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HADAMARD AND FEJÉR-HADAMARD TYPE INTEGRAL INEQUALITIES FOR HARMONICALLY CONVEX FUNCTIONS VIA AN EXTENDED GENERALIZED MITTAG-LEFFLER FUNCTION

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Abstract. In this paper generalized form of some new inequalities of the Hadamard and the Fejér-Hadamard type have been established. Fractional integral operators due to an extended generalized Mittag-Leffler function via harmonically convex functions are utilized to obtain the new results.

Keywords: Harmonically convex function; Hadamard inequality; Mittag-Leffler function; fractional integral operators.

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1. Introduction

Convex functions are equivalently defined by the Hadamard inequality stated in the following theorem.

Theorem 1.1. Let I be an interval of real numbers and $f : I \to \mathbb{R}$ be a convex function on I.

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Then for $a,b \in I$, $a < b$ the following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

holds.

It is always in the focus of researchers especially working in the field of mathematical analysis. Now a days it is under consideration via a variety of fractional integral operators and fractional differential operators (see for example [1, 2, 4, 6, 7, 11] and references there in). In this paper, we are interested to find the Hadamard and the Fejér-Hadamard and related fractional inequalities for harmonically convex functions via fractional integral operators due to an extended generalized Mittag-Leffler function. First we give the definition of harmonically convex function as follows.

Definition 1.1. Let I be an interval of non-zero real numbers. A function $f : I \to \mathbb{R}$ is said to be harmonically convex function, if

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq tf(b)+(1-t)f(a) \tag{1}$$

holds for all $a,b \in I$ and $t \in [0,1]$. If inequality in (1) is reversed, then f is said to be harmonically concave function for more detail one can see [6].

Definition 1.2. [7] A function $h : [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be harmonically symmetric about $\frac{a+b}{2ab}$, if

$$h\left(\frac{1}{x}\right) = h\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)$$

for $x \in [a,b]$.

Now we define the extended generalized Mittag-Leffler function $E^{\gamma,\delta,k,c}_{\mu,\nu,l}(t;p)$ as follows.

Definition 1.3. [3] Let $\mu,\nu,l,\gamma,c \in \mathbb{C}$, $\Re(\mu),\Re(\nu),\Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta+\Re(\mu)$. Then the extended generalized Mittag-Leffler function $E^{\gamma,\delta,k,c}_{\mu,\nu,l}(t;p)$ is defined by

$$E^{\gamma,\delta,k,c}_{\mu,\nu,l}(t;p) = \sum_{n=0}^{\infty}\frac{\beta_p(\gamma+nk,c-\gamma)}{\beta(\gamma,c-\gamma)}\frac{(c)_{nk}}{\Gamma(\mu n+\nu)}\frac{t^n}{(l)_{n\delta}}, \tag{2}$$

where β_p is the generalized beta function defined by

$$\beta_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t(1-t)}}dt$$

and $(c)_{nk}$ is the Pochhammer symbol defined as $(c)_{nk} = \frac{\Gamma(c+nk)}{\Gamma(c)}$.

Remark 1.1. (2) is a generalization of the following functions.

(i) setting $p = 0$, it reduces to the Salim-Faraj function $E^{\gamma,\delta,k,c}_{\mu,\nu,l}(t)$ defined in [10],

(ii) setting $l = \delta = 1$, it reduces the function $E^{\gamma,k,c}_{\mu,\nu}(t;p)$ defined by Rahman et al. in [9],

(iii) setting $p = 0$ and $l = \delta = 1$, it reduces to the Shukla-Prajapati function $E^{\gamma,k}_{\mu,\nu}(t)$ defined in [12] see also [13],

(iv) setting $p = 0$ and $l = \delta = k = 1$, it reduces to the Prabhakar function $E^{\gamma}_{\mu,\nu}(t)$ defined in [8].

The corresponding generalized fractional integral operators $\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega,a^+}f$ and $\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega,b^-}f$ are defined as follows.

Definition 1.4. [3] Let $\omega, \mu, \nu, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\nu), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Let $f \in L_1[a,b]$ and $x \in [a,b]$. Then the generalized fractional integral operators $\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega,a^+}f$ and $\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega,b^-}f$ are defined by

$$\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega,a^+}f\right)(x;p) = \int_a^x (x-t)^{\nu-1} E^{\gamma,\delta,k,c}_{\mu,\nu,l}(\omega(x-t)^{\mu};p)f(t)dt \tag{3}$$

and

$$\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega,b^-}f\right)(x;p) = \int_x^b (t-x)^{\nu-1} E^{\gamma,\delta,k,c}_{\mu,\nu,l}(\omega(t-x)^{\mu};p)f(t)dt. \tag{4}$$

Remark 1.2. (3) and (4) are the generalization of the following fractional integral operators.

(i) setting $p = 0$, it reduces to the fractional integral operators defined by Salim-Faraj in [10],

(ii) setting $l = \delta = 1$, it reduces to the fractional integral operators defined by Rahman et al. in [9],

(iii) setting $p = 0$ and $l = \delta = 1$, it reduces to the fractional integral operators defined by Srivastava-Tomovski in [13],

(iv) setting $p = 0$ and $l = \delta = k = 1$, it reduces to the fractional integral operators defined by Prabhakar in [8],

(v) setting $p = \omega = 0$, it reduces to the right-sided and left-sided Riemann-Liouville fractional integrals.

Definition 1.5. [14] Let $f \in L_1[a,b]$. Then Riemann-Liouville fractional integral operators of f of order ν are defined by

$$I_{a^+}^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t)dt, \ x > a$$

and

$$I_{b_-}^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_x^b (t-x)^{\nu-1} f(t)dt, \ x < b.$$

A lot of authors of this age are working on inequalities involving fractional integral operators for example for Riemann-Liouville, Caputo, Hillfer, Canvati etc [3, 5, 11].

Kunt et al. in [7], produced the following result for harmonically convex functions.

Theorem 1.2. Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a harmonically convex function on $[a,b]$ for $a,b \in I$ with $a < b$. If $f \in L[a,b]$, then the following inequalities for fractional integrals hold

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{\Gamma(\nu+1)}{2^{1-\nu}} \left(\frac{ab}{b-a}\right)^{\nu} \left(I^{\nu}_{\frac{a+b}{2ab}-} f \circ g\left(\frac{1}{b}\right) + I^{\nu}_{\frac{a+b}{2ab}+} f \circ g\left(\frac{1}{a}\right)\right)$$
$$\leq \frac{f(a)+f(b)}{2}$$

where $g(t) = \frac{1}{t}$ for $t \in [\frac{1}{b}, \frac{1}{a}]$.

Chen and Wu in [2] presented the following Fejér-Hadamard inequality for harmonically convex functions.

Theorem 1.3. Let $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function on $[a,b]$ for $a,b \in I$ with $a < b$. If $f \in L[a,b]$ and $g : [a,b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is non-negative integrable as well as harmonically symmetric about $\frac{a+b}{2ab}$, then the following integral inequalities hold

$$f\left(\frac{2ab}{a+b}\right) \int_a^b \frac{g(x)}{x^2} dx \leq \int_a^b \frac{f(x)g(x)}{x^2} dx \leq \frac{f(a)+f(b)}{2} \int_a^b \frac{g(x)}{x^2} dx.$$

In the next section first we prove the Hadamard type inequality for harmonically convex functions via generalized fractional integral operators defined in (3) and (4). Also we produce the Hadamard type inequalities given in [1, 2, 6, 7]. Then we investigate the Fejér-Hadamard type inequalities via fractional integral operators defined in (3) and (4) and reproduce such results given in [1, 2, 6, 7].

2. Main results

In this section we give our results.

Theorem 2.1. Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a function such that $f\in L_1[a,b]$ with $a<b$. If f is a harmonically convex function on $[a,b]$, then the following inequalities for generalized fractional integral operators hold

$$
\begin{aligned}
(5)\qquad & f\left(\frac{2ab}{a+b}\right)\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{1}{a}-}1\right)\left(\frac{1}{b};p\right)\\
&\leq \frac{1}{2}\left(\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{1}{a}-}f\circ g\right)\left(\frac{1}{b};p\right)+\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{1}{b}+}f\circ g\right)\left(\frac{1}{a};p\right)\right)\\
&\leq \frac{f(a)+f(b)}{2}\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{1}{b}+}1\right)\left(\frac{1}{a};p\right),
\end{aligned}
$$

where $\omega'=\omega(\frac{ab}{b-a})^{\mu}$ and $g(t)=\frac{1}{t}$ for $t\in[\frac{1}{b},\frac{1}{a}]$.

Proof. Since f is harmonically convex function on $[a,b]$, therefore for all $x,y\in[a,b]$ $f\left(\frac{2xy}{x+y}\right)\leq\frac{f(x)+f(y)}{2}$. For $x=\frac{ab}{tb+(1-t)a}$ and $y=\frac{ab}{ta+(1-t)b}$, we have

$$
(6)\qquad 2f\left(\frac{2ab}{a+b}\right)\leq f\left(\frac{ab}{tb+(1-t)a}\right)+f\left(\frac{ab}{ta+(1-t)b}\right).
$$

Multiplying (6) by $t^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}(\omega t^{\mu};p)$ on both sides and integrating over $[0,1]$, we have

$$
\begin{aligned}
2f\left(\frac{2ab}{a+b}\right)\int_0^1 t^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}(\omega t^{\mu};p)dt &\leq \int_0^1 t^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}(\omega t^{\mu};p)f\left(\frac{ab}{tb+(1-t)a}\right)dt\\
&+\int_0^1 t^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}(\omega t^{\mu};p)f\left(\frac{ab}{ta+(1-t)b}\right)dt.
\end{aligned}
$$

If we put in above $x=\frac{tb+(1-t)a}{ab}$ and $y=\frac{ta+(1-t)b}{ab}$, then we have the following inequality

$$
\begin{aligned}
(7)\qquad & 2f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{ab}{b-a}\right)^{\nu-1}\left(x-\frac{1}{b}\right)^{\nu-1}\\
&E^{\gamma,\delta,k,c}_{\mu,\nu,l}\left(\omega\left(\frac{ab}{b-a}\right)^{\mu}\left(x-\frac{1}{b}\right)^{\mu};p\right)\left(\frac{ab}{b-a}\right)dx\\
&\leq\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{ab}{b-a}\right)^{\nu-1}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}\left(\omega\left(\frac{ab}{b-a}\right)^{\mu}\left(x-\frac{1}{b}\right)^{\mu};p\right)\\
&f\left(\frac{1}{x}\right)\left(\frac{ab}{b-a}\right)dx+\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{ab}{b-a}\right)^{\nu-1}\left(\frac{1}{a}-y\right)^{\nu-1}\\
&E^{\gamma,\delta,k,c}_{\mu,\nu,l}\left(\omega\left(\frac{ab}{b-a}\right)^{\mu}\left(\frac{1}{a}-y\right)^{\mu};p\right)f\left(\frac{1}{y}\right)\left(\frac{ab}{b-a}\right)dy.
\end{aligned}
$$

After simplification, we get

$$\begin{aligned}
&2f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k,c}\left(\omega'\left(x-\frac{1}{b}\right)^{\mu};p\right)dx\\
&\leq \int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k,c}\left(\omega'\left(x-\frac{1}{b}\right)^{\mu};p\right)f\left(\frac{1}{x}\right)dx\\
&+\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-y\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k,c}\left(\omega'\left(\frac{1}{a}-y\right)^{\mu};p\right)f\left(\frac{1}{y}\right)dy.
\end{aligned}$$

By using Definition 1.4, we get

$$\begin{aligned}
&2f\left(\frac{2ab}{a+b}\right)\left(\varepsilon_{\mu,\nu,l,\omega',\frac{1}{a}-}^{\gamma,\delta,k,c}1\right)\left(\frac{1}{b};p\right)\leq\left(\varepsilon_{\mu,\nu,l,\omega',\frac{1}{a}-}^{\gamma,\delta,k,c}f\circ g\right)\left(\frac{1}{b};p\right)\\
&+\left(\varepsilon_{\mu,\nu,l,\omega',\frac{1}{b}+}^{\gamma,\delta,k,c}f\circ g\right)\left(\frac{1}{a};p\right).
\end{aligned}\tag{8}$$

Again by using that harmonically convexity of f for $t\in[0,1]$, one can have

$$f\left(\frac{ab}{tb+(1-t)a}\right)+f\left(\frac{ab}{ta+(1-t)b}\right)\leq f(a)+f(b).\tag{9}$$

Multiplying (9) by $t^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega t^{\mu};p)$ on both sides and integrating over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 t^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega t^{\mu};p)f\left(\frac{ab}{tb+(1-t)a}\right)dt\\
&+\int_0^1 t^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega t^{\mu};p)f\left(\frac{ab}{ta+(1-t)b}\right)dt\\
&\leq (f(a)+f(b))\int_0^1 t^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega t^{\mu};p)dt.
\end{aligned}\tag{10}$$

By putting in above $x=\frac{tb+(1-t)a}{ab}$ and $y=\frac{ta+(1-t)b}{ab}$, then after simplifications, we have

$$\begin{aligned}
&\left(\varepsilon_{\mu,\nu,l,\omega',\frac{1}{a}-}^{\gamma,\delta,k,c}f\circ g\right)\left(\frac{1}{b};p\right)+\left(\varepsilon_{\mu,\nu,l,\omega',\frac{1}{b}+}^{\gamma,\delta,k,c}f\circ g\right)\left(\frac{1}{a};p\right)\\
&\leq (f(a)+f(b))\left(\varepsilon_{\mu,\nu,l,\omega',\frac{1}{b}+}^{\gamma,\delta,k,c}1\right)\left(\frac{1}{a};p\right).
\end{aligned}\tag{11}$$

Inequalities (8) and (11) provide the required inequality (5). □

Remark 2.2. In Theorem 2.1.

(i) If we put $p=0$, then we get [1, Theorem 3.1].

(ii) If we put $\omega=p=0$, then we get [6, Theorem 4].

Now we prove the next result.

Theorem 2.3. Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a harmonically convex function such that $f \in L_1[a,b]$ with $a < b$. If f is a harmonically symmetric about $\frac{a+b}{2ab}$, then the following inequalities for generalized fractional integral operators hold

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right)&\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}-}1\right)\left(\frac{1}{b};p\right) \\ &\leq \frac{1}{2}\left(\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}+}f\circ g\right)\left(\frac{1}{a};p\right)+\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}-}f\circ g\right)\left(\frac{1}{b};p\right)\right) \\ &\leq \frac{f(a)+f(b)}{2}\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}+}1\right)\left(\frac{1}{a};p\right), \end{aligned} \tag{12}$$

where $\omega' = \omega(\frac{ab}{b-a})^{\mu}$ and $g(t) = \frac{1}{t}$, $t \in [\frac{1}{b}, \frac{1}{a}]$.

Proof. Multiplying (6) by $2t^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}(\omega t^{\mu};p)$ on both sides and integrating over $[0,\frac{1}{2}]$, we have

$$\begin{aligned} 2f\left(\frac{2ab}{a+b}\right)\int_0^{\frac{1}{2}} t^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}(\omega t^{\mu};p)dt &\leq \int_0^{\frac{1}{2}} t^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}(\omega t^{\mu};p)f\left(\frac{ab}{ta+(1-t)b}\right)dt \\ &+\int_0^{\frac{1}{2}} t^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}(\omega t^{\mu};p)f\left(\frac{ab}{tb+(1-t)a}\right)dt. \end{aligned} \tag{13}$$

Putting in above $x = \frac{tb+(1-t)a}{ab}$ that is $\frac{ab}{ta+(1-t)b} = \frac{1}{\frac{1}{a}+\frac{1}{b}-x}$, then we have

$$\begin{aligned} &2f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\nu}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}\left(\omega\left(\frac{ab}{b-a}\right)^{\mu}\left(x-\frac{1}{b}\right)^{\mu};p\right)dx \\ &\leq \int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\nu}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}\left(\omega\left(\frac{ab}{b-a}\right)^{\mu}\left(x-\frac{1}{b}\right)^{\mu};p\right) \\ &f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)dx+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\nu}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}\left(\omega\left(\frac{ab}{b-a}\right)^{\mu}\right) \\ &\left(\left(x-\frac{1}{b}\right)^{\mu};p\right)f\left(\frac{1}{x}\right)dx. \end{aligned}$$

Since f is harmonically symmetric about $\frac{a+b}{2ab}$, we replace $\frac{1}{a}+\frac{1}{b}-x$ by x in first term on R.H.S. of the above inequality and after simplification, we have

$$\begin{aligned}
&2f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(x-\frac{1}{b}\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k,c}\left(\omega'\left(x-\frac{1}{b}\right)^{\mu};p\right)dx\\
&\leq\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\frac{1}{a}-x\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k,c}\left(\omega'\left(\frac{1}{a}-x\right)^{\mu};p\right)f\left(\frac{1}{x}\right)dx\\
&+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(x-\frac{1}{b}\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k,c}\left(\omega'\left(x-\frac{1}{b}\right)^{\mu};p\right)f\left(\frac{1}{x}\right)dx.
\end{aligned}$$

By using Definition 1.4, we get

$$\begin{aligned}
&2f\left(\frac{2ab}{a+b}\right)\left(\varepsilon_{\mu,\nu,l,\omega',\frac{a+b}{2ab}-}^{\gamma,\delta,k,c}1\right)\left(\frac{1}{b};p\right)\\
&\leq\left(\varepsilon_{\mu,\nu,l,\omega',\frac{a+b}{2ab}+}^{\gamma,\delta,k,c}f\circ g\right)\left(\frac{1}{a};p\right)+\left(\varepsilon_{\mu,\nu,l,\omega',\frac{a+b}{2ab}-}^{\gamma,\delta,k,c}f\circ g\right)\left(\frac{1}{b};p\right).
\end{aligned}\tag{14}$$

Now multiplying (9) by $t^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega t^{\mu};p)$ on both sides and integrating over $[0,\frac{1}{2}]$, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}}t^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega t^{\mu};p)f\left(\frac{ab}{tb+(1-t)a}\right)dt\\
&+\int_0^{\frac{1}{2}}t^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega t^{\mu};p)f\left(\frac{ab}{ta+(1-t)b}\right)dt\\
&\leq(f(a)+f(b))\int_0^{\frac{1}{2}}t^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k,c}(\omega t^{\mu};p)dt.
\end{aligned}\tag{15}$$

Putting in above $x=\frac{tb+(1-t)a}{ab}$ that is $\frac{ab}{ta+(1-t)b}=\frac{1}{\frac{1}{a}+\frac{1}{b}-x}$, then we have

$$\begin{aligned}
&\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\nu}\left(x-\frac{1}{b}\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k,c}\left(\omega\left(\frac{ab}{b-a}\right)^{\mu}\left(x-\frac{1}{b}\right)^{\mu};p\right)f\left(\frac{1}{x}\right)dx\\
&+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\nu}\left(x-\frac{1}{b}\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k,c}\left(\omega\left(\frac{ab}{b-a}\right)^{\mu}\left(x-\frac{1}{b}\right)^{\mu};p\right)\\
&f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)dx\leq(f(a)+f(b))\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\nu}\left(x-\frac{1}{b}\right)^{\nu-1}\\
&E_{\mu,\nu,l}^{\gamma,\delta,k,c}\left(\omega\left(\frac{ab}{b-a}\right)^{\mu}\left(x-\frac{1}{b}\right)^{\mu};p\right)dx.
\end{aligned}$$

Since f is harmonically symmetric about $\frac{a+b}{2ab}$, therefore by replacing $\frac{1}{a}+\frac{1}{b}-x$ with x in first term of L.H.S. of above inequality and after simple calculation, we have

$$\begin{aligned}(16)\qquad &\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}+}f\circ g\right)\left(\frac{1}{a};p\right)+\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}-}f\circ g\right)\left(\frac{1}{b};p\right)\\ &\leq (f(a)+f(b))\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}+}1\right)\left(\frac{1}{a};p\right).\end{aligned}$$

Inequalities (14) and (16) provide the required inequality (12). □

Remark 2.4. In Theorem 2.3.

(i) If we put p=0, then we get [1, Theorem 3.3].

(ii) If we put $\omega=p=0$, then we get Theorem 1.2.

To prove our next result first we give the following lemma.

Lemma 2.5. Let $f:[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be integrable and harmonically symmetric about $\frac{a+b}{2ab}$. Then the following equality for generalized fractional integral operators hold

$$\begin{aligned}(17)\qquad &\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+}f\circ g\right)\left(\frac{1}{a};p\right)=\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-}f\circ g\right)\left(\frac{1}{b};p\right)\\ &=\frac{1}{2}\left(\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+}f\circ g\right)\left(\frac{1}{a};p\right)+\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-}f\circ g\right)\left(\frac{1}{b};p\right)\right),\end{aligned}$$

where $g(t)=\frac{1}{t}$ for $t\in[\frac{1}{b},\frac{1}{a}]$.

Proof. Since f is harmonically symmetric about $\frac{a+b}{2ab}$, we have $f(\frac{1}{x})=f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)$. By the definition of generalized fractional integral operators, we have

$$(18)\qquad \left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+}f\circ g\right)\left(\frac{1}{a};p\right)=\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}\left(\omega\left(\frac{1}{a}-t\right)^{\mu};p\right)f\left(\frac{1}{t}\right)dt$$

replace t by $\frac{1}{a}+\frac{1}{b}-x$ in above, we have

$$\begin{aligned}&\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+}f\circ g\right)\left(\frac{1}{a};p\right)\\ &=\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}\left(\omega\left(x-\frac{1}{b}\right)^{\mu};p\right)f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)dx\\ &=\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}\left(\omega\left(x-\frac{1}{b}\right)^{\mu};p\right)f\left(\frac{1}{x}\right)dx.\end{aligned}$$

By using Definition 1.4, we get

$$\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+} f\circ g\right)\left(\frac{1}{a};p\right)=\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-} f\circ g\right)\left(\frac{1}{b};p\right). \tag{19}$$

By adding $\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+} f\circ g\right)(\frac{1}{a};p)$ in both sides of above, we have

$$\begin{aligned} 2\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+} f\circ g\right)\left(\frac{1}{a};p\right)&=\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+} f\circ g\right)\left(\frac{1}{a};p\right)\\ &+\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-} f\circ g\right)\left(\frac{1}{b};p\right)\end{aligned} \tag{20}$$

which is required. □

Theorem 2.6. Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a harmonically convex function with $a<b$. Let $f\in L_1[a,b]$ and also let $g:[a,b]\to\mathbb{R}$ be a non-negative, integrable and harmonically symmetric about $\frac{a+b}{2ab}$, then the following inequalities for generalized fractional integral operators hold

$$\begin{aligned} &f\left(\frac{2ab}{a+b}\right)\left[\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}+} g\circ h\right)\left(\frac{1}{a};p\right)+\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}-} g\circ h\right)\left(\frac{1}{b};p\right)\right]\\ &\leq\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}+} fg\circ h\right)\left(\frac{1}{a};p\right)+\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}-} fg\circ h\right)\left(\frac{1}{b};p\right)\\ &\leq\frac{f(a)+f(b)}{2}\left[\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}+} g\circ h\right)\left(\frac{1}{a};p\right)+\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}-} g\circ h\right)\left(\frac{1}{b};p\right)\right],\end{aligned} \tag{21}$$

where $\omega'=\omega(\frac{ab}{b-a})^{\mu}$ and $h(t)=\frac{1}{t}$ for $t\in[\frac{1}{b},\frac{1}{a}]$.

Proof. Multiplying (6) by $t^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}(\omega t^{\mu};p)g\left(\frac{ab}{tb+(1-t)a}\right)$ on both sides and integrating over $[0,\frac{1}{2}]$, we have

$$\begin{aligned} &2f\left(\frac{2ab}{a+b}\right)\int_0^{\frac{1}{2}}t^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}(\omega t^{\mu};p)g\left(\frac{ab}{tb+(1-t)a}\right)dt\\ &\leq\int_0^{\frac{1}{2}}t^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}(\omega t^{\mu};p)f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt\\ &+\int_0^{\frac{1}{2}}t^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}(\omega t^{\mu};p)f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt.\end{aligned} \tag{22}$$

Putting in above $x=\frac{tb+(1-t)a}{ab}$ that is $\frac{ab}{ta+(1-t)b}=\frac{1}{\frac{1}{a}+\frac{1}{b}-x}$, then we have

$$\begin{aligned}(23)\qquad & 2f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\nu}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}\left(\omega\left(\frac{ab}{b-a}\right)^{\mu}\left(x-\frac{1}{b}\right)^{\mu};p\right)\\ & g\left(\frac{1}{x}\right)dx\leq\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\nu}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}\left(\omega\left(\frac{ab}{b-a}\right)^{\mu}\left(x-\frac{1}{b}\right)^{\mu};p\right)\\ & f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)g\left(\frac{1}{x}\right)dx+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\nu}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}\left(\omega\left(\frac{ab}{b-a}\right)^{\mu}\right)\\ & \left(\left(x-\frac{1}{b}\right)^{\mu};p\right)f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)dx.\end{aligned}$$

Since f is harmonically symmetric about $\frac{a+b}{2ab}$, therefore after simplification (23), becomes

$$\begin{aligned}(24)\qquad & 2f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}\left(\omega'\left(x-\frac{1}{b}\right)^{\mu};p\right)g\left(\frac{1}{x}\right)dx\\ & \leq\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\frac{1}{a}-x\right)^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}\left(\omega'\left(\frac{1}{a}-x\right)^{\mu};p\right)f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)dx\\ & +\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}\left(\omega'\left(x-\frac{1}{b}\right)^{\mu};p\right)f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)dx.\end{aligned}$$

By using Definition 1.4, we get

$$\begin{aligned}& 2f\left(\frac{2ab}{a+b}\right)\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}-}g\circ h\right)\left(\frac{1}{b};p\right)\\ & \leq\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}+}fg\circ h\right)\left(\frac{1}{a};p\right)+\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}-}fg\circ h\right)\left(\frac{1}{b};p\right).\end{aligned}$$

Using Lemma 2.5 in above inequality, we have

$$\begin{aligned}(25)\qquad & f\left(\frac{2ab}{a+b}\right)\left[\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}-}g\circ h\right)\left(\frac{1}{b};p\right)+\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}+}g\circ h\right)\left(\frac{1}{a};p\right)\right]\\ & \leq\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}+}fg\circ h\right)\left(\frac{1}{a};p\right)+\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}-}fg\circ h\right)\left(\frac{1}{b};p\right).\end{aligned}$$

Now multiplying (9) by $t^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}(\omega t^{\mu};p)g\left(\frac{ab}{tb+(1-t)a}\right)$ on both sides and integrating over $[0,\frac{1}{2}]$, we have

$$\begin{aligned}(26)\qquad &\int_0^{\frac{1}{2}} t^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}(\omega t^{\mu};p)f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt\\ &+\int_0^{\frac{1}{2}} t^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}(\omega t^{\mu};p)f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt\\ &\leq (f(a)+f(b))\int_0^{\frac{1}{2}} t^{\nu-1}E^{\gamma,\delta,k,c}_{\mu,\nu,l}(\omega t^{\mu};p)g\left(\frac{ab}{tb+(1-t)a}\right)dt.\end{aligned}$$

By putting $x=\frac{tb+(1-t)a}{ab}$ and using harmonically symmetry of f with respect to $\frac{a+b}{2ab}$ in above then after simplification, we have

$$\begin{aligned}(27)\qquad &\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}+}fg\circ h\right)\left(\frac{1}{a};p\right)+\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}-}fg\circ h\right)\left(\frac{1}{b};p\right)\\ &\leq (f(a)+f(b))\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}+}g\circ h\right)\left(\frac{1}{a};p\right).\end{aligned}$$

Using Lemma 2.5 in (27), we have

$$\begin{aligned}(28)\qquad &\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}+}fg\circ h\right)\left(\frac{1}{a};p\right)+\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}-}fg\circ h\right)\left(\frac{1}{b};p\right)\\ &\leq \frac{(f(a)+f(b))}{2}\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}-}g\circ h\right)\left(\frac{1}{b};p\right)+\left(\varepsilon^{\gamma,\delta,k;c}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}+}g\circ h\right)\left(\frac{1}{a};p\right).\end{aligned}$$

Inequalities (25) and (28) provide the required inequality (21).

□

Remark 2.7. In Theorem 2.6.

(i) If we put p=0, then we get [1, Theorem 3.6].

(ii) If we put $\omega=p=0$ and $\nu=1$, then we get Theorem 1.3.

(iii) If we put $\omega=p=0$ and $g(x)=1$, then we get Theorem 1.2.

Corollary 2.8. In Theorem 2.6, if we put $\omega = p = 0$, then we get the following inequalities via Riemann-Liouville fractional integral operators

$$f\left(\frac{2ab}{a+b}\right)\left[\left(I^{\nu}_{\frac{a+b}{2ab}+}g\circ h\right)\left(\frac{1}{a}\right)+\left(I^{\nu}_{\frac{a+b}{2ab}-}g\circ h\right)\left(\frac{1}{b}\right)\right]$$
$$\leq \left(I^{\nu}_{\frac{a+b}{2ab}+}fg\circ h\right)\left(\frac{1}{a}\right)+\left(I^{\nu}_{\frac{a+b}{2ab}-}fg\circ h\right)\left(\frac{1}{b}\right)$$
$$\leq \frac{f(a)+f(b)}{2}\left[\left(I^{\nu}_{\frac{a+b}{2abb}+}g\circ h\right)\left(\frac{1}{a}\right)+\left(I^{\nu}_{\frac{a+b}{2ab}-}g\circ h\right)\left(\frac{1}{b}\right)\right].$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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