Available online at http://scik.org
J. Math. Comput. Sci. 7 (2017), No. 1, 12-29
ISSN: 1927-5307

EXAMINING MOTION OF A ROBOT END-EFFECTOR VIA THE CURVATURE THEORY OF DUAL LORENTZIAN CURVES

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Abstract. In this paper, we study the motion of a robot end-effector using the curvature theory of a dual Lorentzian unit spherical spacelike curve which corresponds to a timelike ruled surface with spacelike ruling generated by a line fixed in the end-effector. In this way, we determine linear and angular time dependent differential properties of motion such as velocities and accelerations which are important information in kinematics and robot trajectory planning. Moreover, motion of a robot end-effector in Lorentzian space whose generating surface is a helicoid is examined as a practical example.

Keywords: Curvature theory; dual Darboux frame; dual Lorentzian space; dual tool frame; robot end-effector; trajectory planning.

2010 AMS Subject Classification: 53A17, 53A25, 53A35.

1. Introduction

Motion of a robot end-effector is referred to as robot trajectory, and the process of computing this robot trajectory is called robot trajectory planning [14]. Recently, since robot end-effectors can be used in transportation, painting, welding industry, medical science, and etc., accurate

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Received August 15, 2016

trajectory planning of a robot end-effector becomes an important research area of robotics and engineering.

In the research area of robot trajectory planning, two important problems attract author's attention. These are describing path of a robot end-effector and determining linear and angular time-dependent properties of motion. The well-known methods used for these problems in robot trajectory planning are the joint interpolating method and the Cartesian interpolating method [3,10]. These methods have not been sufficient to allow a robot end-effector to follow a smooth and differentiable trajectory and they have not been efficient for computer programming since they have been based on the matrix representation which requires excessive computation. As an alternative to these traditional methods, Ryuh and Pennock proposed a new method based on the curvature theory of a ruled surface generated by a line fixed in the end-effector. It was the first attempt to use the curvature theory of ruled surfaces for robot trajectory planning. They gave a relationship between differential geometry of a ruled surface and kinematics of a robot end-effector.

The research area of robot trajectory planning is not only interesting for researcher who study in Euclidean space but also for researchers who study in Lorentzian space. Ekici et al., for example, used the curvature theory of timelike ruled surface with timelike ruling to study the motion of a robot end-effector in Lorentzian space [4].

As a robot end-effector moves on a specified trajectory in Lorentzian space, a line fixed in the end-effector generates a ruled surface. In this paper, we assume that this ruled surface is a timelike ruled surface with spacelike ruling. First, we introduce three reference frames which can be used to study the motion of a robot end-effector in Lorentzian space. From transference principle (also known as E.Study mapping), a dual Lorentzian unit spherical spacelike curve corresponds to this generating ruled surface. By the aid of this mapping, we are allowed to use the dual Lorentzian curve instead of a ruled surface to study the motion of robot end-effector moves in Lorentzian space. Then, we give dual Darboux frame of the dual Lorentzian curve and define a dual frame called dual tool frame on robot end-effector. By using the relationship between these two frames, we determine time-dependent linear and angular differential properties of motion of a robot end-effector such as velocities and accelerations which play important roles in kinematics and robot trajectory planning.

2. Preliminaries

In this section, we give a brief summary of Lorentzian space, dual space, dual Lorentzian space, and some basic concepts in these spaces.

The Lorentzian space IR_1^3 is the vector space IR^3 provided with the following Lorentzian inner product

$$\langle a,b\rangle = a_1b_1 + a_2b_2 - a_3b_3$$

where $a,b \in IR^3$ [8].

Let $a = (a_1,\ a_2,\ a_3)$ and $b = (b_1,\ b_2,\ b_3)$ be two vectors in IR_1^3, the Lorentzian vector product of a and b can be defined by [18]

$$a\times b = (-a_2b_3 + a_3b_2,\ a_1b_3 - a_3b_1, a_1b_2 - a_2b_1).$$

If $\langle a,a\rangle > 0$ or $a = 0$, then a is called spacelike, if $\langle a,a\rangle < 0$, then a is called timelike, if $\langle a,a\rangle = 0$ and $a \neq 0$, then a is called null (lightlike) vector [8].

Let a and b be spacelike vectors in IR_1^3 and they span a timelike vector subspace, then there is a real number $\theta \geq 0$ such that $|\langle a,b\rangle| = \|a\|\,\|b\|\cosh\theta$, where θ is called the central angle between the vectors a and b [11].

A surface in Lorentzian space IR_1^3 is called a timelike surface if the induced metric on the surface is a Lorentzian metric, i.e., the normal vector on the surface is a spacelike vector [17].

A dual number, as introduced by W. Clifford, can be defined as an ordered pair of real numbers $\bar{a} = (a,a^*)$, where a and a^* are called real part and dual part of the dual number, respectively. The set of all dual numbers is denoted by ID. The dual numbers can also be expressed as $\bar{a} = a + \varepsilon a^*$, where ε is the dual unit which satisfies the condition that $\varepsilon^2 = 0$ [20]. Two inner operations and equality in ID can be defined as follows [1,5]:

Addition : $(a,a^*) + (b,b^*) = (a+b,\ a^* + b^*)$,

Multiplication : $(a,a^*)\,(b,b^*) = (ab,\ ab^* + a^*b)$, Equality : $(a,a^*) = (b,b^*) \Leftrightarrow a = b, \ a^* = b^*$.

The set ID with the above operations is a commutative ring over the real number field.

Since $\varepsilon^n = 0$ for $n > 1$, a function of a dual number $f(\bar{a})$ can be expanded in Maclaurin series as

$$f(\bar{a}) = f(a+\varepsilon a^*) = f(a) + \varepsilon a^* f'(a),$$

where $f'(a)$ is the derivative of $f(a)$ with respect to a [2].

Similar to dual numbers, a dual vector can be defined as an ordered pair of two real vectors (a,a^*), where $a, a^* \in IR^3$. Dual vectors can also be expressed as $\tilde{a}= a+\varepsilon a^*$, where $\varepsilon^2 = 0$ [15]. The set of all dual vectors denoted by ID^3 is a module over the ring ID and is called dual space or ID-module.

For two dual vectors $\tilde{a}= a+\varepsilon a^*$ and $\tilde{b}= b+\varepsilon b^*$, the dual Lorentzian inner product and the dual Lorentzian vector product can be defined, respectively, as [19]

$$\left\langle \tilde{a},\tilde{b} \right\rangle = \langle a,b \rangle + \varepsilon \left(\langle a,b^* \rangle + \langle a^*,b \rangle \right)$$

and

$$\tilde{a} \times \tilde{b} = a \times b + \varepsilon \left(a \times b^* + a^* \times b \right),$$

where $\langle \ , \ \rangle$ and $\times$ denote the Lorentzian inner product and the Lorentzian vector product, respectively. A dual vector $\tilde{a}= a+\varepsilon a^*$ is said to be timelike if a is timelike, spacelike if a is spacelike or $a = 0$ and lightlike (null) if a is lightlike (null) and $a \neq 0$ [19]. The set of all dual Lorentzian vectors is called dual Lorentzian space and it is denoted by ID_1^3 [19].

Let $\tilde{a} = a + \varepsilon a^* \in ID_1^3$. Then $\tilde{a}$ is said to be dual unit timelike (resp. spacelike) vector if the vectors a and a^* satisfy the following equations:

$$\langle a,a \rangle = -1 \ (resp. \ \langle a,a \rangle = 1), \ \langle a\,, a^* \rangle = 0.$$

The set of all dual unit spacelike vectors is called dual Lorentzian unit sphere, and can be denoted by $\tilde{S}_1^2$ [19].

Let $\tilde{a}$ and $\tilde{b}$ be dual spacelike vectors in ID_1^3 that span a dual timelike vector subspace. Then there is a dual number $\bar{\theta} = \theta + \varepsilon \theta^*$ called the dual central angle such that $\left\langle \tilde{a}, \tilde{b} \right\rangle = \|\tilde{a}\| \left\|\tilde{b}\right\| \cosh \bar{\theta}$, where θ and θ^* are central angle and Lorentzian shortest distance between the lines corresponding the dual spacelike vectors $\tilde{a}$ and $\tilde{b}$, respectively [19].

There exists one-to-one correspondence between dual unit vectors in dual space ID^3 and the directed lines in line space IR^3 [7,16]. This correspondence is known as E. Study mapping or transference principle. If we consider 3-dimensional Lorentzian space IR^3_1 instead of IR^3, we have the correspondence which can be stated that: The spacelike unit vectors of the Lorentzian unit sphere $\tilde{S}^2_1$ in the dual Lorentzian space ID^3_1 are in one-to-one correspondence with the directed spacelike lines in Lorentzian space IR^3_1 [19]. Thus, a timelike ruled surface with spacelike ruling in Lorentzian space can be represented by a dual spacelike unit spherical curve lying fully on $\tilde{S}^2_1$.

3. Reference Frames of a Robot End-Effector in Lorentzian Space

In [4], three reference frames were defined to study the motion of a robot end-effector moving in Lorentzian space such that generating ruled surface is a timelike ruled surface with timelike ruling. In this section, similar to the procedure used in [4], we introduce three reference frames for the case of timelike ruled surface with spacelike ruling. These frames are a tool frame, a surface frame and a generator trihedron. The aim of this section is to provide better understanding the geometrical meaning of the dual procedure to be used in Section 4.

Analogously in [4,12], a tool frame defined on robot end-effector consists of three orthonormal vectors: the orientation vector O, the approach vector A, and the normal vector N (see Figure 1). The origin of the tool frame is called tool center point and denoted by TCP.

As a robot end-effector moves on a specified trajectory in Lorentzian space, a line called tool line which passes through TCP and whose direction vector is parallel to the orientation vector O, generates a ruled surface. This ruled surface is assumed to be timelike ruled surface with spacelike ruling in this paper and it can be expressed by

$$X(t,v) = \alpha(t) + v\,R(t)$$

where $\alpha(t)$ is the specified trajectory traced by TCP, $R(t)$ is the spacelike ruling of the ruled surface with constant magnitude which is parallel to the orientation vector O, v is an arbitrary real valued parameter and t is the parameter of time. For simplicity in formulations, a normalized parameter s which is the arc-length parameter of the spherical image curve of R can be used instead of the parameter of time t and it can be obtained as

$$s(t) = \int_0^t \left\| \frac{dR}{dt} \right\| dt.$$

From now on, it is assumed that approach vector A is a spacelike vector during motion. This assumption restricts the motion that a robot end-effector moves in a restricted area such that the approach vector A should be always a spacelike vector. But in general, the approach vector A may be timelike or null vector during the motion. The procedure to be used to study this restricted motion of a robot end-effector can be adapted to the case of the Lorentzian character of A which may be timelike or null. Since A is assumed to be a spacelike vector and the ruled surface is a timelike ruled surface with spacelike ruling, the orientation vector O should be a space vector and the normal vector N should be a timelike vector.

The spacelike approach vector A may not be always perpendicular to the ruled surface in Lorentzian manner as a robot end-effector moves in Lorentzian space. As shown in Figure 1, there may be an angle between the spacelike approach vector A and the spacelike surface normal vector which can be denoted by S_n. This angle is called central spin angle and it can be denoted by η.

A surface frame which consists of the spacelike orientation vector O, the spacelike surface normal vector S_n and the timelike surface binormal vector S_b is defined on the directrix of the ruled surface to describe the orientation of the tool frame relative to the ruled surface. The spacelike surface normal vector S_n can be expressed as

$$S_n = \frac{X_v \times X_s}{\|X_v \times X_s\|}$$

where X_v and X_s are derivations of X with respect to v and s, respectively. The timelike surface binormal vector S_b can also be given as

$$S_b = O \times S_n.$$

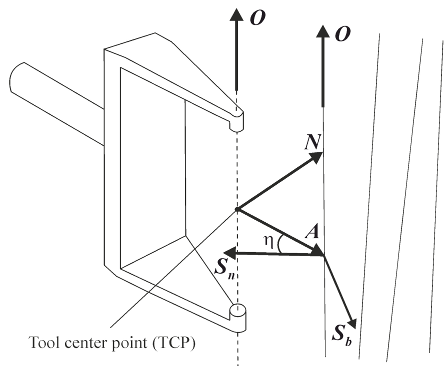

Figure 1 A robot end-effector, its tool frame and the surface frame.

The tool frame and the surface frame have a common vector O, and the relation between these frames can be expressed by

$$\begin{bmatrix} O \\ A \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\eta & \sinh\eta \\ 0 & \sinh\eta & \cosh\eta \end{bmatrix} \begin{bmatrix} O \\ S_n \\ S_b \end{bmatrix} \tag{3.1}$$

where η is a central spin angle which is the angle between A and S_n.

The generator trihedron is defined on the line of striction of the ruled surface which is a unique curve given by $c(s) = \alpha(s) - \mu(s)R(s)$, where $\mu = \langle \alpha', R' \rangle$. The generator trihedron consists of three orthonormal vector: the spacelike generator vector e, the spacelike central normal vector t, and the timelike central tangent vector g. These vectors can be given as, respectively,

$$e = \frac{R}{\|R\|}, \quad t = R', \quad g = e \times t,$$

where the prime indicates differentiation with respect to s. The first-order derivation formulae of the generator trihedron can be expressed in matrix form as

$$\frac{d}{ds}\begin{bmatrix} e \\ t \\ g \end{bmatrix} = \frac{1}{\|R\|}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \gamma \\ 0 & \gamma & 0 \end{bmatrix}\begin{bmatrix} e \\ t \\ g \end{bmatrix},$$

where γ is called geodesic curvature of a timelike ruled surface with spacelike ruling.

The generator vector e is a common vector of the generator trihedron and the surface frame. The vectors t and g of generator trihedron and the vectors S_n and S_b of the surface frame are in the same Lorentzian plane. Thus, the orientation of the surface frame relative to the generator trihedron can be given in matrix form as

$$\begin{bmatrix} O \\ S_n \\ S_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\sigma & \sinh\sigma \\ 0 & \sinh\sigma & \cosh\sigma \end{bmatrix} \begin{bmatrix} e \\ t \\ g \end{bmatrix}, \tag{3.2}$$

where σ is a central angle between S_n and t (see Figure 2).

By substituting equation (3.1) into equation (3.2), the relation between the tool frame and the generator trihedron can be obtained as

$$\begin{bmatrix} O \\ A \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\varphi & \sinh\varphi \\ 0 & \sinh\varphi & \cosh\varphi \end{bmatrix} \begin{bmatrix} e \\ t \\ g \end{bmatrix},$$

where

$$\varphi = \eta + \sigma. \tag{3.3}$$

Figure 2 shows the relations between the three reference frames which are the tool frame, the surface frame and the generator trihedron as a robot end-effector moves on a specified trajectory.

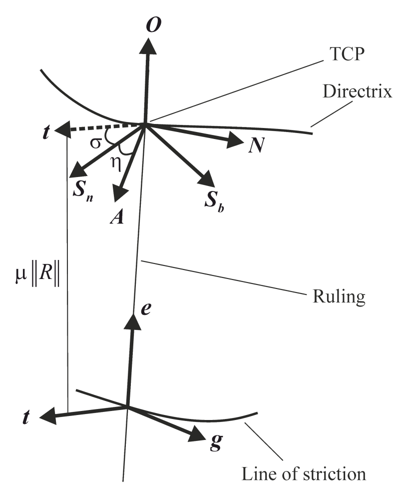

Figure 2 Relations between three reference frames.

4. Differential properties of the motion of a robot end-effector

In [9], the dual representation and the dual Darboux frame of a timelike ruled surface with timelike ruling are given in detail. In this section, similar to the procedure used in [9], we give the dual Darboux frame of a timelike ruled surface with spacelike ruling which corresponds to a dual Lorentzian unit spherical spacelike curve. Then we define a dual tool frame on robot end-effector and determine the time dependent differential properties of the motion of a robot end-effector by using the relationship between the dual tool frame and the dual Darboux frame.

The timelike ruled surface with spacelike ruling which is generated by a line fixed in the end-effector can be given by the equation

$$X(s,v)=\alpha(s)+v\,R(s)$$

where α is the directrix of the ruled surface (specified trajectory of robot end-effector), R is the ruling with constant magnitude and s is the arc-length parameter of spherical image curve of R. From E. Study mapping (or transference principle), this ruled surface corresponds to a dual spacelike curve on dual Lorentzian unit sphere which can be expressed as $\tilde{e}(s)=e(s)+\varepsilon e^*(s)$, where e is the spacelike generator vector of the ruled surface as defined in Section 3, e^* is the moment vector of e about the origin and it can be determined as $e^*=c\times e$, where c is the line of striction of the ruled surface.

The dual Darboux frame of a dual Lorentzian unit spherical spacelike curve consists of three orthonormal dual unit vectors. The first is the dual curve itself, i.e., $\tilde{e}(s)$. Dual arc-length of the dual curve $\tilde{e}$ can be given by [9]

$$\bar{s}=\int_0^s \parallel \tilde{e}'(u)\parallel du=\int_0^s(1-\varepsilon\Delta)du=s-\varepsilon\int_0^s\Delta du,$$

where $\Delta=\det(c',e,t)$. The second and the third dual unit vectors of the dual Darboux frame can be given, respectively, as

$$\tilde{t}=\frac{d\tilde{e}}{d\bar{s}}=\frac{\tilde{e}'}{\bar{s}'}=\frac{\tilde{e}'}{1-\varepsilon\Delta}=t+\varepsilon(c\times t).$$

and

$$\tilde{g}=\tilde{e}\times\tilde{t}=g+\varepsilon(c\times g).$$

where t and g are the spacelike central normal vector and the timelike central tangent vector, respectively, as defined in Section 3.

The derivation formulae of the dual Darboux frame of a dual curve lying on dual Lorentzian unit sphere can also be expressed in matrix form as

$$\frac{d}{d\bar{s}}\begin{bmatrix}\tilde{e}\\ \tilde{t}\\ \tilde{g}\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & \bar{\gamma}\\ 0 & \bar{\gamma} & 0\end{bmatrix}\begin{bmatrix}\tilde{e}\\ \tilde{t}\\ \tilde{g}\end{bmatrix}, \tag{4.1}$$

where $\bar{\gamma}$ is called dual geodesic curvature.

Dual tool frame to be used to study the motion of a robot end-effector can be defined by three orthonormal dual unit vectors which correspond to three lines which pass through TCP of robot end-effector. These lines can be called orientation line, approach line and normal line, and their direction vectors are the spacelike orientation vector O, the spacelike approach vector A and the timelike normal vector N, respectively (see Figure 3). From E.Study mapping, the orientation line, the approach line and the normal line correspond to three dual unit vectors which may be denoted by $\tilde{O}$, $\tilde{A}$ and $\tilde{N}$ and called dual spacelike orientation vector, the dual spacelike approach vector and the dual timelike normal vector, respectively. It is pointed that since A is assumed to be a spacelike vector, dual approach vector $\tilde{A}$ is also a spacelike vector. In general motion, it may be dual timelike or lightlike. For these cases, formulations to be required to determine the differential properties of the motion of a robot end-effector can be adapted to the case of Lorentzian casual character of $\tilde{A}$.

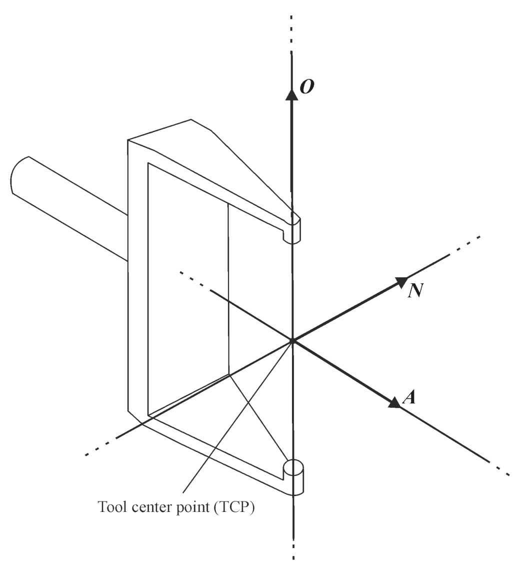

Figure 3 The dual tool frame of a robot end-effector.

Let the dual central angle between the dual unit vectors $\tilde{A}$ and $\tilde{t}$ be $\bar{\varphi} = \varphi + \varepsilon\,\varphi^*$, where φ is the Lorentzian angle and φ^* is the Lorentzian shortest distance between the corresponding lines of $\tilde{A}$ and $\tilde{t}$. From Section 3, we know that φ^* is the distance from line of striction to directrix of generating ruled surface, i.e., $\varphi^* = \mu\|R\|$. Thus, the dual tool frame relative to the dual Darboux frame can be given in matrix form as

$$\begin{bmatrix} \tilde{O} \\ \tilde{A} \\ \tilde{N} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\bar{\varphi} & \sinh\bar{\varphi} \\ 0 & \sinh\bar{\varphi} & \cosh\bar{\varphi} \end{bmatrix} \begin{bmatrix} \tilde{e} \\ \tilde{t} \\ \tilde{g} \end{bmatrix}. \tag{4.2}$$

By differentiating equation (4.2) and substituting equation (4.1) into the result, we have

$$\begin{bmatrix} \tilde{O}' \\ \tilde{A}' \\ \tilde{N}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\cosh\bar{\varphi} & \bar{\delta}\sinh\bar{\varphi} & \bar{\delta}\cosh\bar{\varphi} \\ -\sinh\bar{\varphi} & \bar{\delta}\cosh\bar{\varphi} & \bar{\delta}\sinh\bar{\varphi} \end{bmatrix} \begin{bmatrix} \tilde{e} \\ \tilde{t} \\ \tilde{g} \end{bmatrix},$$

where $\bar{\delta} = \bar{\varphi}' + \bar{\gamma}$ and the prime denotes derivation with respect to the dual arc-length parameter $\bar{s}$. By using equation (4.2), the derivation formulas of the dual tool frame in terms of itself can be obtained in matrix form as

$$\begin{bmatrix} \tilde{O}' \\ \tilde{A}' \\ \tilde{N}' \end{bmatrix} = \begin{bmatrix} 0 & \cosh\bar{\varphi} & -\sinh\bar{\varphi} \\ -\cosh\bar{\varphi} & 0 & \bar{\delta} \\ -\sinh\bar{\varphi} & \bar{\delta} & 0 \end{bmatrix} \begin{bmatrix} \tilde{O} \\ \tilde{A} \\ \tilde{N} \end{bmatrix}.$$

The dual instantaneous rotation vector of the dual tool frame can be found as

$$\tilde{w}_O = \bar{\delta}\,\tilde{O} + \sinh\bar{\varphi}\,\tilde{A} - \cosh\bar{\varphi}\,\tilde{N}.$$

By using equation (4.2), the dual instantaneous rotation vector of the dual tool frame can be expressed in terms of the dual Darboux frame as

$$\tilde{w}_O = \bar{\delta}\,\tilde{e} - \tilde{g}. \tag{4.3}$$

The dual instantaneous rotation vector $\tilde{w}_O = w_O + \varepsilon w_O^*$ plays the same role to the dual Pfaff vector which is dual velocity vector in dual spherical motion [6]. Thus, dual vector $\tilde{w}_O$ can be considered as dual velocity vector of the motion of a robot end-effector. The dual tool frame rotates along the axis $\dfrac{\tilde{w}_O}{\|\tilde{w}_O\|}$ with the dual angle $\|\tilde{w}_O\|$. This dual Lorentzian motion includes both rotational and translational motion in real Lorentzian space. w_O and w_O^* correspond to the instantaneous angular velocity vector and the instantaneous linear velocity vector, respectively. By separating equation (4.3) into the real and dual parts, we get

$$w_O = \delta e - g \tag{4.4}$$

and

$$w_O^* = \delta e^* + \delta^* e - g^* \tag{4.5}$$

respectively.

The instantaneous angular and linear velocity vector given in equations (4.4) and (4.5) are obtained in terms of s which is the arc-length parameter of spherical image curve of R. In order to determine the time dependent differential properties (angular and linear velocities) of motion, these vectors should be associated with t which is the parameter of time. Now, we can give the following corollary concerning time dependent linear and angular velocities of a robot end-effector which moves on a specified trajectory in Lorentzian space.

Corollary 4.1. *Let a robot end-effector moves on a specified trajectory in Lorentzian space such that tool line fixed in the end-effector generates a timelike ruled surface with spacelike ruling. Linear and angular velocities of robot end-effector can be given, respectively, as*

$$v_L = w_O^* \, \dot{s} \tag{4.6}$$

and

$$v_A = w_O \, \dot{s} \tag{4.7}$$

where w_O^* and w_O are given by equations (4.5) and (4.4), respectively, and the dot indicates differentiation with respect to time, i.e., $\dot{s} = \dfrac{ds}{dt}$.

By differentiating equation (4.3) and using equation (4.1), derivation of the dual instantaneous rotation vector of dual tool frame can be obtained as

$$\tilde{w}_O' = \bar{\delta}' \, \tilde{e} + \bar{\varphi}' \, \tilde{t}. \tag{4.8}$$

By separating equation (4.8) into real and dual parts, the instantaneous angular acceleration vector and the instantaneous linear acceleration vector can be found as

$$w_O^{'} = \delta' \, e + \varphi' \, t \tag{4.9}$$

and

$$w_O^{*'} = \delta' \, e^* + \delta^{*'} e + \varphi' \, t^* + \varphi^{*'} t. \tag{4.10}$$

By relating the parameter of time, we can give the following corollary concerning time dependent linear and angular accelerations of a robot end-effector which moves on a specified trajectory in Lorentzian space.

Corollary 4.2. *Let a robot end-effector moves on a specified trajectory in Lorentzian space such that tool line fixed in the end-effector generates a timelike ruled surface with spacelike ruling. Linear and angular accelerations of the robot end-effector can be given, respectively, as*

$$a_L = w_O^* \, \ddot{s} + w_O^{*'} \dot{s}^2 \tag{4.11}$$

and

$$a_A = w_O \, \ddot{s} + w_O^{'} \dot{s}^2 \tag{4.12}$$

where $w_O^{*'}$ and $w_O^{'}$ are given by equations (4.10) and (4.9), respectively, and $\ddot{s} = \dfrac{d^2 s}{dt^2}$.

Now, let us consider some special cases which may be encountered in motion of robot end-effector. A robot end-effector can move on a specified trajectory such that the central spin angle η may be constant during the motion, or more specifically, η may be equal to zero, namely, approach vector of robot end-effector may be always perpendicular in Lorentzian manner to generating ruled surface. For this special case, we can give the following corollaries concerning time dependent linear and angular velocities and accelerations of a robot end-effector.

Corollary 4.3. *Let a robot end-effector moves on a specified trajectory in Lorentzian space such that tool line fixed in the end-effector generates a timelike ruled surface with spacelike ruling. If the central spin angle η is constant, then linear and angular velocities of the robot end-effector can be expressed as*

$$v_L = ((\sigma' + \gamma) e^* + \delta^* e - g^*) \, \dot{s}$$

and

$$v_A = ((\sigma' + \gamma)e - g)\,\dot{s}$$

respectively.

Corollary 4.4. *Let a robot end-effector moves on a specified trajectory in Lorentzian space such that tool line fixed in the end-effector generates a timelike ruled surface with spacelike ruling. If the central spin angle* η *is constant, then the linear and angular accelerations of the robot end-effector can be expressed as*

$$a_L = ((\sigma' + \gamma)e^* + \delta^* e - g^*)\,\ddot{s} + ((\sigma'' + \gamma')\,e^* + \delta^{*'} e + \sigma'\,t^* + \varphi^{*'} t)\dot{s}^2$$

and

$$a_A = ((\sigma' + \gamma)e - g)\,\ddot{s} + ((\sigma'' + \gamma')\,e + \varphi'\,t)\,\dot{s}^2$$

respectively.

Now, let us consider another special case of motion of robot end-effector. A robot end-effector may follow a specified trajectory which may also be line of striction of ruled surface, in other words, directrix and line of striction of ruled surface can be the same curve. For this special case, we can give the following corollaries concerning time dependent linear and angular velocities and accelerations of a robot end-effector.

Corollary 4.5. *Let a robot end-effector moves on a specified trajectory in Lorentzian space such that tool line fixed in the end-effector generates a timelike ruled surface with spacelike ruling. If the specified trajectory is also the line of striction of the ruled surface which robot end-effector moves on, then the linear and angular velocities of the robot end-effector can be expressed as*

$$v_L = ((\eta' + \gamma)e^* + \gamma^* e - g^*)\,\dot{s}$$

and

$$v_A = ((\eta' + \gamma)e - g)\,\dot{s}$$

respectively.

Corollary 4.6. *Let a robot end-effector moves on a specified trajectory in Lorentzian space such that tool line fixed in the end-effector generates a timelike ruled surface with spacelike* *ruling. If the specified trajectory is also the line of striction of the ruled surface which robot end-effector moves on, then the linear and angular accelerations of the robot end-effector can be expressed as*

$$a_L = ((\eta' + \gamma)e^* + \gamma^* e - g^*)\,\ddot{s} + ((\eta'' + \gamma')\,e^* + \gamma^{*'} e + \eta'\,t^*)\dot{s}^2$$

and

$$a_A = ((\eta' + \gamma)e - g)\,\ddot{s} + ((\eta'' + \gamma')\,e + \eta'\,t)\,\dot{s}^2$$

respectively.

5. Example

As seen in Figure 4, let a robot end-effector moves on a specified trajectory in Lorentzian space such that generating surface is a helicoid given by the equation $X(t,v) = (v\cos t,\ v\sin t,\ ct)$, where t is the parameter of time, $c \neq 0$ and η be a central spin angle during the motion. Directrix and ruling of the helicoid are $\alpha(t) = (0,\ 0,\ ct)$ and $R(t) = (\cos t,\ \sin t,\ 0)$, respectively. Since $\mu = \langle \alpha', R' \rangle = 0$, directrix is also line of striction of the ruled surface, i.e., $c = \alpha$. From transference principle, the helicoid corresponds to a dual Lorentzian curve which can be expressed as

$$\tilde{e}(s) = (\cos s,\ \sin s,\ 0) + \varepsilon(cs\sin s,\ -cs\cos s,\ 0)$$

where s is the arc-length parameter of the spherical image curve of R. The second and third elements of dual Darboux frame of the dual Lorentzian curve can be found as

$$\tilde{t}(s) = (-\sin s,\ \cos s,\ 0) + \varepsilon(cs\cos s,\ cs\sin s, 0)$$

and

$$\tilde{g}(s) = (0,\ 0,\ 1)$$

respectively. By using equation (4.1), dual geodesic curvature can be found as $\bar{\gamma} = \gamma + \varepsilon\gamma^* = 0 + \varepsilon 0$. Since dual geodesic curvature equals to zero and the directrix is also the line of striction, we have $\varphi^* = 0$, $\sigma = 0$, $\bar{\varphi} = \eta$ and so $\bar{\delta} = \eta'$. Thus, the dual instantaneous rotation vector of the dual tool frame can be found as

$$\tilde{w}_O = w_O + \varepsilon w_O^* = (\eta' \cos s,\ \eta' \sin s,\ -1) + \varepsilon(cs \sin s,\ -cs \cos s,\ 0).$$

The angular and linear velocities of the robot end-effector can be obtained by substituting w_O and w_O^* into equations (4.7) and (4.8), respectively. By differentiating the dual instantaneous rotation vector of the dual tool frame, we get

$$\tilde{w}_O' = w_O' + \varepsilon w_O^{*'} = (\eta'' \cos s - \eta' \sin s,\ \eta'' \sin s + \eta' \cos s,\ 0)$$
$$+ \varepsilon(c \sin s + cs \cos s,\ -c \cos s + cs \sin s,\ 0)$$

The angular and linear accelerations of the robot end-effector can be obtained by substituting w_O' and $w_O^{*'}$ into equations (4.12) and (4.11), respectively.

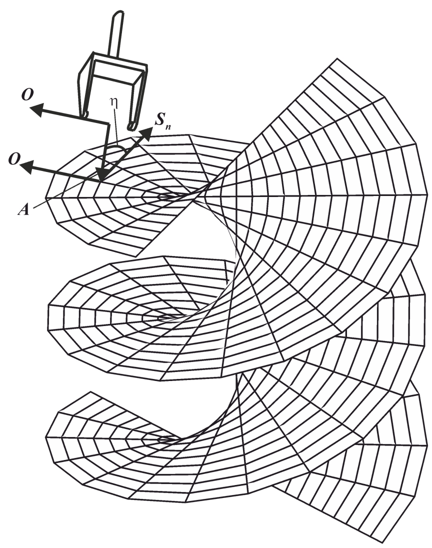

Figure 4 Motion of a robot end-effector whose generating surface is a helicoid.

6. Conclusions

In this paper, it is shown that dual curvature theory of dual Lorentzian unit spherical spacelike curve which corresponds to a ruled surface generated by a line fixed in the robot end-effector can be used to study the motion of a robot end-effector. Linear and angular time-dependent differential properties of motion such as velocities and accelerations of robot end-effector which are important in robot trajectory planning are determined. Dual curvature theory of a ruled surface is much simpler in expression than curvature theory in real space and it reduces redundant parameters in formulations. This paper is believed to be an original contribution to the research area of robot trajectory planning. We would like to emphasize that examining of motion in this paper is a restricted study such that generating ruled surface is a timelike ruled surface with spacelike ruling. Studying of motion such that generating ruled surface are spacelike ruled surfaces and timelike ruled surfaces with timelike rulings which correspond to dual Lorentzian unit spherical timelike curves and the dual hyperbolic unit spherical curves are not included in this paper. They are the subject of ongoing research works.

Conflict of Interests

The authors declare that there is no conflict of interests.

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