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STABILITY OF BRESSE SYSTEM WITH INTERNAL DISTRIBUTED DELAY

LAMINE BOUZETTOUTA[1], SALAH ZITOUNI[2], KHALED ZENNIR[3,*], AMAR GUESMIA[4]

[1]Department of Mathematics, University Badji Mokhtar, Annaba, Algeria

[2]Department of Mathematics, University of Souk Ahras, Algeria

[3]Department of Mathematics, Al-Rass, Qassim University, Kingdom Of Saudi Arabia

[4]Department of Mathematics, University 20 Aout 1955, Skikda, Algeria

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Abstract. In bounded domain, we consider a Bresse system with delay terms in the internal feedbacks acting in the first, third equations and distributed delay term in the second equation. We prove the global existence of its solution in Sobolev spaces by means of semigroup theory. Furthermore, we study the stability of solutions using the well known multiplier method.

Keywords: Bresse system; delay terms; decay rate; multiplier method; distributed delay.

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1. Introduction and related results

Originally the Bresse system consists of three wave equations where the main variables describing the longitudinal, vertical and shear angle displacements, which can be represented as

*Corresponding author

E-mail address: k.zennir@qu.edu.sa

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(see [4]):

$$(1)\qquad \begin{cases} \rho_1\varphi_{tt} = Q_x + lN + F_1 \\ \rho_2\psi_{tt} = M_x - Q + F_2 \\ \rho_1 w_{tt} = N_x - lQ + F_3, \end{cases}$$

where

$$(2)\qquad N = k_0\left(w_x - l\varphi\right), Q = k\left(\varphi_x + lw + \psi\right), M = b\psi_x$$

We use N, Q and M to denote the axial force, the shear force and the bending moment. By w, φ and ψ we are denoting the longitudinal, vertical and shear angle displacements. Here $\rho_1 = \rho A = \rho I, k_0 = EA, k = k'GA$ and $l = R^{-1}$. To material properties, we use ρ for density, E for the modulus of elasticity, G for the shear modulus, K for the shear factor, A for the cross-sectional area, I for the second moment of area of the cross-section and R for the radius of curvature and we assume that all this quantities are positives. Also by F_i we are denoting external forces.

System (1) is an undamped system and its associated energy remains constant when the time t evolves. To stabilize system (1), many damping terms have been considered by several authors. (see [1], [3], [6], [11], [12])

By considering a damping terms as infinite memories acting in the three equations, the system (1) have been recently studied by [6] in

$$(3)\qquad \begin{cases} \rho_1\varphi_{tt} - Gh\left(\varphi_x + lw + \psi\right)_x - Ehl\left(w_x - l\varphi\right) + \int_0^\infty g_1(s)\varphi_{xx}(t-s)ds = 0, \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh\left(\varphi_x + lw + \psi\right) + \int_0^\infty g_2(s)\psi_{xx}(t-s)ds = 0 \\ \rho_1 w_{tt} - Eh\left(w_x - l\varphi\right)_x + lGh\left(\varphi_x + lw + \psi\right) + \int_0^\infty g_3(s)w_{xx}(t-s)ds = 0 \end{cases}$$

where $(x,t) \in]0, L[\times \mathbb{R}_+, g_i : \mathbb{R}_+ \to \mathbb{R}_+, i = 1,2,3$ are given functions. The authors proved, under suitable conditions on the initial data and the memories g_i, that the system is well-posed and its energy converges to zero when time goes to infinity, and we provide a connection between the decay rate of energy and the growth of g_i at infinity. The proof is based on the semigroups theory for the well-posedness, and the energy method and the approach introduced in [5], for the stability.

In [3], the authors considered the Bresse system in bounded domain with delay terms in the internal feedbacks

$$(4)\qquad \begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + lw + \psi)_x - Ehl(w_x - l\varphi) + \mu_1\varphi_t + \mu_2\varphi_t(x,t-\tau_1) = 0, \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + lw + \psi) + \widetilde{\mu_1}\psi_t + \widetilde{\mu_2}\psi_t(x,t-\tau_2) = 0 \\ \rho_1 w_{tt} - Eh(w_x - l\varphi)_x + lGh(\varphi_x + lw + \psi) + \widetilde{\widetilde{\mu_1}}w_t + \widetilde{\widetilde{\mu_2}}w_t(x,t-\tau_3) = 0 \end{cases}$$

where $(x,t) \in (0,L)\times(0,+\infty)$, $\tau_i > 0$ $(i = 1,2,3)$ is a time delay, $\mu_1, \mu_2, \widetilde{\mu_1}, \widetilde{\mu_2}, \widetilde{\widetilde{\mu_1}}, \widetilde{\widetilde{\mu_2}}$ are positive real numbers. This system is subjected to the Dirichlet boundary conditions and to the initial conditions which belong to a suitable Sobolev space. First, the author proved the global existence of its solutions in Sobolev spaces by means of semigroup theory under a condition between the weight of the delay terms in the feedbacks and the weight of the terms without delay. Furthermore, they studied the asymptotic behavior of solutions using multiplier method.

The Bresse system (1), is more general than the well-known Timoshenko system where the longitudinal displacement ω is not considered $l = 0$. There are a number of publications concerning the stabilization of Timoshenko system with different kinds of damping, in this regard, we note the next references (see [9], [13], [14], [15], [21], [22] and [23]).

In the present paper we are concerned at the Bresse system with a distributed delay term,

$$(5)\qquad \begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + lw + \psi)_x - Ehl(w_x - l\varphi) + \mu_1\varphi_t + \mu_2\varphi_t(x,t-\tau_1) = 0, \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + lw + \psi) + \mu_0\psi_t + \int_{\tau_1}^{\tau_2}\mu(s)\psi_t(x,t-s)\,ds = 0, \\ \rho_1 w_{tt} - Eh(w_x - l\varphi)_x + lGh(\varphi_x + lw + \psi) + \widetilde{\mu_1}w_t + \widetilde{\mu_2}w_t(x,t-\tau_2) = 0, \end{cases}$$

where $(x,t) \in]0,L[\times\mathbb{R}_+$, with the Dirichlet conditions:

$$(6)\qquad \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = w(0,t) = w(L,t) = 0, t > 0$$

and the initial conditions

$$(7)\qquad \begin{cases} \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), \psi(x,0) = \psi_0(x) \\ \psi_t(x,0) = \psi_1(x), w(x,0) = w_0(x), w_t(x,0) = w_1(x), x \in (0,L) \\ \varphi_t(x,t-\tau_1) = \widetilde{f_0}(x,t-\tau_1) \text{ in } (0,L)\times(0,\tau_1) \\ \psi_t(x,t) = f_0(x,t) \text{ in } (0,L)\times(0,\tau_1) \\ w_t(x,t-\tau_2) = \widetilde{\widetilde{f_0}}(x,t-\tau_2) \text{ in } (0,L)\times(0,\tau_2) \end{cases}.$$

τ_1 and τ_2 are two real numbers with $0 \leq \tau_1 < \tau_2$, μ_0 is a positive constant, $\mu : [\tau_1, \tau_2] \to \mathbb{R}$ is an L^∞ function, $\mu \geq 0$ almost everywhere, and the initial data $(u_0, u_1, v_0, v_1, f_0)$ belong to a suitable space (see below).

Here, we will prove the well-posedness an and the stability results for problem (5)-(7), under the assumption

$$\mu_0 \geq \int_{\tau_1}^{\tau_2} \mu(s) ds \tag{8}$$

Concerning the distributed delay, Nicaise and Pignotti [19] considered wave equation with linear frictional damping and internal distributed delay

$$u_{tt} - \Delta u + \mu_1 u_t + a(x) \int_{\tau_1}^{\tau_2} \mu_2(s) u_t(x, t-s) ds = 0,$$

in $\Omega \times (0, \infty)$, with initial and mixed Dirichlet-Neumann boundary conditions and a is a function chosen in an appropriate space. They established exponential stability of the solution under the assumption that

$$\|a\|_\infty \int_{\tau_1}^{\tau_2} \mu_2(s) ds < \mu_1.$$

Regarding the similar result concerning boundary distributed delay see [2, 16, 17].

The aim of this article is to study the well-posedness and asymptotic stability of system (5)-(7). The paper is organized as follows. The well-posedness of the problem is analyzed in Section 2 using the semigroup theory (see [8], [20]). In Section 3, we prove the exponential decay of the energy when time tends to infinity (see [10]).

2. Preliminaries and Well-posedness

First assume the following hypotheses:

$$|\mu_2| < |\mu_1|, |\widetilde{\mu_2}| < |\widetilde{\mu_1}|$$

and state some lemmas which will be needed later.

Lemma 2.1. (Sobolev-Poincare's inequality). *Let q be a number with $2 \leq q < +\infty$. Then there is a constant $c_* = c_*((0,L), q)$ such that*

$$\|\psi\|_q \leq c_* \|\psi_x\|_2 \text{ for } \psi \in H_0^1((0,L)). \tag{9}$$

Lemma 2.2. *([7], [10]). Let $E:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ be a non increasing function and assume that there are two constants $\alpha>-1$ and $\beta>0$ such that*

$$\int_s^{\infty} E^{\alpha+1}(t)dt \le \frac{1}{\beta}E(0)^{\alpha}E(s), \quad 0\le s<+\infty \tag{10}$$

Then we have

$$E(t)=0, \quad \forall t\ge \frac{E(0)^{\alpha}}{\beta\left|\alpha\right|} \text{ if } -1<\alpha<0 \tag{11}$$

$$E(t)\le E(0)\left(\frac{1+\alpha}{1+\alpha\beta t}\right)^{\frac{1}{\alpha}}, \ \forall t\ge 0, \quad \text{if } \alpha>0 \tag{12}$$

$$E(t)\le E(0)e^{1-\beta t}, \ \forall t>0, \ \text{if } \alpha=0 \tag{13}$$

First of all, note that assumption (8) implies that there exists a positive constant c_0 such that,

$$\mu_0-\int_{\tau_1}^{\tau_2}\mu(s)ds-\frac{c_0}{2}\left(\tau_2-\tau_1\right)>0 \tag{14}$$

We will prove that systems (5)-(7) are well posed using semigroup theory by introduce the following new variable (see [19])

$$\begin{aligned} z_1(x,\rho,t) &= \varphi_t\left(x,t-\tau_1\rho\right), x\in(0,L), \rho\in(0,1), t>0,\\ z(x,\rho,t,s) &= \psi_t\left(x,t-\rho s\right), x\in(0,L), \rho\in(0,1), s\in(\tau_1,\tau_2), t>0,\\ z_2(x,\rho,t) &= w_t\left(x,t-\tau_2\rho\right), x\in(0,L), \rho\in(0,1), t>0, \end{aligned} \tag{15}$$

Then, we have

$$\tau_i z_{it}(x,\rho,t)+z_{i\rho}(x,\rho,t)=0 \text{ in } (0,L)\times(0,1)\times(0,\infty) \text{ for } i=1,2 \tag{16}$$

and

$$sz_t(x,\rho,t,s)+z_\rho(x,\rho,t,s)=0, in\,(0,L)\times(0,L)\times(0,\infty)\times(\tau_1,\tau_2) \tag{17}$$

Therefore, problem (5) takes the form

$$
\left\{
\begin{array}{l}
\rho_1 \varphi_{tt} - Gh\left(\varphi_x + lw + \psi\right)_x - lEh\left(w_x - l\varphi\right) + \mu_1 \varphi_t\left(x,t\right) + \mu_2 z_1\left(x,1,t\right) = 0 \\
\tau_1 z_{1t}(x,\rho,t) + z_{1\rho}(x,\rho,t) = 0 \\
\rho_2 \psi_{tt} - EI\psi_{xx} + Gh\left(\varphi_x + lw + \psi\right) + \mu_0 \psi_t + \int_{\tau_1}^{\tau_2} \mu(s) z(x,,1,t,s)\, ds = 0 \\
s z_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0 \\
\rho_1 w_{tt} - Eh\left(w_x - l\varphi\right)_x + lGh\left(\varphi_x + lw + \psi\right) + \mu_1 w_t\left(x,t\right) + \mu_2 z_2\left(x,1,t\right) = 0 \\
\tau_2 z_{2t}(x,\rho,t) + z_{2\rho}(x,\rho,t) = 0 \\
s z_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0, \text{ in } (0,L)\times(0,L)\times(0,\infty)\times(\tau_1,\tau_2)
\end{array}
\right.
\tag{18}
$$

With the Dirichlet conditions:

$$
\varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = w(0,t) = w(L,t) = 0, t > 0
$$

and the initial conditions:

$$
\left\{
\begin{array}{l}
\varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), \psi(x,0) = \psi_0(x),\ x \in (0,L) \\
\psi_t(x,0) = \psi_1(x), w(x,0) = w_0(x), w_t(x,0) = w_1(x), x \in (0,L) \\
\varphi_t(x,-t) = f_0(x,t) \text{ in } (0,L)\times(0,\tau_2), \\
z(x,0,t,s) = \psi_t(x,t) \text{ on } (0,L)\times(0,\infty)\times(\tau_1,\tau_2), \\
z(x,\rho,0,s) = f_0(x,\rho,s) \text{ in } (0,L)\times(0,L)\times(0,\tau_2) \\
z_1\left(x,1,t\right) = f_1\left(x,t-\tau_1\right) \text{ in } (0,L)\times(0,\tau_1) \\
z_2\left(x,1,t\right) = f_2\left(x,t-\tau_2\right) \text{ in } (0,L)\times(0,\tau_2)
\end{array}
\right.
\tag{19}
$$

Let $\xi_1$and ξ_2 be positive constants such that:

$$
\left\{
\begin{array}{l}
\tau_1\left|\mu_2\right| < \xi_1 < \tau_1\left(2\mu_1 - \left|\mu_2\right|\right) \\
\tau_2\left|\widetilde{\mu_2}\right| < \xi_2 < \tau_2\left(2\widetilde{\mu_1} - \left|\widetilde{\mu_2}\right|\right)
\end{array}
\right.
$$

where, τ_1 and τ_2 are two real numbers with $0 \le \tau_1 < \tau_2,\ \mu_0, \mu_1, \mu_2$ is a positive constant, $\mu :$ $[\tau_1, \tau_2]$ is an L^∞ function, $\mu \ge 0$ almost everywhere, and the initial data $(\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1, f_0, f_1, f_2)$ belong to a suitable space (see below).

If we set $U = \left(\varphi, \varphi_t, z_1, \psi, \psi_t, z, w, w_t, z_2\right)^T$, then

$U' = (\varphi_t, \varphi_{tt}, z_{1t}, \psi_t, \psi_{tt}, z_t, w_t, w_{tt}, z_{2t})^T$ .Therefore, problem (18)-(19) can be written as

$$\begin{cases} U' = AU, \\ U(0) = (\varphi_0, \varphi_1, f_1(.-,\tau_1), \psi_0, \psi_1, f_0, w_0, w_1, f_2(.-,\tau_2)), \end{cases} \tag{20}$$

where the operator A is defined by

$$A\begin{pmatrix} \varphi \\ u \\ z_1 \\ \psi \\ v \\ z \\ w \\ \omega \\ z_2 \end{pmatrix} = \begin{pmatrix} u \\ \frac{Gh}{\rho_1}(\varphi_x + lw + \psi)_x + \frac{lEh}{\rho_1}(w_x - l\varphi) - \frac{\mu_1}{\rho_1}u - \frac{\mu_2}{\rho_1}z_1(.,1) \\ -\left(\frac{1}{\tau_1}\right) z_{1\rho} \\ v \\ \frac{EI}{\rho_2}\psi_{xx} - \frac{Gh}{\rho_2}(\varphi_x + lw + \psi) - \frac{\mu_0}{\rho_2}v - \int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)\,ds \\ -s^{-1}z_\rho \\ \omega \\ \frac{Eh}{\rho_1}(w_x - l\varphi)_x - \frac{lGh}{\rho_1}(\varphi_x + lw + \psi) - \frac{\widetilde{\mu_1}}{\rho_1}\omega - \frac{\widetilde{\mu_2}}{\rho_1}z_2(.,1) \\ -\left(\frac{1}{\tau_2}\right) z_{2\rho} \end{pmatrix}$$

with domain

$$\begin{aligned} D(A) \;=\; & \Big\{ (\varphi, u, z_1, \psi, v, z, w, \omega, z_2)^T \in \left(H^2(0,L) \cap H_0^1(0,L) \times H_0^1(0,L)\right)^3 \\ & \times L^2\left((0,L), (\tau_1,\tau_2); H_0^1(0,1)\right), \\ & u = z_1(.,0), \omega = z_2(.,0), v(x) = z(x,0,s) \text{ in } (0,L) \Big\}. \end{aligned}$$

Note further that, for $U = (\varphi, u, , z_1, \psi, v, z, w, \omega, z_2)^T \in D(A)$, since $z(.,1,s)$ is in $L^2(0,L)$.Denote by H the Hilbert space

$$H := \left(H^2(0,L) \cap H_0^1(0,L) \times H_0^1(0,L)\right)^3 \times L^2\left((0,L), (\tau_1,\tau_2); H_0^1(0,1)\right)$$

We will show that A generates a C_0 semigroup on H; under the assumption (8).

Let us define on the Hilbert space H the inner product, for

$$U = (\varphi, u, , z_1, \psi, v, z, w, \omega, z_2)^T \;, \overline{U} = (\overline{\varphi}, \overline{u}, \overline{z_1}, \overline{\psi}, \overline{v}, \bar{z}, \overline{w}, \overline{\omega}, \overline{z_2})^T$$

$$
\begin{aligned}
\left\langle U,\overline{U}\right\rangle_H &= \int_0^L [\rho_1 u\overline{u}+\rho_2 v\overline{v}+\rho_1\omega\overline{\omega}+EI\psi_x\overline{\psi}_x+Gh(\varphi_x+\psi+lw)(\overline{\varphi}_x+\overline{\psi}+l\overline{w}) \\
&\quad +Eh\left(w_x-l\varphi\right)\left(\overline{w}_x-l\overline{\varphi}\right)+\sum_{i=1}^{2}\xi_i\int_0^1 z_i\overline{z_i}d\rho\Bigg]dx \\
&\quad +\int_0^L\int_{\tau_1}^{\tau_2} s\mu(s)\int_0^1 z\left(x,\rho,s\right)\overline{z}\left(x,\rho,s\right)d\rho dsdx
\end{aligned}
$$

Theorem 2.3. *Let $(\varphi_0,\varphi_1,u,\psi,v,w,\omega,f_0)\in H$. Assume that the hypothesis (8) holds. Then, for any initial datum $U_0\in H$ there exists a unique solution $U\in C\left([0,\infty),H\right)$ for problem (20) . Moreover, if $U_0\in D(A)$, then $U\in C([0,\infty),D(A))\cap C^1[0,\infty),H)$.*

Proof We show that the operator A generates a C_0-semigroup in H. In this step, we prove that the operator A is dissipative. Let $U=(\varphi,u,,z_1,\psi,v,z,w,\omega,z_2)^T$,

$$
\langle AU,U\rangle =
$$

$$
\left\langle \begin{pmatrix}
u \\
\frac{Gh}{\rho_1}\left(\varphi_x+lw+\psi\right)_x+\frac{lEh}{\rho_1}\left(w_x-l\varphi\right)-\frac{\mu_1}{\rho_1}u-\frac{\mu_2}{\rho_1}z_1\left(.,1\right) \\
-\left(\frac{1}{\tau_1}\right)z_{1\rho} \\
v \\
\frac{EI}{\rho_2}\psi_{xx}-\frac{Gh}{\rho_2}\left(\varphi_x+lw+\psi\right)-\frac{\mu_0}{\rho_2}v-\int_{\tau_1}^{\tau_2}\mu(s)z\left(x,1,t,s\right)ds \\
-s^{-1}z_\rho \\
\omega \\
\frac{Eh}{\rho_1}\left(w_x-l\varphi\right)_x-\frac{lGh}{\rho_1}\left(\varphi_x+lw+\psi\right)-\frac{\widetilde{\mu_1}}{\rho_1}\omega-\frac{\widetilde{\mu_2}}{\rho_1}z_2\left(.,1\right) \\
-\left(\frac{1}{\tau_2}\right)z_{2\rho}
\end{pmatrix},
\begin{pmatrix}
\varphi \\ u \\ z_1 \\ \psi \\ v \\ z \\ w \\ \omega \\ z_2
\end{pmatrix}\right\rangle
$$

$$
\begin{aligned}
&= Gh\int_0^L\left(\varphi_x+lw+\psi\right)_x udx+Ehl\int_0^L\left(w_x-l\varphi\right)udx-\mu_0\int_0^L u^2dx \\
&\quad -\int_0^L\int_{\tau_1}^{\tau_2} u\mu(s)z(x,1,t,s)dsdx+EI\int_0^L\psi_{xx}vdx \\
&\quad -Gh\int_0^L\left(\varphi_x+lw+\psi\right)vdx+\rho_1\int_0^L\omega wdx+EI\int_0^L v_x\psi_x dx
\end{aligned}
$$

$$+Gh\int_0^L (u_x+v+l\omega)(\varphi_x+lw+\psi)\,dx+Eh\int_0^L(\omega_x-lu)(w_x-l\varphi)\,dx$$
$$-\int_0^L\int_{\tau_1}^{\tau_2}\mu(s)\int_0^1 z_\rho(x,\rho,s)\,z(x,\rho,s)\,d\rho ds dx$$
$$-\sum_1^2\xi_i\int_0^L\int_0^1\left(\frac{1}{\tau_i}\right)z_iz_{i\rho}d\rho dx$$

Integrating by parts, we obtain

$$\begin{aligned}\langle AU,U\rangle &= Gh\int_0^L(\varphi_{xx}+lw_x+\psi_x)\,udx+Ehl\int_0^L(w_x-l\varphi)\,udx-\mu_0\int_0^L u^2dx\\
&-\int_0^L\int_{\tau_1}^{\tau_2}u\mu(s)z(x,1,t,s)dsdx+EI\int_0^L\psi_{xx}vdx\\
&-Gh\int_0^L(\varphi_x+lw+\psi)\,vdx+Eh\int_0^L(w_{xx}-l\varphi_x)\,\omega dx\\
&-Ghl\int_0^L(\varphi_x+lw+\psi)\,\omega dx+EI\int_0^L v_x\psi_xdx\\
&+Gh\int_0^L(u_x+v+l\omega)(\varphi_x+lw+\psi)\,dx+Eh\int_0^L(\omega_x-lu)(w_x-l\varphi)\,dx\\
&-\int_0^L\int_{\tau_1}^{\tau_2}\mu(s)\int_0^1 z_\rho(x,\rho,s)\,z(x,\rho,s)\,d\rho dsdx\\
&-\sum_1^2\xi_i\int_0^L\int_0^1\left(\frac{1}{\tau_i}\right)z_iz_{i\rho}d\rho dx\end{aligned}$$

Then

$$\begin{aligned}\langle AU,U\rangle &= -\mu_1\int_0^L u^2dx-\widetilde{\mu_1}\int_0^L\omega^2dx-\mu_2\int_0^L z_1(x,1)udx-\widetilde{\mu_2}\int_0^L z_2(x,1)\omega dx\\
&-\sum_1^2\frac{\xi_i}{\tau_i}\int_0^L\int_0^1 z_i(x,\rho)z_{i\rho}(x,\rho)d\rho dx\\
&-\int_0^L\int_{\tau_1}^{\tau_2}v\mu(s)z(x,1,t,s)dsdx\\
&-\int_0^L\int_{\tau_1}^{\tau_2}\mu(s)\int_0^1 z_\rho(x,\rho,s)\,z(x,\rho,s)\,d\rho dsdx\end{aligned}\tag{21}$$

Integrating by parts in ρ , we have

$$\begin{aligned}\int_0^1 z_\rho(x,\rho,s)\,z(x,\rho,s)\,d\rho &= \frac{1}{2}\int_0^1\frac{\partial}{\partial\rho}z^2(x,\rho,s)\,d\rho\\
&= \frac{1}{2}\left\{z^2(x,1,s)-z^2(x,0,s)\right\}\end{aligned}$$

that is

$$\begin{aligned}
&\int_0^L \int_{\tau_1}^{\tau_2} \mu(s) \int_0^1 z\rho\,(x,\rho,s)\, z\,(x,\rho,s)\, d\rho ds dx \\
&= \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} \mu(s) \left\{ z^2(x,1,s) - z^2(x,0,s) \right\} ds dx
\end{aligned} \tag{22}$$

Therefore, from (21) and (22),

$$\begin{aligned}
\langle AU,U\rangle &= -\mu_1 \int_0^L u^2 dx - \widetilde{\mu_1}\int_0^L \omega^2 dx - \mu_2 \int_0^L z_1(x,1) u dx - \widetilde{\mu_2}\int_0^L z_2(x,1)\omega dx \\
&\quad -\sum_1^2 \frac{\xi_i}{\tau_i}\int_0^L \int_0^1 z_i(x,\rho) z_{i\rho}(x,\rho) d\rho dx \\
&\quad -\int_0^L \int_{\tau_1}^{\tau_2} v(x)\mu(s) z(x,1,s) ds dx \\
&\quad -\int_0^L \int_{\tau_1}^{\tau_2} \mu(s) \int_0^1 z_\rho\,(x,\rho,s)\, z\,(x,\rho,s)\, d\rho ds dx \\
&= -\mu_1 \int_0^L u^2 dx - \widetilde{\mu_1}\int_0^L \omega^2 dx - \mu_2 \int_0^L z_1(x,1) u dx - \widetilde{\mu_2}\int_0^L z_2(x,1)\omega dx \\
&\quad -\sum_1^2 \frac{\xi_i}{\tau_i}\int_0^L \int_0^1 z_i(x,\rho) z_{i\rho}(x,\rho) d\rho dx \\
&\quad -\int_0^L v(x)\left(\int_{\tau_1}^{\tau_2} v(x)\mu(s) z(x,1,s) ds\right) dx \\
&\quad -\frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} \mu(s) z^2(x,1,s) ds dx + \frac{1}{2}\int_{\tau_1}^{\tau_2} \mu(s) ds \int_0^L v^2(x) dx
\end{aligned}$$

Now, using Cauchy-Schwarz's inequality, we can estimate,

$$\begin{aligned}
&\left| \int_0^L v(x)\left(\int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s) ds\right) dx \right| \\
&\leq \frac{1}{2}\int_0^L v^2(x)\left(\int_{\tau_1}^{\tau_2} \mu(s) ds\right) dx + \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} \mu(s) z^2(x,1,s) ds dx
\end{aligned} \tag{23}$$

Therefore, from the assumption (8) we have

$$\begin{aligned}\langle AU,U\rangle \;\leq\; & -\mu_1\int_0^L u^2dx-\widetilde{\mu_1}\int_0^L \omega^2dx\\ & -\mu_2\int_0^L z_1(x,1)udx-\widetilde{\mu_2}\int_0^L z_2(x,1)\omega dx\\ & -\sum_1^2\frac{\xi_i}{\tau_i}\int_0^L\int_0^1 z_i(x,\rho)z_{i\rho}(x,\rho)d\rho dx\\ & +\left(-\mu_0+\int_{\tau_1}^{\tau_2}\mu(s)ds\right)\int_0^L v^2(x)dx \;\leq\; 0\end{aligned}$$

that is, the operator A is dissipative.

Now, we will prove that the operator $\lambda I-A$ is surjective for $\lambda>0$. For this purpose, let $(f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8,f_9)^T\in H$, we seek $U=(\varphi,u,,z_1,\psi,v,z,w,\omega,z_2)^T\in D(A)$ solution of the following system of equations

$$(24)\qquad \begin{cases}\lambda\varphi-u=f_1\\ \lambda u-\frac{Gh}{\rho_1}(\varphi_x+lw+\psi)_x-\frac{lEh}{\rho_1}(w_x-l\varphi)+\frac{\mu_1}{\rho_1}u+\frac{\mu_2}{\rho_1}z_1(.,1)=f_2\\ \lambda z_1+\left(\frac{1}{\tau_1}\right)z_{1\rho}=f_3\\ \lambda\psi-v=f_4\\ \lambda v-\frac{EI}{\rho_2}\psi_{xx}+\frac{Gh}{\rho_2}(\varphi_x+lw+\psi)+\frac{\mu_0}{\rho_2}v+\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)\,ds=f_5\\ \lambda z+s^{-1}z_\rho=f_6\\ \lambda w-\omega=f_7\\ \lambda\omega-\frac{Eh}{\rho_1}(w_x-l\varphi)_x+\frac{lGh}{\rho_1}(\varphi_x+lw+\psi)+\frac{\widetilde{\mu_1}}{\rho_1}\omega+\frac{\widetilde{\mu_2}}{\rho_1}z_2(.,1)=f_8\\ \lambda z_2+\left(\frac{1}{\tau_2}\right)z_{2\rho}=f_9\end{cases}$$

Suppose that we have found φ,ψ and w. Therefore, the first, the fourth and the seventh equation in (24) give

$$(25)\qquad \begin{cases}u=\lambda\varphi-f_1,\\ v=\lambda\psi-f_4,\\ \omega=\lambda w-f_7,\end{cases}$$

It is clear that $u\in H_0^1(0,L), v\in H_0^1(0,L)$ and $w\in H_0^1(0,L)$. and we can find,

$$(26)\qquad z(x,0,s)=v(x),\text{ for } x\in(0,L), s\in(\tau_1,\tau_2).$$

and furthermore, by (24) we can find z_i $(i=1,2)$ as

$$z_1(x,0)=u(x), z_2(x,0)=\omega(x) \text{ for } x\in(0,L).$$

Following the same approach as in [18], we obtain, by using equations for z in (25)

$$\lambda z(x,\rho,s)+s^{-1}z_\rho(x,\rho,s)=f_6(x,\rho,s), \text{ for } x\in(0,L), s\in(\tau_1,\tau_2) \tag{27}$$

and

$$z_1(x,\rho)=u(x)e^{-\lambda\tau_1\rho}+\tau_1 e^{-\lambda\tau_1\rho}\int_0^\rho f_3(x,s)e^{\lambda\tau_1\rho}ds$$

$$z_2(x,\rho)=\omega(x)e^{-\lambda\tau_2\rho}+\tau_2 e^{-\lambda\tau_2\rho}\int_0^\rho f_9(x,s)e^{\lambda\tau_2\rho}ds$$

Then by (26) and (27)

$$z(x,\rho,s)=e^{-\lambda\rho s}v(x)+se^{-\lambda\rho s}\int_0^\rho f_6(x,\sigma,s)e^{\lambda\sigma s}d\sigma$$

So, from (24) on $(0,L)\times(0,1)\times(\tau_1,\tau_2)$

$$z(x,\rho,s)=\lambda\psi(x)e^{-\lambda\rho s}-f_2(x)e^{-\lambda\rho s}+se^{-\lambda\rho s}\int_0^\rho f_6(x,\sigma,s)e^{\lambda\sigma s}d\sigma \tag{28}$$

and from (25)

$$z_1(x,\rho)=\lambda\varphi(x)e^{-\lambda\tau_1\rho}-f_1e^{-\lambda\tau_1\rho}+\tau_1 e^{-\lambda\tau_1\rho}\int_0^\rho f_3(x,s)e^{\lambda\tau_1\rho}ds \tag{29}$$

$$z_2(x,\rho)=\lambda w(x)e^{-\lambda\tau_2\rho}-f_7e^{-\lambda\tau_2\rho}+\tau_2 e^{-\lambda\tau_2\rho}\int_0^\rho f_9(x,s)e^{\lambda\tau_2\rho}ds \tag{30}$$

By using (8) and (24) the functions φ, ψ and w satisfying the following system,

$$\begin{cases} \lambda^2\varphi-\frac{Gh}{\rho_1}(\varphi_x+lw+\psi)_x-\frac{lEh}{\rho_1}(w_x-l\varphi)+\frac{\mu_1}{\rho_1}u+\frac{\mu_2}{\rho_1}z_1(.,1)=f_2+\lambda f_1 \\ \lambda^2\psi-\frac{EI}{\rho_2}\psi_{xx}+\frac{Gh}{\rho_2}(\varphi_x+lw+\psi)+\frac{\mu_0}{\rho_2}v+\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)\,ds=f_5+\lambda f_4 \\ \lambda^2 w-\frac{Eh}{\rho_1}(w_x-l\varphi)_x+\frac{lGh}{\rho_1}(\varphi_x+lw+\psi)+\frac{\widetilde{\mu_1}}{\rho_1}\omega+\frac{\widetilde{\mu_2}}{\rho_1}z_2(.,1)=f_8+\lambda f_7 \end{cases} \tag{31}$$

Solving system (31) is equivalent to finding $(\varphi,\psi,w)\in(H^2\cap H_0^1(0,L))^3$ such that,

$$
(32)\qquad \begin{cases}
\int_0^L(\rho_1\lambda^2\varphi\varepsilon-Gh(\varphi_x+lw+\psi)\,\varepsilon_x-Eh(w_x-l\varphi)\,\varepsilon\\
+\mu_1u\varepsilon+\mu_2z_1(.,1)\varepsilon)dx=\int_0^L\rho_1(f_2+\lambda f_1)\varepsilon dx\\
\int_0^L(\rho_2\lambda^2\psi\zeta-EI\psi_x\zeta_x+Gh(\varphi_x+lw+\psi)\,\zeta+\mu_0v\zeta\\
+\rho_2\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)\,\zeta ds)dx=\int_0^L\rho_2(f_5+\lambda f_4)\zeta dx\\
\int_0^L(\rho_1\lambda^2w\eta-Eh(w_x-l\varphi)\,\eta_x-lGh(\varphi_x+lw+\psi)\,\eta\\
+\widetilde{\mu_1}\omega\eta+\widetilde{\mu_2}z_2(.,1)\,\eta)dx=\int_0^L\rho_1(f_8+\lambda f_7)\eta dx
\end{cases}
$$

for all $(\varepsilon,\zeta,\eta)\in H_0^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L)$. From (28), (29) and (30) we have

$$z_1(x,1)=\lambda\varphi(x)e^{-\lambda\tau_1}-f_1e^{-\lambda\tau_1}+\tau_1e^{-\lambda\tau_1}\int_0^1 f_3(x,s)e^{\lambda\tau_1}ds$$

$$z_2(x,1)=\lambda w(x)e^{-\lambda\tau_2}-f_7e^{-\lambda\tau_2}+\tau_2e^{-\lambda\tau_2}\int_0^1 f_9(x,s)e^{\lambda\tau_2}ds$$

$$z(x,1,s)=\lambda\psi(x)e^{-\lambda s}-f_2(x)e^{-\lambda s}+se^{-\lambda s}\int_0^1 f_6(x,\sigma,s)\,e^{\lambda\sigma s}d\sigma$$

Consequently, problem (32) is equivalent to the problem

$$(33)\qquad a((\varphi,\psi,w),(\varepsilon,\zeta,\eta))=L(\varepsilon,\zeta,\eta)$$

where the bilinear form $a:\left[H_0^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L)\right]^2\longrightarrow\mathbb{R}$ and the linear form $L:H_0^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L)\longrightarrow\mathbb{R}$ are defined by

$$
\begin{aligned}
a((\varphi,\psi,w),(\varepsilon,\zeta,\eta)) \;=\; & \int_0^L\left(\rho_1\lambda^2\varphi\varepsilon+Gh(\varphi_x+lw+\psi)(\varepsilon_x+\zeta+\eta)\right)dx\\
&+\int_0^L(\rho_1\lambda^2\psi\zeta+(w_x-l\varphi)(\eta_x-lw)\,dx\\
&+\int_0^L\lambda\varphi\left(\mu_1+\mu_2e^{-\lambda\tau_1}\right)\varepsilon dx\\
&+\int_0^L\lambda w\left(\widetilde{\mu_1}+\widetilde{\mu_2}e^{-\lambda\tau_2}\right)\eta dx\\
&+\int_0^L\lambda\psi\mu_0\zeta dx+\int_0^L\lambda\psi\zeta\int_{\tau_1}^{\tau_2}\mu(s)e^{-\lambda s}dsdx
\end{aligned}
$$

and

$$\begin{aligned} L(\varepsilon,\zeta,\eta) &= \int_0^L (\mu_1 f_1 - \mu_2 N_1)\,\varepsilon dx + \int_0^L \rho_1 (f_2 + \lambda f_1)\,\varepsilon dx \\ &+ \int_0^L \rho_2 (f_5 + \lambda f_4)\,\zeta dx + \int_0^L \left(\widetilde{\mu_1} f_7 - \widetilde{\mu_2} N_2\right)\eta dx \\ &+ \int_0^L \varepsilon \int_{\tau_1}^{\tau_2} \mu(s) z_0(x,s) ds dx + \int_0^L \rho_1 (f_8 + \lambda f_7)\,\eta dx \end{aligned}$$

$\forall \varepsilon \in H_0^1(0,L)$. It is easy to verify that a is continuous and coercive, and L is continuous. So applying the Lax-Milgram theorem, we deduce that for all $(\varepsilon,\zeta,\eta) \in H_0^1(0,L) \times H_0^1(0,L) \times H_0^1(0,L)$ problem (33) admits a unique solution $(\varphi,\psi,w) \in H_0^1(0,L) \times H_0^1(0,L) \times H_0^1(0,L)$ .Applying the classical elliptic regularity, it follows from (32) that $(\varphi,\psi,w) \in H^2(0,L) \times H^2(0,L) \times H^2(0,L)$. Therefore, the operator $\lambda I - A$ is surjective for any $\lambda > 0$. Consequently, the existence result of theorem follows from the Hille-Yosida theorem.

3. Stability results

We define the energy associated to the solution of the problem (5) by the following formula:

$$\begin{aligned} E(t) &= \frac{\rho_1}{2}\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{\rho_1}{2}\|w_t\|_2^2 + \frac{EI}{2}\|\psi_x\|_2^2 \\ &+ \frac{Gh}{2}\|\varphi_x + \psi + lw\|_2^2 + \frac{Eh}{2}\|w_x - l\varphi\|_2^2 \\ &+ \sum_{i=1}^{2}\frac{\xi_i}{2}\int_0^1 \|z_i(x,\rho,t)\|_2^2 d\rho \\ &+ \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} s\left[\mu(s) + c_0\right]\int_0^1 \psi_t^2(x,t-\rho s)\, d\rho ds dx \end{aligned} \tag{34}$$

We can prove that the energy is decreasing. More precisely, we have the following result.

Theorem 3.1. *Let (φ,ψ,w) be the solution of (5)-(7) and the assumption (8). Then there exist two positive constants c and κ such that*

$$E(t) \le cE(0)e^{-\kappa t}, t \ge 0.$$

Lemma 3.2. *Let $(\varphi,\psi,w,z_1,z,z_2)$ be a solution of the problem (5). Then, the energy functional defined by (34) satisfies*

$$\begin{aligned}
E'(t) \quad \leq \quad & -\left(\mu_1-\frac{\xi_1}{2\tau_1}-\frac{|\mu_2|}{2}\right)\|\varphi_t\|_2^2-\left(\widetilde{\mu_1}-\frac{\xi_2}{2\tau_2}-\frac{\left|\widetilde{\mu_2}\right|}{2}\right)\|w_t\|_2^2 \\
& -\left(\frac{\xi_1}{2\tau_1}-\frac{|\mu_2|}{2}\right)\|z_1(x,1,t)\|_2^2-\left(\frac{\xi_2}{2\tau_2}-\frac{\left|\widetilde{\mu_2}\right|}{2}\right)\|z_2(x,1,t)\|_2^2 \\
& \left(-\mu_0+\int_{\tau_1}^{\tau_2}\mu(s)ds+\frac{c_0}{2}(\tau_2-\tau_1)\right)\int_0^L\psi_t^2(x,t)\,dx \\
& -\frac{c_0}{2}\int_0^L\int_{\tau_1}^{\tau_2}\psi_t^2(x,t-s)\,dsdx
\end{aligned}$$

Proof Multiplying the first equation in (5) by φ_t, the third equation by ψ_t, the fifth equation by w_t, integrating over $(0,L)$ and using integration by parts, we get

$$(35)\qquad \begin{cases}
\rho_1\int_0^L\varphi_{tt}\varphi_t dx-Gh\int_0^L(\varphi_x+\psi+lw)_x\varphi_t dx-lEh\int_0^L(w_x-l\varphi)\varphi_t dx \\
+\mu_1\int_0^L\varphi_t^2 dx+\mu_2\int_0^L\varphi_t^2(x,t-\tau_1)\,dx=0 \\
\rho_2\int_0^L\psi_{tt}\psi_t dx-EI\int_0^L\psi_{xx}\psi_t dx-El\int_0^L(\varphi_x+\psi+lw)\psi_t dx \\
+\rho_1\int_0^L\int_{\tau_1}^{\tau_2}\mu(s)\psi_t^2(x,t-s)\,dsdx=0 \\
\rho_2\int_0^L w_{tt}w_t dx-EI\int_0^L(w_x-l\varphi)_x w_x dx+lGh\int_0^L(\varphi_x+\psi+lw)w_x dx \\
+\widetilde{\mu_1}\int_0^L w_t^2 dx+\widetilde{\mu_2}\int_0^L w_t^2(x,t-\tau_1)\,dx=0
\end{cases}$$

Then

$$\begin{cases}
\frac{1}{2}\rho_1\frac{d}{dt}\|\varphi_t\|_2^2-Gh\int_0^L(\varphi_x+\psi+lw)_x\varphi_t dx-lEh\int_0^L(w_x-l\varphi)\varphi_t dx \\
+\mu_1\|\varphi_t\|_2^2+\mu_2\int_0^L z_1(x,1,t)\varphi_t dx=0 \\
\frac{1}{2}\rho_2\frac{d}{dt}\|\psi_t\|_2^2+\frac{EI}{2}\frac{d}{dt}\|\psi_x\|_2^2+Gh\int_0^L(\varphi_x+\psi+lw)\psi_t dx \\
+\int_0^L\int_{\tau_1}^{\tau_2}\mu(s)\psi_t(x,t-s)\psi_t(x,t)\,dsdx=0 \\
\frac{1}{2}\rho_1\frac{d}{dt}\|w_t\|_2^2-Eh\int_0^L(w_x-l\varphi)_x w_t dx+lGh\int_0^L(\varphi_x+\psi+lw)w_t dx \\
+\widetilde{\mu_1}\|w_t\|_2^2+\widetilde{\mu_2}\int_0^L z_2(x,1,t)w_t dx=0=0
\end{cases}$$

We obtain

$$\begin{aligned}
&\tfrac{d}{dt}\Big(\tfrac{1}{2}\rho_1\|\varphi_t\|_2^2+\tfrac{1}{2}\rho_2\|\psi_t\|_2^2+\tfrac{1}{2}\rho_1\|w_t\|_2^2+\tfrac{1}{2}EI\|\psi_x\|_2^2+\tfrac{1}{2}Gh\|\varphi_x+\psi+lw\|_2^2\\
&+\tfrac{1}{2}Eh\|w_x-l\varphi\|_2^2\Big)+\mu_1\|\varphi_t\|_2^2+\widetilde{\mu_1}\|w_t\|_2^2\\
&+\mu_2\textstyle\int_0^L z_1(x,1,t)\,\varphi_t dx+\widetilde{\mu_2}\int_0^L z_2(x,1,t)\,w_t dx\\
&+\textstyle\int_0^L\int_{\tau_1}^{\tau_2} s\mu(s)\int_0^1\psi_t(x,t-\rho s)\,\psi_{tt}(x,t-\rho s)\,d\rho ds dx=0
\end{aligned} \tag{36}$$

Multiplying the equation in (16) by $\xi_i z_i$ and integrating over $(0,L)\times(0,1)$, obtain:

$$\begin{aligned}
2\xi_i\frac{d}{dt}\int_0^L\int_0^1 z_i^2(x,\rho,t)\,d\rho dx &= -\frac{\xi_i}{\tau_i}\int_0^L\int_0^1 z_i z_{i\rho}\,d\rho dx\\
&= \frac{2\xi_i}{\tau_i}\int_0^L\left(z_i^2(x,0,t)-z_i^2(x,1,t)\right)dx\\
&= \frac{2\xi_i}{\tau_i}\left[\|z_i(x,0,t)\|_2^2-\|z_i(x,1,t)\|_2^2\right]
\end{aligned} \tag{37}$$

$$\frac{d}{dt}2\xi_1\int_0^L\int_0^1 z_1^2(x,\rho,t)\,d\rho dx-\frac{2\xi_1}{\tau_1}\|\varphi_t\|_2^2+\frac{2\xi_1}{\tau_1}\|z_1(x,1,t)\|_2^2=0$$

$$\frac{d}{dt}2\xi_2\int_0^L\int_0^1 z_2^2(x,\rho,t)\,d\rho dx-\frac{2\xi_2}{\tau_2}\|w_t\|_2^2+\frac{2\xi_2}{\tau_2}\|z_2(x,1,t)\|_2^2=0$$

where $z_1(x,0,t)=\varphi_t(x,t)$ and $z_2(x,0,t)=w_t(x,t)$. From (34), (36) and (37) and using Young inequality we get

$$\begin{aligned}
\mu_2\int_0^L z_1(x,1,t)\,\varphi_t dx &\le \frac{\mu_2}{2}\int_0^L z_1^2(x,1,t)\,dx+\frac{\mu_2}{2}\int_0^L\varphi_t^2 dx\\
&\le \frac{\mu_2}{2}\int_0^L z_1^2(x,1,t)\,dx+\frac{\mu_2}{2}\|\varphi_t\|_2^2
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\mu_2}\int_0^L z_2(x,1,t)\,w_t dx &\le \frac{\widetilde{\mu_2}}{2}\int_0^L z_2^2(x,1,t)\,dx+\frac{\widetilde{\mu_2}}{2}\int_0^L w_t^2 dx\\
&\le \frac{\widetilde{\mu_2}}{2}\int_0^L z_2^2(x,1,t)\,dx+\frac{\widetilde{\mu_2}}{2}\|w_t\|_2^2
\end{aligned}$$

Then

$$
\begin{aligned}
E'(t) \leq & -\left(\mu_1 - \frac{\mu_2}{2} - \frac{2\xi_1}{\tau_1}\right)\|\varphi_t\|_2^2 - \left(\frac{2\xi_1}{\tau_1} - \frac{\mu_2}{2}\right)\|z_1(x,1,t)\|_2^2 \\
& -\left(\widetilde{\mu_1} - \frac{\widetilde{\mu_2}}{2} - \frac{2\xi_2}{\tau_2}\right)\|w_t\|_2^2 - \left(\frac{2\xi_2}{\tau_2} - \frac{\widetilde{\mu_2}}{2}\right)\|z_2(x,1,t)\|_2^2 \\
& + \int_0^L \int_{\tau_1}^{\tau_2} s(\mu(s)+c_0) \int_0^1 \psi_t(x,t-\rho s)\,\psi_{tt}(x,t-\rho s)\,d\rho ds dx. \qquad (38)
\end{aligned}
$$

Now, observe that

$$-s\psi_t(x,t-\rho s) = \psi_\rho(x,t-\rho s)$$

and

$$s^2\psi_{tt}(x,t-\rho s) = \psi_{\rho\rho}(x,t-\rho s)$$

Therefore,

$$\int_0^1 \psi_t(x,t-\rho s)\,\psi_{tt}(x,t-\rho s)\,d\rho = -\int_0^1 s^{-3}\psi_\rho(x,t-\rho s)\,\psi_{\rho\rho}(x,t-\rho s)\,d\rho$$

from which follows, integrating by parts in

$$
\begin{aligned}
& \int_0^L \int_{\tau_1}^{\tau_2} s(\mu(s)+c_0) \int_0^1 \psi_t(x,t-\rho s)\,\psi_{tt}(x,t-\rho s)\,d\rho ds dx \\
= & \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} (\mu(s)+c_0)\left[\psi_t^2(x,t) - \psi_t^2(x,t-s)\right] ds dx \\
= & \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} (\mu(s)+c_0)\psi_t^2(x,t)\,ds dx - \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} (\mu(s)+c_0)\psi_t^2(x,t-s)\,ds dx
\end{aligned}
$$

Now, from Cauchy-Schwarz's inequality,

$$
\begin{aligned}
& \left|\int_0^L \psi_t(t) \int_{\tau_1}^{\tau_2} \mu(s)\psi_t(t-s) ds dx\right| \\
\leq & \int_0^L |\psi_t(t)| \int_{\tau_1}^{\tau_2} \mu(s)\,|\psi_t(t-s)|\,ds dx \\
\leq & \int_0^L |\psi_t(t)| \left(\int_{\tau_1}^{\tau_2} \mu(s) ds\right)^{\frac{1}{2}} \left(\int_{\tau_1}^{\tau_2} \mu(s)\psi_t^2(x,t-s) ds\right)^{\frac{1}{2}} dx \\
\leq & \frac{1}{2}\int_0^L \psi_t^2(t)\left(\int_{\tau_1}^{\tau_2} \mu(s) ds\right) dx \\
& + \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} \mu(s)\psi_t^2(x,t-s) ds dx \qquad (39)
\end{aligned}
$$

So, from (38) and (39) we obtain

$$\begin{aligned} E'(t) \leq & -\left(\mu_1 - \frac{\mu_2}{2} - \frac{2\xi_1}{\tau_1}\right)\|\varphi_t\|_2^2 - \left(\frac{2\xi_1}{\tau_1} - \frac{\mu_2}{2}\right)\|z_1(x,1,t)\|_2^2 \\ & -\left(\widetilde{\mu_1} - \frac{\widetilde{\mu_2}}{2} - \frac{2\xi_2}{\tau_2}\right)\|w_t\|_2^2 - \left(\frac{2\xi_2}{\tau_2} - \frac{\widetilde{\mu_2}}{2}\right)\|z_2(x,1,t)\|_2^2 \\ & \left(-\mu_0 + \int_{\tau_1}^{\tau_2}\mu(s)ds + \frac{c_0}{2}(\tau_2 - \tau_1)\right)\int_0^L \psi_t^2(x,t)\,dx \\ & -\frac{c_0}{2}\int_0^L\int_{\tau_1}^{\tau_2}\psi_t^2(x,t-s)\,dsdx \end{aligned}$$

This completes the proof of the Lemma 3.2.

Lemma 3.3. *There exists a positive constant C such that the following inequality holds for every* $(\varphi,\psi,w) \in (H_0^1(0,L))^3$

$$\begin{aligned} \int_0^L \left(|\varphi_x|^2 + |\psi_x|^2 + |w_x|^2\right)dx & \leq C\int_0^L \left(EI|\psi_x|^2 + Gh|\varphi_x + \psi + lw|^2 + Eh|w_x - l\varphi|^2\right)dx \\ & \leq E(t) \end{aligned} \tag{40}$$

Proof We will argue by contradiction. Indeed, let us suppose that (40) is not true. So, we can find a sequence $\{\varphi_\upsilon,\psi_\upsilon,w_\upsilon\}_{\upsilon\in\mathbb{N}}$ in $(H_0^1(0,L))^3$ satisfying

$$\int_0^L \left(EI|\psi_{\upsilon x}|^2 + Gh|\varphi_{\upsilon x} + \psi_\upsilon + lw_\upsilon|^2 + Eh|w_{\upsilon x} - l\varphi_\upsilon|^2\right)dx \leq \frac{1}{\upsilon} \tag{41}$$

and

$$\int_0^L \left(|\varphi_{\upsilon x}|^2 + |\psi_{\upsilon x}|^2 + |w_{\upsilon x}|^2\right)dx = 1 \tag{42}$$

From (42), the sequence $\{\varphi_\upsilon,\psi_\upsilon,w_\upsilon\}_{\upsilon\in\mathbb{N}}$ is bounded in $(H_0^1(0,L))^3$. Since the embedding $H_0^1(0,L) \hookrightarrow L^2(0,L)$ is compact, then the sequence $\{\varphi_\upsilon,\psi_\upsilon,w_\upsilon\}_{\upsilon\in\mathbb{N}}$ converges strongly in $(H_0^1(0,L))^3$.

From (41)

$$\psi_{\upsilon x} \longrightarrow 0 \text{ strongly in } L^2(0,L)$$

Using Poincaré's inequality we can conclude that

$$\psi_\upsilon \longrightarrow 0 \text{ strongly in } L^2(0,L)$$

Now, setting

$$\varphi_\upsilon \longrightarrow \varphi \text{ and } w_\upsilon \longrightarrow w \text{ strongly in } L^2(0,L)$$

From (41), we have

$$\varphi_{\upsilon x} + \psi_\upsilon + l w_\upsilon \longrightarrow 0 \text{ strongly in } L^2(0,L)$$

Then

$$\varphi_{\upsilon x} + \psi_\upsilon + l w_\upsilon = \varphi_{\upsilon x} + \psi_\upsilon + l(w_\upsilon - w) + lw \longrightarrow 0 \text{ strongly in } L^2(0,L)$$

which implies that

$$\varphi_{\upsilon x} \longrightarrow -lw \text{ strongly in } L^2(0,L)$$

Then, $\{\varphi_\upsilon\}_n$ is a Cauchy sequence in $H^1(0,L)$. Therefore $\{\varphi_\upsilon\}_n$ converges to a function $\varphi_1$in $H^1(0,L)$.Consequently $\{\varphi_\upsilon\}_n$ converges to $\varphi_1$in $H^1(0,L)$. Thus by the uniqueness of the limit $\varphi_1 = \varphi$.Moreover $\varphi \in H_0^1(0,L)$.

From (39) we deduce that

$$\varphi_x + lw = 0 \text{ a.e } x \in (0,L) \tag{43}$$

Similarly, we have

$$w_x - l\varphi = 0 \text{ a.e } x \in (0,L) \tag{44}$$

and $w \in H_0^1(0,L)$. (43) and (44) provides us $\varphi = w = 0$, contradicting (42). This completes the proof of the Lemma 3.3.

Proof of Theorem 3.1. From now on, we denote by c various positive constants which may be different at different occurrences. Multiplying the first equation in (5) by $E^q\varphi$, the third equation by $E^q\psi$ and the fifth equation by $E^q w$ we obtain

$$\begin{aligned} 0 &= \int_S^T E^q \int_0^L \varphi(\rho_1\varphi_{tt} - Gh(\varphi_x + lw + \psi)_x - Ehl(w_x - l\varphi) \\ &\quad + \mu_1\varphi_t + \mu_2 z_1(x,1,t))dxdt \end{aligned}$$

$$\begin{aligned} 0 &= \left[E^q\rho_1\int_0^L \varphi\varphi_t dx\right]_S^T - \int_S^T \rho_1 qE'E^{q-1}\int_0^L \varphi\varphi_t dxdt \\ &-\rho_1\int_S^T E^q\|\varphi_t\|_2^2 dt - \int_S^T E^q\int_0^L \varphi_x Gh(\varphi_x+lw+\psi)\,dxdt \\ &-\int_S^T E^q\int_0^L \varphi(lEh)(w_x-l\varphi)\,dxdt + \mu_1\int_S^T E^q\int_0^L \varphi_t\varphi dxdt \\ &+\mu_2\int_S^T E^q\int_0^L \varphi z_1(x,1,t)dxdt, \end{aligned}$$

$$\begin{aligned} 0 &= \int_S^T E^q\int_0^L \psi(\rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x+lw+\psi) \\ &+\mu_0\psi_t + \rho_2\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s))dsdxdt \end{aligned}$$

$$\begin{aligned} 0 &= \left[E^q\rho_2\int_0^L \psi\psi_t dx\right]_S^T - \int_S^T \rho_2 qE'E^{q-1}\int_0^L \psi\psi_t dxdt - \rho_2\int_S^T E^q\|\psi_t\|_2^2 dt \\ &+\int_S^T E^q EI\|\psi_x\|_2^2 dt + \int_S^T E^q\int_0^L \psi_x Gh(\varphi_x+lw+\psi)\,dxdt \\ &+\rho_2\int_S^T E^q\int_0^L \psi\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)\,dsdxdt \end{aligned}$$

$$\begin{aligned} 0 &= \int_S^T E^q\int_0^L w(\rho_1 w_{tt} - Eh(w_x-l\varphi)_x + lGh(\varphi_x+lw+\psi) \\ &+\widetilde{\mu_1}w_t + \widetilde{\mu_2}z_2(x,1,t))dxdt, \end{aligned} \tag{45}$$

$$\begin{aligned} 0 &= \left[E^q\rho_1\int_0^L ww_t dx\right]_S^T - \int_S^T \rho_1 qE'E^{q-1}\int_0^L ww_t dxdt \\ &-\rho_1\int_S^T E^q\|w_t\|_2^2 dt + \int_S^T E^q\int_0^L Ehw_x(w_x-l\varphi)\,dxdt \\ &+\int_S^T E^q\int_0^L w(lGh)(\varphi_x+lw+\psi)\,dxdt \\ &+\widetilde{\mu_1}\int_S^T E^q\int_0^L w_t w dxdt + \widetilde{\mu_2}\int_S^T E^q\int_0^L wz_2(x,1,t)dxdt. \end{aligned}$$

Taking their sum, we obtain

$$\begin{aligned}
0 \;=\;& \left[E^q\rho_1\int_0^L \varphi\varphi_t dx\right]_S^T + \left[E^q\rho_2\int_0^L \psi\psi_t dx\right]_S^T + \left[E^q\rho_1\int_0^L ww_t dx\right]_S^T \\
&-\int_S^T \rho_1 qE'E^{q-1}\int_0^L (\rho_1\varphi\varphi_t+\rho_2\psi\psi_t+\rho_1 ww_t)\,dxdt \\
&-2\rho_1\int_S^T E^q\|\varphi_t\|_2^2 dt - 2\rho_2\int_S^T E^q\|\psi_t\|_2^2 dt - 2\rho_1\int_S^T E^q\|w_t\|_2^2 dt \\
&+\int_S^T E^q(\rho_1\|\varphi_t\|_2^2+\rho_2\|\psi_t\|_2^2+\rho_1\|w_t\|_2^2 \\
&+Gh\|\varphi_x+lw+\psi\|_2^2+EI\|\psi_x\|_2^2+Eh\|w_x-l\varphi\|_2^2)dt \\
&+\mu_1\int_S^T E^q\int_0^L \varphi_t\varphi dxdt+\mu_2\int_S^T E^q\int_0^L \varphi z_1(x,1,t)\,dxdt \\
&+\widetilde{\mu_1}\int_S^T E^q\int_0^L w_t w dxdt+\widetilde{\mu_2}\int_S^T E^q\int_0^L wz_2(x,1,t)\,dxdt \\
&+\rho_1\int_S^T E^q\int_0^L \varphi\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)\,dsdxdt. \qquad (46)
\end{aligned}$$

Similarly, we multiply the equation of (17) by $E^q e^{-2\rho}z(x,\rho,t,s)$ and get

$$\begin{aligned}
0 \;=\;& \int_S^T E^q\int_0^L\int_0^1 e^{-2\rho}z(sz_t+z_\rho)d\rho dxdt \\
=\;& \left[\frac{s}{2}E^q\int_0^L\int_0^1 e^{-2\rho}z^2 d\rho dx\right]_S^T \\
&-\frac{s}{2}\int_S^T qE^{q-1}E'\int_0^L\int_0^1 e^{-2\rho}z^2 d\rho dxdt \\
&+\int_S^T E^q\int_0^L\int_0^1 \frac{e^{-2\rho}}{2}\frac{d}{d\rho}(z^2)d\rho dxdt \qquad (47)
\end{aligned}$$

$$\begin{aligned}
0 \;=\;& \left[\frac{s}{2}E^q\int_0^L\int_0^1 e^{-2\rho}z^2 d\rho dx\right]_S^T \\
&-\frac{s}{2}\int_S^T qE^{q-1}E'\int_0^L\int_0^1 e^{-2\rho}z^2 d\rho dxdt \\
&+\frac{1}{2}\int_S^T E^q\int_0^L\int_0^1 \left[\frac{d}{d\rho}\left(e^{-2\rho}z^2\right)+2e^{-2\rho}z^2\right]d\rho dxdt
\end{aligned}$$

$$\begin{aligned}0 &= \left[\frac{s}{2}E^{q}\int_{0}^{L}\int_{0}^{1}e^{-2\rho}z^{2}d\rho dx\right]_{S}^{T}\\&-\frac{s}{2}\int_{S}^{T}qE^{q-1}E'\int_{0}^{L}\int_{0}^{1}e^{-2\rho}z^{2}d\rho dxdt\\&+\frac{1}{2}\int_{S}^{T}E^{q}\int_{0}^{L}\left[e^{-2}z^{2}(x,1,t)-z^{2}(x,1,t)\right]dxdt\\&+\int_{S}^{T}\int_{0}^{L}\int_{0}^{1}e^{-2\rho}z^{2}d\rho dxdt\end{aligned}$$

And also multiplying the equation of (16) by $E^{q}\xi_i e^{-2\tau_i\rho}z_i(x,\rho,t)$ we get

$$\begin{aligned}0 &= \int_{S}^{T}E^{q}\int_{0}^{L}\int_{0}^{1}e^{-2\tau_i\rho}\xi_i z_i(\tau_i z_{it}+z_{i\rho})d\rho dxdt\\&= \left[\frac{1}{2}\xi_i\tau_i E^{q}\int_{0}^{L}\int_{0}^{1}e^{-2\tau_i\rho}z_i^{2}d\rho dx\right]_{S}^{T}\\&-\frac{\tau_i\xi_i}{2}\int_{S}^{T}qE^{q-1}E'\int_{0}^{L}\int_{0}^{1}e^{-2\tau_i\rho}z_i^{2}d\rho dxdt\\&+\int_{S}^{T}E^{q}\xi_i\int_{0}^{L}\int_{0}^{1}\frac{e^{-2\tau_i\rho}}{2}\frac{d}{d\rho}(z_i^{2})d\rho dxdt,\end{aligned} \tag{48}$$

$$\begin{aligned}0 &= \left[\frac{1}{2}\xi_i\tau_i E^{q}\int_{0}^{L}\int_{0}^{1}e^{-2\tau_i\rho}z_i^{2}d\rho dx\right]_{S}^{T}\\&-\frac{\tau_i\xi_i}{2}\int_{S}^{T}qE^{q-1}E'\int_{0}^{L}\int_{0}^{1}e^{-2\tau_i\rho}z_i^{2}d\rho dxdt\\&+\frac{\xi_i}{2}\int_{S}^{T}E^{q}\int_{0}^{L}\int_{0}^{1}\left[\frac{d}{d\rho}\left(e^{-2\tau_i\rho}z_i^{2}\right)+2\tau_i e^{-2\tau_i\rho}z_i^{2}\right]d\rho dxdt,\end{aligned}$$

$$\begin{aligned}0 &= \left[\frac{1}{2}\xi_i\tau_i E^{q}\int_{0}^{L}\int_{0}^{1}e^{-2\tau_i\rho}z_i^{2}d\rho dx\right]_{S}^{T}\\&-\frac{\tau_i\xi_i}{2}\int_{S}^{T}qE^{q-1}E'\int_{0}^{L}\int_{0}^{1}e^{-2\tau_i\rho}z_i^{2}d\rho dxdt\\&+\frac{\xi_i}{2}\int_{S}^{T}E^{q}\int_{0}^{L}\left[e^{-2\tau_i}z_i^{2}(x,1,t)-z_i^{2}(x,1,t)\right]dxdt\\&+\xi_i\tau_i\int_{S}^{T}\int_{0}^{L}\int_{0}^{1}e^{-2\tau_i\rho}z_i^{2}d\rho dxdt\end{aligned}$$

Recalling the de definition of E and from (46), (47),and (48) we get

$$
\begin{aligned}
A\int_S^T E^{q+1}dt \leq & -\left[\rho_1 E^q\int_0^L \varphi\varphi_t dx\right]_S^T-\left[\rho_2 E^q\int_0^L \psi\psi_t dx\right]_S^T-\left[\rho_1 E^q\int_0^L ww_t dx\right]_S^T \\
& +\int_S^T qE'E^{q-1}\int_0^L (\rho_1\varphi\varphi_t+\rho_2\psi\psi_t+\rho_1 ww_t)\,dxdt \\
& +2\int_S^T E^q\left(\rho_1\|\varphi_t\|_2^2+\rho_2\|\psi_t\|_2^2+\rho_1\|w_t\|_2^2\right)dt \\
& -\mu_0\int_S^T E^q\int_0^L \psi\psi_t dxdt+\rho_2\int_S^T E^q\int_0^L \psi\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)\,dsdxdt \\
& -\left[\frac{1}{2}E^q\int_0^L\int_0^1 e^{-2\rho}z^2 d\rho dx\right]_S^T \\
& +\frac{1}{2}\int_S^T qE^{q-1}E'\int_0^L\int_0^1 e^{-2\rho}z^2 d\rho dxdt \\
& -\frac{1}{2}\int_S^T E^q e^{-2}\int_0^L z^2(x,1,t)\,dxdt \\
& +\frac{1}{2}\int_S^T \left\|z^2(x,0,t)\right\|_2^2 dt \\
& -\mu_1\int_S^T E^q\int_0^L \varphi_t\varphi dxdt-\mu_2\int_S^T E^q\int_0^L \varphi z_1(x,1,t)\,dxdt \\
& -\widetilde{\mu_1}\int_S^T E^q\int_0^L w_t w dxdt-\widetilde{\mu_2}\int_S^T E^q\int_0^L wz_2(x,1,t)\,dxdt \\
& -\sum_{i=1}^{2}\left[\frac{\xi_i\tau_i}{2}E^q\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2 d\rho dx\right]_S^T \\
& +\sum_{i=1}^{2}\frac{\xi_i\tau_i}{2}\int_S^T qE^{q-1}E'\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2 d\rho dx \\
& -\sum_{i=1}^{2}\frac{\xi_i}{2}\int_S^T E^q e^{-2\tau_i}\int_0^L z_i^2(x,1,t)\,dxdt \\
& +\sum_{i=1}^{2}\frac{\xi_i}{2}\int_S^T E^q\|z_i(x,0,t)\|_2^2 dt
\end{aligned}
$$

where $A=2\min\left\{1,e^{-2},2\tau_1 e^{-2\tau_1},2\tau_2 e^{-2\tau2}\right\}$. Using the Young and Sobolev- Poincare inequalities and Lemma , we find that

$$
\begin{aligned}
\left[E^q\int_0^L \varphi\varphi_t dx\right]_S^T &= E^q(S)(S)\int_0^L \varphi(S)\varphi_t(S)dx-E^q(T)\int_0^L \varphi(T)\varphi_t(T)dx \\
&\leq CE^{q+1}(S)
\end{aligned}
$$

$$\begin{aligned}\left|\int_S^T\left(qE'E^{q-1}\right)\int_0^L(\rho_1\varphi\varphi_t+\rho_2\psi\psi_t+\rho ww_t)dxdt\right| &\leq c\int_S^T(-E')E^q dt\\ &\leq cE^{q+1}(S)\end{aligned}$$

$$\left|\frac{1}{2}\xi_i\tau_i E^q\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2 dxd\rho\right|\leq cE^{q+1}(S),\forall t\geq s,$$

$$\int_S^T E^q\int_0^L\varphi_t^2 dxdt\leq c\int_S^T(-E')E^q dt\leq cE^{q+1}(S)$$

$$\int_S^T E^q\xi_i\int_0^L e^{-2\tau_i}z_i^2(x,1,t)\,dxdt\leq c\int_S^T(-E')E^q dt\leq cE^{q+1}(S)$$

$$\frac{1}{2}\int_S^T E^q\xi_i\int_0^L z_i^2(x,0,t)\,dxdt=\int_S^T E^q\xi_i\int_0^L\varphi'^2 dxdt\leq cE^{q+1}(S)$$

$$\left|\frac{\tau_i\xi_i}{2}\int_S^T qE^{q-1}E'\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2 dxd\rho dt\right|\leq c\int_S^T(-E')E^q dt\leq cE^{q+1}(S) \tag{49}$$

$$\begin{aligned}\left|\int_S^T E^q\int_0^L\varphi\varphi_t dxdt\right| &\leq \varepsilon_1\int_S^T E^q\int_0^L\varphi^2 dxdt+c(\varepsilon_1)\int_S^T E^q\int_0^L\varphi_t^2 dxdt\\ &\leq \varepsilon_1 c\int_S^T E^{q+1}dt+c(\varepsilon_1)\int_S^T E^q\int_0^L\varphi_t^2 dxdt\\ &\leq \varepsilon_1 c\int_S^T E^{q+1}dt+c(\varepsilon_1)\int_S^T E^q\left(-E'\right)dt\\ &\leq \varepsilon_1 c\int_S^T E^{q+1}dt+c(\varepsilon_1)E^{q+1}(S) \qquad (50)\end{aligned}$$

and

$$
\begin{aligned}
\left|\int_S^T E^q \int_0^L \varphi z_1(x,1,t)dxdt\right| &\leq \varepsilon_2 \int_S^T E^q \int_0^L \varphi^2 dxdt + c(\varepsilon_2)\int_S^T E^q \int_0^L z_1(x,1,t)^2 dxdt \\
&\leq \varepsilon_2 c \int_S^T E^{q+1}dt + c(\varepsilon_2)\int_S^T E^q \int_0^L z_1(x,1,t)^2 dxdt \\
&\leq \varepsilon_2 c \int_S^T E^{q+1}dt + c(\varepsilon_2)\int_S^T E^q \left(-E'\right)dt \\
&\leq \varepsilon_2 c \int_S^T E^{q+1}dt + c(\varepsilon_2) E^{q+1}(S). \qquad (51)
\end{aligned}
$$

$$
\left|\int_S^T E^q \int_0^L \psi\psi_t dxdt\right| \leq \varepsilon_1' c \int_S^T E^{q+1}dt + c(\varepsilon_1')E^{q+1}(S), \qquad (52)
$$

$$
\left|\int_S^T E^q \int_0^L w w_t dxdt\right| \leq \varepsilon_2' c \int_S^T E^{q+1}dt + c(\varepsilon_2')E^{q+1}(S), \qquad (53)
$$

$$
\begin{aligned}
\left|\int_S^T E^q \int_0^L w z_2(x,1,t)dxdt\right| &\leq \varepsilon_3 \int_S^T E^q \int_0^L w^2 dxdt + c(\varepsilon_3)\int_S^T E^q \int_0^L z_2(x,1,t)^2 dxdt \\
&\leq \varepsilon_3 c \int_S^T E^{q+1}dt + c(\varepsilon_3)\int_S^T E^q \int_0^L z_2(x,1,t)^2 dxdt \\
&\leq \varepsilon_3 c \int_S^T E^{q+1}dt + c(\varepsilon_3)\int_S^T E^q \left(-E'\right)dt \\
&\leq \varepsilon_3 c \int_S^T E^{q+1}dt + c(\varepsilon_3) E^{q+1}(S).
\end{aligned}
$$

Using (23) we obtain

$$
\begin{aligned}
\left|\int_S^T E^q \int_0^L \varphi \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,t,s)\, dsdxdt\right| &\leq \varepsilon_3' c \int_S^T E^q(-E')dt \\
&\leq \varepsilon_3' c E^{q+1}(S). \qquad (54)
\end{aligned}
$$

Choosing $\varepsilon_1, \varepsilon_1', \varepsilon_2, \varepsilon_2', \varepsilon_3$ and ε_3' small enough, we deduce from (49), (50), (51), (52), (53) and (54) that

$$
\int_S^T E^{q+1}dt \leq cE^{q+1}(S)
$$

where c is a positive constant independent of $E(0)$. We choose $q=0$. Hence, we deduce from Lemma that

$$E(t) \leq cE(0)e^{-\kappa t}, t \geq 0.$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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