}^2)w_1'''+(w_1'L_{12}^2+w_2'L_{22}^2)w_2''' \\ & =\delta(v_1'L_{11}^1+v_2'L_{12}^1)v_1'''+\delta(v_1'L_{12}^1+v_2'L_{22}^1)v_2'''+\frac{c_{16}}{4} \end{aligned} \tag{4.28}$$

where

$$
\begin{aligned}
c_{16} = \ & (v_1')^4 \langle P_{1111}, N_2\rangle + 4(v_1')^3 v_2' \langle P_{1112}, N_2\rangle + 6(v_1')^2(v_2')^2 \langle P_{1122}, N_2\rangle \\
& + (v_2')^4 \langle P_{2222}, N_2\rangle + 4v_1'(v_2')^3 \langle P_{1222}, N_2\rangle + 6(v_1')^2 v_1'' \langle P_{111}, N_2\rangle \\
& + 6(v_2')^2 v_2'' \langle P_{222}, N_2\rangle + 6(2v_1'v_2'v_1'' + (v_1')^2 v_2'') \langle P_{112}, N_2\rangle \\
& + 6(v_1''(v_2')^2 + 2v_1'v_2'v_2'') \langle P_{122}, N_2\rangle + 3\delta(v_1'')^2 L_{11}^1 + 6\delta v_1''v_2'' L_{12}^1 \\
& + 3\delta(v_2'')^2 L_{22}^1 - (w_1')^4 \langle Q_{1111}, N_2\rangle - 4(w_1')^3 w_2' \langle Q_{1112}, N_2\rangle \\
& - 6(w_1')^2(w_2')^2 \langle Q_{1122}, N_2\rangle - (w_2')^4 \langle Q_{2222}, N_2\rangle - 3(w_1'')^2 L_{11}^2 \\
& - 4w_1'(w_2')^3 \langle Q_{1222}, N_2\rangle - 6(w_1')^2 w_1'' \langle Q_{111}, N_2\rangle - 6w_1''w_2'' L_{12}^2 \\
& - 6(w_2')^2 w_2'' \langle Q_{222}, N_2\rangle - 6(2w_1'w_2'w_1'' - (w_1')^2 w_2'') \langle Q_{112}, N_2\rangle \\
& - 6(w_1''(w_2')^2 - 2w_1'w_2'w_2'') \langle Q_{122}, N_2\rangle - 3(w_2'')^2 L_{22}^2
\end{aligned} \tag{4.29}
$$

Since

$$\left\langle \alpha', \alpha''' \right\rangle = -\kappa^2$$

which can be written as

(4.30)

$$
\begin{aligned}
(v_1' g_{11} + v_2' g_{12}) v_1''' + (v_1' g_{12} + v_2' g_{22}) v_2''' = -(\ & \kappa^2 + (v_1')^4 \langle P_{111}, P_1\rangle + 3(v_1')^3 v_2' \langle P_{112}, P_1\rangle \\
& + 3(v_1')^2(v_2')^2 \langle P_{122}, P_1\rangle + v_1'(v_2')^3 \langle P_{222}, P_1\rangle \\
& 3(v_1'v_2'v_1'' + (v_1')^2 v_2'') \langle P_{12}, P_1\rangle + 3(v_1')^2 v_1'' \langle P_{11}, P_1\rangle \\
& + 3v_1'v_2'v_2'' \langle P_{22}, P_1\rangle + (v_1')^3 v_2' \langle P_{111}, P_2\rangle \\
& + 3(v_1')^2(v_2')^2 \langle P_{112}, P_2\rangle + 3v_1'(v_2')^3 \langle P_{122}, P_2\rangle \\
& + (v_2')^4 \langle P_{222}, P_2\rangle + 3v_1'v_2'v_1'' \langle P_{11}, P_2\rangle \\
& + 3(v_1''(v_2')^2 + v_1'v_2'v_2'') \langle P_{12}, P_2\rangle + 3(v_2')^2 v_2'' \langle P_{22}, P_2\rangle)
\end{aligned}
$$

We can compute v_1''', v_2''', w_1''' and w_2''' by solving (4.26), (4.28) and (4.30).

The third derivative vector and the torsion of the tangential intersection curves of two parametric surfaces $P(v_1, v_2)$ and $Q(w_1, w_2)$ can be computed by using (2.14) and (2.7), respectively.

The third derivative vector and the torsion of the tangential self-intersection curves of the surface $R(u_1, u_2)$ are given by replacing the surfaces $P(v_1, v_2)$ and $Q(w_1, w_2)$ to $R(v_1, v_2)$ and $R(w_1, w_2)$, respectively.

5. Examples

Example **1.** Consider the two parametric surfaces

$$P = (1+\sin v_2, \cos v_2, v_1)$$
$$Q = (2\cos w_1 \cos w_2, 2\sin w_1 \cos w_2, 2\sin w_2) \tag{5.1}$$

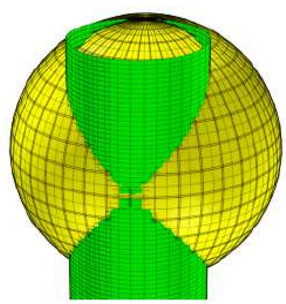

FIGURE 2. Fig 5.1

Transversal intersection: Using (3.9) and (5.1), we obtain

$$t = \frac{(\cos v_2 \sin w_2, -\sin v_2 \sin w_2, -\cos w_2 \cos(v_2+w_1))}{\sqrt{1-\cos^2 w_2 \sin^2(w_1+v_2)}}. \tag{5.2}$$

Using (3.19) and (5.1), we obtain

$$\alpha'' = (\frac{a_1}{a_4}, \frac{a_2}{a_4}, \frac{a_3}{a_4}), \tag{5.3}$$

where

$$
\begin{aligned}
a_1 = & \cos v_2 \cos^3(v_2+w_1)\cos^3 w_2 + 2\sin w_1 \cos(v_2+w_1)\cos^4 w_2 \\
& -\cos v_2 \cos(v_2+w_1)\cos^3 w_2 - 2\sin w_1 \cos(v_2+w_1)\cos^2 w_2 \\
& +\cos v_2 \cos(v_2+w_1)\cos w_2 + 2\sin v_2 \cos^4 w_2 - 4\sin v_2 \cos^2 w_2 + 2\sin v_2
\end{aligned}
$$

$$
\begin{aligned}
a_2 = & -\sin v_2 \cos^3(v_2+w_1)\cos^3 w_2 - 2\cos w_1 \cos(v_2+w_1)\cos^4 w_2 \\
& -\sin v_2 \cos(v_2+w_1)\cos w_2 + 2\cos v_2 \cos^4 w_2 - 4\cos v_2 \cos^2 w_2 \\
& +\sin v_2 \cos(v_2+w_1)\cos^3 w_2 + 2\cos w_1 \cos(v_2+w_1)\cos^2 w_2 + 2\cos v_2
\end{aligned} \tag{5.4}
$$

$$
a_3 = \sin w_2 \begin{pmatrix} \cos^2(v_2+w_1)\cos^2 w_2 - 2\cos v_2 \cos w_2 \sin w_1 \\ -2\cos w_1 \cos w_2 \sin v_2 + 2\cos v_2 \cos^3 w_2 \sin w_1 \\ +2\cos w_1 \cos^3 w_2 \sin v_2 - \cos^2 w_2 + 1 \end{pmatrix}
$$

$$
a_4 = \frac{1}{8}\begin{pmatrix} \cos(2v_2+2w_1-2w_2) + \cos(2v_2+2w_1+2w_2) \\ -2\cos 2w_2 + 2\cos(2v_2+2w_1) + 6 \end{pmatrix}
$$

Using (3.21) and (5.1) we obtain

$$
\begin{bmatrix} x_1''' \\ x_2''' \\ x_3''' \end{bmatrix} = \begin{bmatrix} \frac{\cos v_2 \sin w_2}{\sqrt{1-\cos^2 w_2 \sin^2(w_1+v_2)}} & \frac{-\sin v_2 \sin w_2}{\sqrt{1-\cos^2 w_2 \sin^2(w_1+v_2)}} & \frac{-\cos w_2 \cos(v_2+w_1)}{\sqrt{1-\cos^2 w_2 \sin^2(w_1+v_2)}} \\ \sin v_2 & \cos v_2 & 0 \\ \cos w_1 \cos w_2 & \sin w_1 \cos w_2 & \sin w_2 \end{bmatrix}^{-1} \tag{5.5}
$$

$$
\times \begin{bmatrix} -\frac{(a_1)^2 + (a_2)^2 + (a_3)^2}{(a_4)^2} \\ -3v_2' v_2'' \\ 3(w_1')^2 w_2' \sin 2w_2 - 6w_1' w_1'' \cos^2 w_2 - 6w_2' w_2'' \end{bmatrix}
$$

Tangentially intersection: The surfaces are intersecting tangentially at the point $p(2,0,0)$ as shown in Fig. (5.1). The first Equation in the system (4.2) vanishes at the point $p(2,0,0)$, then by using (4.7) at $p(2,0,0)$, we obtain $\Delta > 0$, this means that the point $p(2,0,0)$ is a branch point, $\lambda = \pm 1$. Using (4.13)and (5.1) at the point $p(2,0,0)$, we obtain

(5.6) $$t = (0, \pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}).$$

Using (4.20), (4.24), (2.4), (2.13) and (5.1) at the point $p(2,0,0)$, we obtain

(5.7) $$n = (-1,0,0), \quad \kappa = \tfrac{1}{2}.$$

(5.8) $$b = (0, -\frac{1}{2}\sqrt{2}, \pm\frac{1}{2}\sqrt{2})$$

Using (4.26), (4.28), (4.30), (2.7), (2.14) and (5.1) at the point $p(2,0,0)$, we obtain

$$\alpha'''(s) = (0, \pm\tfrac{1}{32}\sqrt{2}, -\tfrac{9}{32}\sqrt{2}) \quad \tau = \mp\tfrac{5}{8}.$$

Example **2:** Consider the surface

(5.9) $$R(u_1,u_2) = (u_1 - \frac{u_1^3}{3} + u_1u_2^2, -u_2 - u_1^2u_2 + \frac{u_2^3}{3}, u_1^2 - u_2^2); \quad -5 < u_1, u_2 < 5.$$

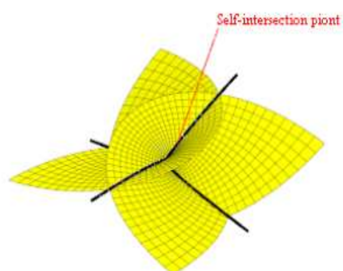

FIGURE 3. Fig 5.2

Let us find the Frenet vectors, the curvature and the torsion of the transversal self-intersection curve at the transversal self-intersection point $p(\frac{28}{3}\sqrt{2}, 0, -7) = R(\sqrt{2}, 3) = R(\sqrt{2}, -3) \in R(u_1,u_2)$. Using (3.12) and (5.2) at the point $p(\frac{28}{3}\sqrt{2}, 0, -7)$, we obtain

(5.10) $$t = (\frac{5}{9}\sqrt{3}, 0, -\frac{1}{9}\sqrt{6}).$$

Using (3.19), (2.4), (2.13) and (5.2) at the point $p(\frac{28}{3}\sqrt{2}, 0, -7)$, we obtain

(5.11) $$\kappa n = (\tfrac{1}{972}\sqrt{2}, 0, \tfrac{5}{972}), \quad \kappa = \tfrac{1}{324}\sqrt{3}, \quad n = (\tfrac{1}{9}\sqrt{6}, 0, \tfrac{5}{9}\sqrt{3}), \quad b = (0,-1,0).$$

Using (3.21), (2.7), (2.14) and (5.2) at the point $p(\frac{28}{3}\sqrt{2},0,-7)$, we obtain

(5.12) $$\alpha''' = (-\tfrac{23}{314928}\sqrt{3},0,-\tfrac{11}{78732}\sqrt{6}), \quad \tau = 0.$$

6. Conclusion

Algorithms for computing all the differential geometry properties of, self-intersection curves of a parametric surface and the intersection curves of two parametric surfaces in $\mathbb{R}^3$, for transversal and tangential intersection. This paper is an extension to the works of Ye and Maekawa (1999). They gave an example of implicit-parametric surfaces intersection and they computed the tangent vector field and refired to how obtain the curvature vector, the curvature, the torsion and the higher derivatives of the intersection curves by using they method. But this paper introduce a direct formulas to compute all the properties. The types of singularity on the intersection curve are characterized. The questions of how to exploit and extend these algorithms to compute the differential geometry properties of intersection curves between three surfaces in $\mathbb{R}^4$, can be topics of future research.

Conflicts of Interests

The authors declare that there is no conflict of interests.

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