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GENERALIZED DERIVATIONS WITH POWER VALUES IN RINGS AND BANACH ALGEBRAS

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Abstract. Let R be a prime ring, U the Utumi quotient ring of R, $C = Z(U)$ the extended centroid of R and F a generalized derivation with associated derivation d of R. Suppose that $(F([x,y]))^m = [x,y]^n$ for all $x,y \in I$, a nonzero ideal of R, where $m \geq 1$ and $n \geq 1$ are fixed integers, then one of the following holds:

(1) R is commutative;

(2) there exists $a \in C$ such that $F(x) = ax$ for all $x \in R$ with $a^m = 1$. Moreover, in this case if $m \neq n$, then either char $(R) = 2$ or char $(R) = 2^{|m-n|} - 1$.

We also extend the result to the one sided case for $m = n$. Finally as an application we obtain a range inclusion result of continuous generalized derivations on Banach algebras.

Keywords: prime ring; generalized derivation; extended centroid; utumi quotient ring.

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1. Introduction

Let R be a prime ring with center $Z(R)$ and extended centroid C, U the Utumi quotient ring of R (for more details see [3]). We denote by $[x,y] = xy - yx$ the simple commutator of the elements $x,y \in R$ and by $x \circ y = xy + yx$ the simple anti-commutator of x,y. Recall that a ring

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R is prime if for any $a,b \in R$, $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$ and is semiprime if for any $a \in R$, $aRa = \{0\}$ implies that $a = 0$. An additive mapping $d : R \to R$ is called a derivation, if $d(xy) = d(x)y + xd(y)$ holds for all $x,y \in R$. In particular, d is an inner derivation induced by an element $a \in R$, if $d(x) = [a,x]$ for all $x \in R$. In [6], Brešar introduced the definition of generalized derivation. An additive mapping $F : R \to R$ is called a generalized derivation, if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x,y \in R$. Hence, the concept of generalized derivations covers both the concepts of a derivation and a left multiplier (i.e. an additive mapping satisfying $f(xy) = f(x)y$ for all $x,y \in R$). Basic examples of generalized derivations are mappings of type $x \to ax + xb$ for some $a,b \in R$. These maps are called inner generalized derivations.

In [23], Lee proved that every generalized derivation can be uniquely extended to a generalized derivation of U and thus all generalized derivations of R implicitly assumed to be defined on the whole of U. In particular, Lee proved the follwing: Let R be a semiprime ring. Then every generalized derivation F on a dense right ideal of R can be uniquely extended to U and assumes the form $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U.

On the other hand Ashraf and Rehman [2, Theorem 4.1] proved that if R is a prime ring, I a nonzero ideal of R and d is a derivation of R such that $d(x \circ y) = (x \circ y)$ for all $x,y \in I$, then R is commutative. Then Argac and Inceboz [1] generalized the above result by considering some power values. They proved that if R is a prime ring, I a nonzero ideal of R, n a fixed positive integer and d a derivation of R such that $(d(x \circ y))^n = (x \circ y)$ for all $x,y \in I$, then R is commutative. Quadri et al. [27] proved that if R is a prime ring, I a nonzero ideal of R and F a generalized derivation associated with a nonzero derivation d such that $F(x \circ y) = x \circ y$ for all $x,y \in I$, then R is commutative.

In [9, Theorem 2], Daif and Bell proved that if R is a semiprime ring with a nonzero ideal I and d is a derivation of R such that $d([x,y]) = [x,y]$ for all $x,y \in I$, then $I \subseteq Z(R)$. In particular, if R is prime ring, then R must be commutative. Later in [27], Quadri et al. discussed the commutativity of prime rings with generalized derivations. More precisely, they proved that if R is a prime ring, I a nonzero ideal of R and F a generalized derivation associated with a nonzero derivation d such that $F([x,y]) = [x,y]$ for all $x,y \in I$, then R is commutative. Further, this result of Quadri et al. is studied in semiprime ring by Dhara in [12]. Recently in [11], De Filippis and Huang studied the situation $(F([x,y]))^n = [x,y]$ for all $x,y \in I$, where I is a nonzero ideal in a prime ring R, F a generalized derivation of R and $n \geq 1$ fixed integer. In this case they conclude that either R is commutative or $n = 1$, $d = 0$ and $F(x) = x$ for all $x \in R$.

Motivated by these results, we will investigate the situation when a prime ring R satisfies $(F([x,y]))^m = [x,y]^n$ for all x,y in some suitable subsets of R, where F is a generalized derivation of R associated with a derivation d. More precisely, we shall prove the following results:

Theorem 1.1. *Let R be a prime ring, F a generalized derivation of R and I a nonzero ideal of R. Suppose that $(F([x,y]))^m = [x,y]^n$ for all $x,y \in I$, where $m \geq 1$ and $n \geq 1$ are fixed integers. Then one of the following holds:*
(i) R is commutative;
(ii) there exists $a \in C$ such that $F(x) = ax$ for all $x \in R$ with $a^m = 1$. Moreover, in this case if $m \neq n$, then either char $(R) = 2$ or char $(R) = 2^{|m-n|} - 1$.

Theorem 1.2. *Let R be a prime ring, I a nonzero right ideal of R and F a generalized derivation of R. If $(F([x,y]))^n = [x,y]^n$ for all $x,y \in I$, where $n \geq 1$ is fixed integer, then one of the following holds:*
(i) $[I,I]I = 0$;
(ii) there exists $a \in U$ and $\alpha \in C$ such that $F(x) = ax$ for all $x \in R$, with $(a-\alpha)I = 0$ and $\alpha^n = 1$.

Theorem 1.3. *Let R be a semiprime ring and F a generalized derivation of R. If $(F([x,y]))^m = [x,y]^n$ for all $x,y \in R$, where $m \geq 1$ and $n \geq 1$ are fixed integers, then R is commutative or $F(x) = ax + d(x)$ for all $x \in R$, where $a \in C$ and d is a derivation of R such that $d(R) \subseteq Z(R)$.*

In the last section of this paper we will consider A as a Banach algebras with Jacobson radical rad(A). The classical result of Singer and Wermer [29] says that any continuous derivation on a noncommutative Banach algebra has the range in the Jacobson radical of the algebra. Singer and Wermer also formulated the conjecture that the continuity assumption can be removed. In 1988 Thomas verified this conjecture in [30]. Of course the same result of Singer and Wermer does not hold in noncommutative Banach algebras (because of inner derivation). Hence in this context a very interesting question is how to obtain the noncommutative version of the Singer-Wermer theorem. A first answer to this problem was obtained by Sinclair [28]. He proved that every continuous derivation of a Banach algebra leaves primitive ideals of the algebra invariant. Since then many authors obtained more information about derivations satisfying certain suitable conditions in Banach algebras.

In [20], Kim proved that if d is a continuous linear Jordan derivation in a Banach algebra A, such that $[d(x),x]d(x)[d(x),x] \in rad(A)$, for all $x \in A$, then d maps A into $rad(A)$. More recently, Park [26] proved that if d is a continuous linear derivation of a noncommutative Banach algebra A such that $[[d(x),x],d(x)] \in rad(A)$ for all $x \in A$ then $d(A) \subseteq rad(A)$. In [10] De Filippis extended the Park's result to generalized derivations.

In the last section, finally we obtain a range inclusion result about continuous generalized derivations on Banach algebras which is as follows:

Theorem 1.4. *Let A be a noncommutative Banach algebra. Let $F = L_a + d$ be a continuous generalized derivation of A, where L_a denote the left multiplication by some element $a \in A$ and d is a derivation on A. If $(F([x,y]))^m - [x,y]^n \in rad(A)$ for all $x,y \in A$, then $d(A) \subseteq rad(A)$.*

2. The Results on Ideals

We begin with the following:

Lemma 2.1. *Let R be a prime ring with extended centroid C, I a nonzero ideal of R and $a,b \in R$. Suppose that $(a[x,y]+[x,y]b)^m = [x,y]^n$ for all $x,y \in I$, where $m \geq 1, n \geq 1$ are fixed integers. Then one of the following holds:*

(i) R is commutative;

(ii) $a,b \in C$ with $(a+b)^m = 1$. (In this case if $m \neq n$ and $m+n$ is odd, then char $(R) = 2$ and if $m \neq n$ and $m+n$ is even, then char $(R) = 2^{|m-n|} - 1$.

Proof. If R is commutative, the conclusion (1) is obtained. So, we assume that R is noncommutative. Then by assumption, I satisfies the generalized polynomial identity

$$F(x,y) = (a[x,y]+[x,y]b)^m - [x,y]^n.$$

By Chuang [8, Theorem 2], this generalized polynomial identity (GPI) is also satisfied by U. We assume either $a \notin C$ or $b \notin C$ and prove that a number of contradictions follows. In this case $F(x,y) = 0$ is a nontrivial GPI for U. In case C is infinite, we have $F(x,y) = 0$ for all $x,y \in U \bigotimes_C \overline{C}$ where $\overline{C}$ is the algebraic closure of C. Since both U and $U \bigotimes_C \overline{C}$ are prime and centrally closed [13], we may replace R by U or $U \bigotimes_C \overline{C}$ according to C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed and $f(x,y) = 0$ for all $x,y \in R$. By Martindale's Theorem [24], R is then a primitive ring having nonzero $soc(R)$ with C as the associated division ring. Hence by Jacobson's Theorem [17], R is isomorphic to a dense ring of linear transformations of a vector space V over C. Assume first that $\dim_C V = k$, then the density of R on V implies that $R \cong M_k(C)$. Since R is noncommutative, $k \geq 2$.

In this case assuming $x = e_{ij}$ and $y = e_{jj}$, $i \neq j$, so that $[x,y] = e_{ij}$, we obtain that

$$(ae_{ij} + e_{ij}b)^m = (e_{ij})^n.$$

Left and right multiplying by e_{ij} respectively, above relation yields $0 = e_{ij}(ae_{ij})^m = a_{ji}^m e_{ij}$ implying $a_{ji} = 0$ and $0 = (e_{ij}b)^m e_{ij} = b_{ji}^m e_{ij}$ implying $b_{ji} = 0$. Thus both a and b are diagonal matrices in $M_k(C)$. Let $a = \sum\limits_{i=1}^{k} a_{ii}e_{ii}$ and $b = \sum\limits_{i=1}^{k} b_{ii}e_{ii}$. Now since for any automorphism φ of R, $(\varphi(a)[x,y] + [x,y]\varphi(b))^m = [x,y]^n$ holds for all $x,y \in M_k(C)$, we can write by above arguments that $\varphi(a)$ and $\varphi(b)$ are diagonal. Hence for each $j \neq 1$, we have $(1+e_{1j})a(1-e_{1j}) = \sum\limits_{i=1}^{k} a_{ii}e_{ii} + (a_{jj} - a_{11})e_{1j}$ and $(1+e_{1j})b(1-e_{1j}) = \sum\limits_{i=1}^{k} b_{ii}e_{ii} + (b_{jj} - b_{11})e_{1j}$ both diagonal. Therefore, $a_{jj} = a_{11}$ and $b_{jj} = b_{11}$ that is, $a,b \in F.I_k$. Thus we have $a,b \in C$, a contradiction.

Next assume that $\dim_C V = \infty$. Since $a \notin C$ or $b \notin C$, they do not centralize the nonzero ideal $H = soc(R)$ and hence there exist $h,h' \in H$ such that $[a,h] \neq 0$ or $[b,h'] \neq 0$. Moreover, because of the infinite dimensionality, H does not satisfy the polynomial $[x,y]$, that is, there exist $h_1,h_2 \in H$ such that $[h_1,h_2] \neq 0$. By Litoff's theorem [14], there exists idempotent $e^2 = e \in H$ such that $ah, ha, bh', h'b, h, h', h_1, h_2 \in eRe$, moreover eRe is a central simple algebra finite dimensional over its center. Since R satisfies $(a[x,y] + [x,y]b)^m = [x,y]^n$, replacing x with e and y with $ex(1-e)$ we have that R satisfies $(aex(1-e) + ex(1-e)b)^m = (ex(1-e))^n$. Left multiplying by $(1-e)$, we get $(1-e)(aex(1-e))^m = 0$ for all $x \in R$, that is $((1-e)aex)^{m+1} = 0$

for all $x \in R$. Then by Levitzki's lemma [15, Lemma 1.1], we conclude that $(1-e)aex = 0$ for all $x \in R$ and so $(1-e)ae = 0$. Since R satisfies generalized identity $e\{(a[exe,eye]+[exe,eye]b)^m - [exe,eye)]^n\}e = 0$, the subring eRe satisfies $(eae[x,y]+[x,y]ebe)^m - [x,y]^n = 0$. Since $[h_1,h_2] \neq 0$, eRe is not commutative and so $eRe \cong M_k(C)$ for $k \geq 2$. Then by the above finite dimensional case, eae and ebe are central element of eRe. Thus $ah = eaeh = heae = ha$ and $bh' = ebeh' = h'ebe = h'b$, which contradicts our assumption.

In light of previous argument, we have that both $a,b \in C$ and then our identity reduces to $(a+b)^m[x,y]^m = [x,y]^n$ for all $x,y \in R$. By [21, Lemma 2] it follows that there exists a field F such that $R \subseteq M_k(F)$, the ring of all $k \times k$ matrix over the field F, and moreover $M_k(F)$ satisfies the same generalized polynomial identity as R. that is, $(a+b)^m[x,y]^m = [x,y]^n$ for all $x,y \in M_k(F)$. Let $k \geq 2$. Then $[x,y] = [e_{12}, e_{21}] = e_{11} - e_{22}$. We have $(a+b)^m(e_{11}-e_{22})^m = (e_{11}-e_{22})^n$. Right multiplying by e_{11} in the above equation, we get $(a+b)^m e_{11} = e_{11}$. This imples that $((a+b)^m - 1)e_{11} = 0$. Which gives that $(a+b)^m = 1$. Thus our identity reduces to $[x,y]^m = [x,y]^n$ for all $x,y \in M_k(F)$. If $m = n$, then the identity is trivial and then the proof is done.

Now we assume that $m \neq n$. Then we consider the following two cases: (i) Let $m+n$ be odd. We have $[x,y]^m - [x,y]^n = 0$ for all $x,y \in M_k(F)$. In this case replacing y with $-y$, we have $[x,y]^m + [x,y]^n = 0$ for all $x,y \in M_k(F)$. By addition of above two identities, we get $2[x,y]^m = 0$ for all $x,y \in M_k(F)$. Replacing $x = e_{12}$ and $y = e_{21}$, we have $2(e_{11}-e_{22})^m = 0$. Right multiplying by e_{11} in the above equation, we get $2e_{11} = 0$. This leads a contradiction, unless char $(R) = 2$. (ii) Let $m+n$ be even. Choose $x = e_{11}$ and $y = e_{12}+e_{21}$. Then $[x,y] = e_{12} - e_{21}$. Take $m = 3$ and $n = 1$, also $[x,y]^3 = e_{21} - e_{12}$. So $[x,y]^3 = [x,y]$. We have $(e_{21}-e_{12}) = (e_{12}-e_{21})$. Right multiplying by e_{21} in the above equation, we get $2e_{11} = 0$. This leads a contradiction, unless char $(R) = 2$.

Proof of Theorem 1.1. If $F = 0$, then $[x,y]^n = 0$ for all $x,y \in I$. Hence R is commutative, by Herstein [16, Theorem 2]. If $F \neq 0$, then by hypothesis we have,

$$(F([x,y]))^m = [x,y]^n \text{ for all } x,y \in I. \tag{2.1}$$

By [23], we have that for all $x \in U$, $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d of U. Hence I satisfies the differential identity

$$(a[x,y] + [d(x),y] + [x,d(y)])^m = [x,y]^n. \tag{2.2}$$

By Chuang [8, Theorem 2], this generalized polynomial identity (GPI) is also satisfied by U, we have $(a[x,y] + [d(x),y] + [x,d(y)]])^m = [x,y]^n$ for all $x,y \in U$. By Kharchenko [19], we divide the proof into two cases:

Assume first that d is inner derivation of U, that is, there exists $p \in U$ such that $d(x) = [p,x]$ for all $x \in U$. Then

$$(b[x,y] + [[p,x],y] + [x,[p,y]])^m = [x,y]^n,$$

for all $x,y \in U$, that is

$$((b+p)[x,y] - [x,y]p)^m = [x,y]^n,$$

for all $x,y \in U$. By Lemma 2.1, one of the following holds: (i) R is commutative and so conclusion (1) is obtained. (ii) $b+p$, $p \in C$ and $b^m = 1$, which is our conclusion (2). Next assume that d is not U-inner. From (2.2), we have U satisfies

$$(b[x,y] + [d(x),y] + [x,d(y)])^m = [x,y]^n. \tag{2.3}$$

Then by Kharchenko's theorem [19], we have

$$(b[x,y] + [z,y] + [x,t])^m = [x,y]^n \tag{2.4}$$

for all $x,y,z,t \in U$. In particular, for $y = 0$, we have $[x,t]^m = 0$ for all $x,t \in U$. This implies that R is commutative, again by Herstein [16, Theorem 2]. Hence the theorem is proved.

Corollary 2.1. *Let R be a prime ring, F a generalized derivation of R associated to a nonzero derivation of R and I a nonzero ideal of R. Suppose that $(F([x,y]))^m = [x,y]^n$ for all $x,y \in I$, where $m,n \geq 1$ are fixed integers. Then R is commutative.*

Example. *Let Z be the ring of integers. Set $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a,b,c \in Z \right\}$ and $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in Z \right\}$. Define the following maps: $F\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 2b \\ 0 & 0 \end{pmatrix}$, $d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ Then it is easy to see that F is a generalized derivation with associated derivation d of R.*

Moreover, it is straightforward to check that F satisfies the property $(F([x,y]))^m = [x,y]^n$ for all $x,y \in I$, where $m \geq 1$, $n \geq 1$ are fixed integer. But R is not commutative, so we conclude that the primeness hypothesis in Theorem 1.1 is not superfluous.

3. The Results on Right Ideals

In this section we will prove our next Theorem for right ideals.

To prove this theorem, we need the following Lemma:

Lemma 3.1. *Let R be a prime ring with extended centroid C and I a nonzero right ideal of R. If for some $a,b \in R$, $(a[x,y]+[x,y]b)^n - ([x,y])^n = 0$ for all $x,y \in I$, then R satisfy a nontrivial generalized polynomial identity or there exists $\alpha \in C$ such that $(a-\alpha)I = 0$, $b \in C$ with $(b+\alpha)^n = 1$.*

Proof. By our hypothesis, for any $x_0 \in I$, R satisfies the following generalized identity

$$(a[x_0x, x_0y] + [x_0x, x_0y]b)^n - [x_0x, x_0y)^n = 0. \tag{3.1}$$

We assume that this is a trivial (GPI) for R, for otherwise we are done. If there exists $x_0 \in I$ such that $\{x_0, ax_0\}$ is linearly C-independent, then from above we have that R satisfies

$$a[x_0x, x_0y](a[x_0x, x_0y] + [x_0x, x_0y]b)^{n-1} = 0. \tag{3.2}$$

Again, since $\{x_0, ax_0\}$ is linearly C-independent, we have from above relation that R satisfies

$$(a[x_0x, x_0y])^2(a[x_0x, x_0y] + [x_0x, x_0y]b)^{n-2} = 0 \tag{3.3}$$

and hence $a[x_0x, x_0y]^n = 0$, which is nontrivial, a contradiction. Thus $\{x_0, ax_0\}$ is linearly dependent over C for all $x_0 \in I$, that is $(a-\alpha)I = 0$ for some $\alpha \in C$. Then (3.1) becomes

$$[x_0x, x_0y](b+\alpha))^n - [x_0x, x_0y]^n = 0. \tag{3.4}$$

Since this is trivial identity for R, we have that $b+\alpha \in C$, that is $b \in C$. Thus identity reduces to

$$((b+\alpha)^n - 1)[x_0x, x_0y]^n = 0. \tag{3.5}$$

Since this is trivial identity for R, we conclude $(b+\alpha)^n = 1$.

Lemma 3.2. *Let R be a prime ring with extended centroid C and I be a right ideal of R. Let F be an inner generalized derivation of R. If $(F([x,y]))^n = [x,y]^n$ for all $x,y \in I$, then one of the following holds:*
(i)$[I,I]I = 0$;
(ii) there exists $a \in U$ and $\alpha, \beta \in C$ such that $F(x) = (a+\beta)x$ for all $x \in R$, with $(a-\alpha)I = 0$ and $(\beta+\alpha)^n = 1$.

Proof. Since F is inner, there exist $a, b \in U$ such that $F(x) = ax + xb$ for all $x \in R$. If R does not satisfy any nontrivial (GPI), then by Lemma 3.1, we conclude that there exists $\alpha \in C$ such that $(a-\alpha)I = 0$, $b \in C$, $(b+\alpha)^n = 1$. In this case $F(x) = ax + xb = (a+b)x$ for all $x \in R$, where $(a-\alpha)I = 0$, $b \in C$ with $(b+\alpha)^n = 1$. This is our conclusion (2).
So we assume that R satisfies a nontrivial (GPI). If $I = R$, then by Lemma 2.1, either R is commutative or $a, b \in C$ with $(a+b)^m = 1$. In the last case we have $F(x) = \lambda x$ for all $x \in R$, with $\lambda^m = 1$. Thus conclusions (1) and (2) are obtained.

Now let $I \neq R$. In this case we want to prove that either $[I,I]I = 0$ or there exist $\alpha, \beta \in C$ such that $(a-\alpha)I = 0$ and $(b-\beta)I = 0$. To prove this, by contradiction, we suppose that there exist $c_1, c_2, \cdots, c_5 \in I$ such that

- $[c_1, c_2]c_3 \neq 0$;
- $(a-\alpha)c_4 \neq 0$ for all $\alpha \in C$ or $(b-\beta)c_5 \neq 0$ for all $\beta \in C$.

Now we show that this assumption leads a number of contradictions. Since R satisfies nontrivial (GPI), by [24], RC is a primitive ring having a nonzero socle H with a nonzero right ideal $J = IH$. Notice that H is simple, $J = JH$ and J satisfies the same basic conditions as I. Thus we replace R by H and I by J.

Then since R is a regular ring, for $c_1, c_2, \cdots, c_5 \in I$ there exists $e^2 = e \in R$ such that

$$eR = c_1R + c_2R + c_3R + c_4R + c_5R.$$

Then $e \in I$ and $ec_i = c_i$ for $i = 1, \cdots, 5$. Let $x \in R$. Then by our hypothesis we have

$$(a[e, ex(1-e)] + [e, ex(1-e)]b)^n = [e, ex(1-e)]^n \tag{3.6}$$

Left multiplying by $(1-e)$ we have $((1-e)aex)^n(1-e)=0$, that is $((1-e)aex)^{n+1}=0$ for all $x\in R$. By Levitzki's lemma [15, Lemma 1.1], we have $(1-e)aeR=0$ implying $(1-e)ae=0$. Analogously, right multiplying by e, we get $(1-e)be=0$. Therefore $ae=eae$ and $be=ebe$. Moreover, since R satisfies

$$e\{(a[x,y]+[x,y]b)^n-[x,y]^n\}e=0,$$

eRe satisfies

$$(eae[x,y]+[x,y]ebe)^n-[x,y]^n=0.$$

Then by Lemma 2.1, one of the following holds: (1) $[eRe,eRe]=0$, (2) $eae,ebe\in Ce$. Now $[eRe,eRe]=0$ implies $[eR,eR]eR=0$ which contradicts with the choices of c_1,c_2,c_3. Thus $eae=ae\in Ce$ and $ebe=be\in Ce$. Therefore, there exist $\alpha,\beta\in C$ such that $(a-\alpha)e=0$ and $(b-\beta)e=0$. This gives $(a-\alpha)eR=0$ and $(b-\beta)eR=0$. In any case this contradicts with the choices of c_4 and c_5.

In case $[I,I]I=0$, conclusion (1) is obtained. Let $(a-\alpha)I=0$ and $(b-\beta)I=0$ for some $\alpha,\beta\in C$. Then our hypothesis

$$(a[x,y]+[x,y]b)^n-[x,y]^n=0 \tag{3.7}$$

for all $x,y\in I$ gives

$$([x,y](b+\alpha))^n-[x,y]^n=0 \tag{3.8}$$

for all $x,y\in I$ and so

$$[x,y]^n(\beta+\alpha)^{n-1}(b+\alpha)-[x,y]^n=0 \tag{3.9}$$

for all $x,y\in I$. Right multiplying by $[x,y]$, (3.9) reduces to

$$[x,y]^n(\beta+\alpha)^n-[x,y]^n=0 \tag{3.10}$$

and hence $\{(\beta+\alpha)^n-1\}[x,y]^n=0$ for all $x,y\in I$. This implies either $(\beta+\alpha)^n=1$ or $[x,y]^n=0$ for all $x,y\in I$. The last relation implies $[x,y]z=0$ for all $x,y,z\in I$ (see [7, Lemma 2 (II)]). Now we assume first that $(\beta+\alpha)^n=1$. Then multiplying $\beta+\alpha$ in (3.9), we have

$$[x,y]^n(b+\alpha)-(\beta+\alpha)[x,y]^n=0 \tag{3.11}$$

for all $x,y \in I$, that is $[x,y]^n(b-\beta) = 0$ for all $x,y \in I$. Then again by [7], this relation yields either $b-\beta = 0$ that is $b = \beta \in C$ or $[x,y]z = 0$ for all $x,y,z \in I$. This implies that $[I,I]I = 0$,which is our conclusion (1). In case $(a-\alpha)I = 0$, $b = \beta \in C$ and $(\beta+\alpha)^n = 1$, we can write $F(x) = ax+xb = (a+\beta)x$ for all $x \in R$, with $(a-\alpha)I = 0$ and $(\beta+\alpha)^n = 1$. This is our conclusion (2).

Now we are in a position to prove our main theorem for right ideals.

Proof of Theorem 1.2. If F is inner generalized derivation of R, then by Lemma 3.2, we are done. Now let F be not inner. By [23], we have $F(x) = ax+d(x)$ for some $a \in U$ and a derivation d on U. Let $x,y \in I$. Then by [8], U satisfies

$$(a[xX,yY]+d([xX,yY]))^n = [xX,yY]^n$$

that is

$$(a[xX,yY]+[d(x)X+xd(X),yY]+[xX,d(y)Y+yd(Y)])^n-[xX,yY]^n = 0.$$

Since F is not inner, d is also not inner derivation. Then by Kharchenko's Theorem [19], U satisfies

$$(a[xX,yY]+[d(x)X+xZ_1,yY]+[xX,d(y)Y+yZ_2])^n-[xX,yY]^n = 0. \tag{3.12}$$

In particular for $X = 0$, we have $[xZ_1,yY]^n = 0$ for all $Z_1,Y \in U$. In particular, $[x,y]^n = 0$ for all $x,y \in I$. Then by [7, Lemma 2 (II)], $[x,y]z = 0$ for all $x,y,z \in I$. We conclude that $[I,I]I = 0$, which is our conclusion (1).

Example. *Let* $R = \begin{pmatrix} GF(2) & GF(2) \\ 0 & GF(2) \end{pmatrix}$. *We define maps* $F,d : R \to R$, *by* $F\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ *and* $d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. *Then* F *is a generalized derivation associated with the derivation* d *of* R. *Note that* R *is not prime for* $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$. *We see that for* $n = 2$ *and* $I = R$, $(F([x,y]))^n = [x,y]^n$ *for all* $x,y \in I$. *Since* $[I,I]I \neq 0$ *and* $F(x) \neq \pm x$ *for all* $x \in R$, *we conclude that the primeness hypothesis in Theorem 1.2 is not superfluous.*

4. The Results on Semiprime Rings and Banach Algebras

Now we prove our rest theorems in semiprime rings and Banach algebras.

Let R be a semiprime ring and U be its right Utumi quotient ring. It is well known that any derivation of a semiprime ring R can be uniquely extended to a derivation of U and so any derivation of R can be defined on the whole of U [22, Lemma 2].

By the standard theory of orthogonal completions for semiprime rings, we have the following lemma.

Lemma 4.1. ([4, Lemma 1 and Theorem 1] or [22, p.31-32]) *Let R be a 2-torsion free semiprime ring and P a maximal ideal of C. Then PU is a prime ideal of U invariant under all derivations of U. Moreover, $\bigcap\{PU \| P$ is a maximal ideal of C with U/PU 2-torsion free $\} = 0$.*

Proof of Theorem 1.3. We known the fact that any derivation of a semiprime ring R can be uniquely extended to a derivation of its right Utumi quotient ring U and so any derivation of R can be defined on the whole of U [22, Lemma 2]. Moreover R and U satisfy the same GPIs (see [8]) as well as same differential identities (see [22]). Thus by [23], we have $F(x) = ax + d(x)$ for some $a \in U$, a derivation d on U and hence $(a[x,y] + d([x,y]))^m = [x,y]^n$ for all $x, y \in U$.

Let $M(C)$ be the set of all maximal ideals of C and $P \in M(C)$. Now by the standard theory of orthogonal completions for semiprime rings (see [22, p.31-32]), we have PU is a prime ideal of U invariant under all derivations of U. Moreover, $\bigcap\{PU \mid P \in M(C)\} = 0$. Set $\overline{U} = U/PU$. Then derivation d canonically induces a derivation $\overline{d}$ on $\overline{U}$ defined by $\overline{d}(\overline{x}) = \overline{d(x)}$ for all $x \in U$. Therefore,

$$(\overline{a}[\overline{x},\overline{y}] + \overline{d}([\overline{x},\overline{y}]))^m = [\overline{x},\overline{y}]^n$$

for all $\overline{x},\overline{y} \in \overline{U}$. By Theorem 1.1 for prime ring case, we have for each $P \in M(C)$, either $[U,U] \subseteq PU$ or $[a,U] \subseteq PU$ and $d(U) \subseteq PU$. This gives that $[a,U][U,U] \subseteq PU$ for all $P \in M(C)$ and $d(U)[U,U] \subseteq PU$ for all $P \in M(C)$. Since $\bigcap\{PU \mid P \in M(C)\} = 0$, $[a,U][U,U] = 0$ and $d(U)[U,U] = 0$. In particular, $[a,R][R,R] = 0$ and $d(R)[R,R] = 0$. First case implies $a \in C$ and second case implies $d(R) \subseteq Z(R)$. Hence $F(x) = ax + d(x)$ for all $x \in R$, where $a \in C$ and d is a derivation of R such that $d(R) \subseteq Z(R)$.

By a Banach algebra, we shall mean a complex normed algebra A whose underlying vector space is Banach space. The Jacobson radical of A is the intersection of all primitive ideals of A and is denoted by $rad(A)$. Now we prove our theorem for Banach algebras.

Proof of Theorem 1.4. By hypothesis F is continuous generalized derivation. Since we know that left multiplication map is continuous, we get that d is continuous. In [28], Sinclair proved that any continuous derivation of a Banach algebra leaves the primitive ideals invariant. Hence, for any primitive ideal P of A, it is obvious that $F(P) \subseteq aP + d(P) \subseteq P$. It means that continuous generalized derivation F leaves the primitive ideals invariant. Denote $A/P = \bar{A}$ for any primitive ideal P. Thus we can define the generalized derivation $F_p : \bar{A} \to \bar{A}$ by $F_p(\bar{x}) = F_p(x+P) = F(x) + P = ax + d(x) + P$ for all $\bar{x} \in \bar{A}$, where $A/P = \bar{A}$ is a factor Banach algebra. Since P is primitive ideal, the factor algebra $\bar{A}$ is primitive and so it is prime and semisimple. The hypothesis $(F([x,y])^m - [x,y]^n \in rad(A)$ yields that $(F_p[\bar{x},\bar{y}])^m - [\bar{x},\bar{y}]^n = \bar{0}$ for all $\bar{x},\bar{y} \in \bar{A}$. By Theorem 1.1, we have either $\bar{A}$ is commutative or $\bar{d} = \bar{0}$.

Assume first that $\bar{A}$ is commutative. By a result of Johnson and Sinclair [18] every linear derivation on a semisimple Banach algebra is continuous. Thus $\bar{d}$ is continuous in $\bar{A}$. In [29], Singer and Wermer proved that any continuous linear derivation on a commutative Banach algebra maps the algebra into the radical. Hence $\bar{d} = \bar{0}$ in $\bar{A}$.

Therefore, in any case we have that $\bar{d} = \bar{0}$ in $\bar{A}$, that is $d(A) \subseteq P$ for any primitive ideal P of A and hence we get $d(A) \subseteq rad(A)$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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References

[1] N. Argac and H. G. Inceboz, Derivations of prime and semiprime rings, *J. Korean Math. Soc.* 46 (5) (2009), 997-1005.

[2] M. Ashraf and N. Rehman, On commutativity of rings with derivations, *Result. Math.* 42 (2002), 3-8.

[3] K. I. Beidar, W. S. Martindale and A. V. Mikhalev, Rings with Generalized Identities, Monographs and Textbooks in Pure and Applied Mathematics, 196. Marcel Dekker, Inc., New York, 1996.

[4] K. I. Beidar, Rings of quotients of semiprime rings, *Vestnik Moskov. Univ. Ser I Math. Meh. (Engl. Transl:. Moscow Univ. Math. Bull.)* 33 (1978), 36-42.

[5] M. Brešar and J. Vukman, On left derivations and related mappings, *Proc. Amer. Math. Soc.* 110 (1990), 7–16.

[6] M. Brešar, On the distance of the composition of the two derivations to be the generalized derivations, *Glasgow Math. J.* 33 (1) (1991), 89-93.

[7] C. M. Chang, Power central values of derivations on multilinear polynomials, *Taiwanese J. Math.*, 7 (2) (2003), 329-338.

[8] C. L. Chuang, GPIs having coefficients in Utumi quotient rings *Proc. Amer. Math. Soc.* 103 (1988), 723-728.

[9] M. N. Daif and H. E. Bell, Remarks on derivations on semiprime rings, *Internat. J. Math. & Math. Sci.*, 15 (1) (1992), 205-206.

[10] V. De Filippis, Generalized derivations in prime rings and noncommutative Banach algebras, *Bull. Korean Math. Soc.,* 45 (2008), 621-629.

[11] V. De Filippis and S. Huang, Generalized derivations on semi prime rings, *Bull. Korean Math. Soc.* 48 (6) (2011),1253-1259.

[12] B. Dhara, Remarks on generalized derivations in prime and semiprime rings, *Internat. J. Math. & Math. Sc.* Volume 2010, Article ID 646587, 6 pages.

[13] T. S. Erickson, W. S. Martindale III and J. M. Osborn, Prime nonassociative algebras, *Pacific J. Math.* 60 (1975), 49-63.

[14] C. Faith and Y. Utumi, On a new proof of Litoff's theorem, *Acta Math. Acad. Sci. Hung.* 14 (1963), 369-371.

[15] I. N. Herstein, *Topics in ring theory*, Univ. of Chicago Press, Chicago, (1969).

[16] I. N. Herstein, *Center-like elements in prime rings*, J. Algebra 60 (1979) 567-574.

[17] N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Pub. 37. Providence, RI: Amer. Math. Soc. (1964).

[18] B. E. Johnson and A. M. Sinclair, Continuity of derivations and a problem of Kaplansky, *Amer. J. Math.* 90 (1968), 1067-1073.

[19] V. K. Kharchenko, Differential identity of prime rings, *Algebra and Logic* 17 (1978), 155-168.

[20] B. Kim, On the derivations of semiprime rings and noncommutative Banach algebras, *Acta Math. Sinica* 16 (1) (2000), 21-28.

[21] C. Lanski, An engel condition with derivation, *Proc. Amer. Math. Soc.* 118 (3) (1993), 731-734.

[22] T. K. Lee, Semiprime rings with differential identities, *Bull. Inst. Math. Acad. Sinica*, 20 (1) (1992), 27-38.

[23] T. K. Lee, Generalized derivations of left faithful rings, *Comm. Algebra*, 27 (8) (1999), 4057-4073.

[24] W. S. Martindale III, Prime rings satisfying a generalized polynomial identity, *J. Algebra* 12 (1969), 576-584.

[25] M. Mathieu, Properties of the product of two derivations of a C^*-algebra, *Canad. Math. Bull.* 32 (1989), 490–497.

[26] K. H. Park, On derivations in noncommutative semiprime rings and Banach algebras, *Bull. Korean Math. Soc.,* 42 (2005), 671-678.

[27] M. A. Quadri, M. S. Khan and N. Rehman, Generalized derivations and commutativity of prime rings *Indian J. Pure Appl. Math.* 34 (9) (2003), 1393-1396.

[28] A. M. Sinclair, Continuous derivations on Banach algebras, *Proc. Amer. Math. Soc.* 20 (1969), 166-170.

[29] I. M. Singer and J. Wermer, Derivations on commutative normed algebras, *Math. Ann.* 129 (1955), 260–264.

[30] M. Thomas, The image of a derivation is contained in the radical, *Ann. Math.* 128 (1988), 435–460.

[31] J. Vukman, On derivations in prime rings and Banach algebras, *Proc. Amer. Math. Soc.* 116 (4) (1992), 877–884.